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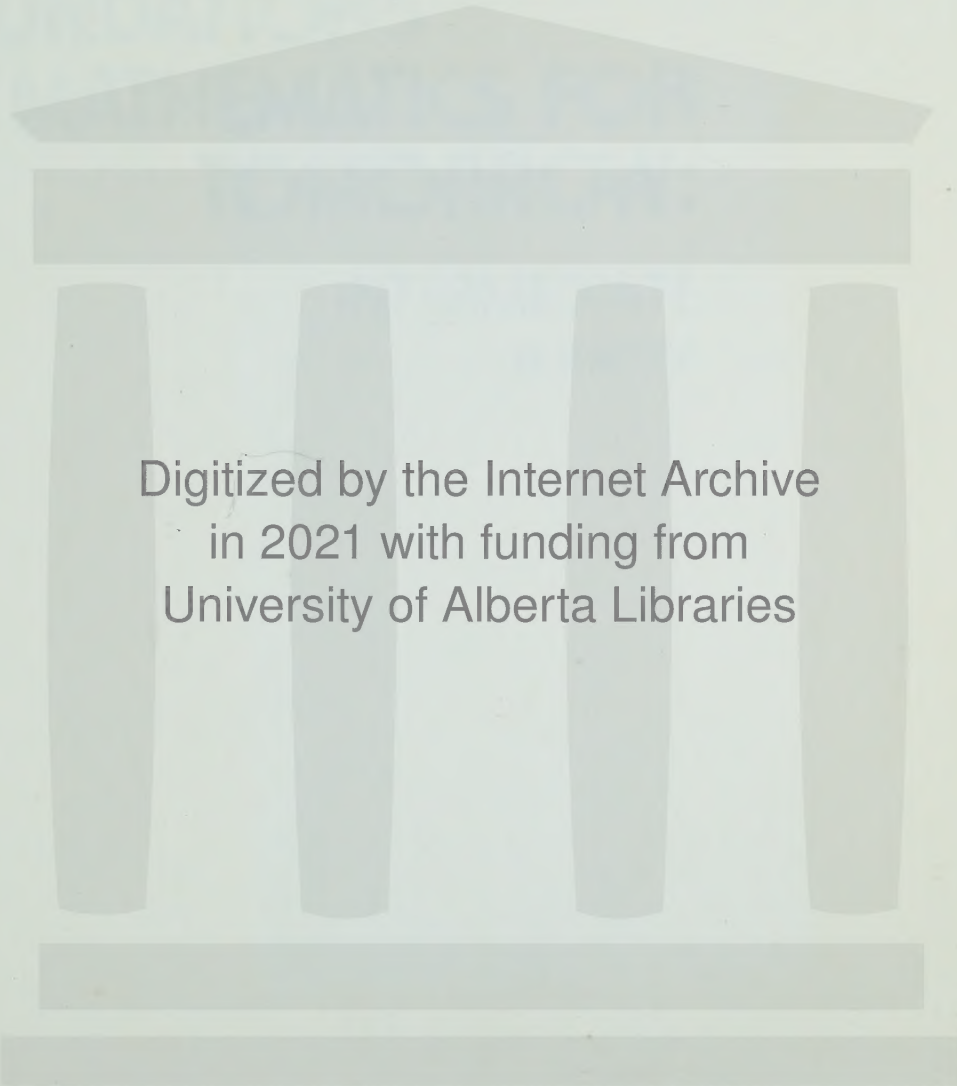
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FOUNDATIONS OF MATHEMATICS FOR TOMORROW:

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FOUNDATIONS OF MATHEMATICS FOR TOMORROW:
INTERMEDIATE
SI METRIC EDITION

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FOUNDATIONS OF MATHEMATICS FOR TOMORROW:
INTERMEDIATE

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FOUNDATIONS OF MATHEMATICS FOR TOMORROW: INTERMEDIATE

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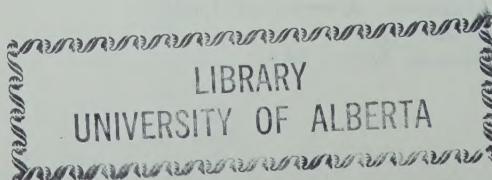
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REVIEW AND PREVIEW TO CHAPTER 1

EXERCISE 1

ADDITION AND SUBTRACTION

1. Add:

- (a) $3w - 4x + 7y$; $4w - 3x + 8y$
- (b) $3x - 5y + 4z$; $5x + 6y - 3z$; $7x - 8y + 11z$
- (c) $2xy - 3yz + 4xz$; $6xy + 12yz - 6xz$; $8xy - 5yz - xz$
- (d) $5r - 4s + 11t$; $6s - 7t + 5r$; $3t - r - s$
- (e) $4x^2 - 3x + 7$; $2x - 3x^2 + 6$; $7 - 2x^2 - x$
- (f) $m - 3m^2 - 2$; $4m^2 - 5 + 3m$; $6m^2 - 4$
- (g) $11xy - 4yz + 6x$; $5x - 5yz + 11xy$; $3x$
- (h) $3m - 2n$; $4x - 7y$; 8
- (i) $3x^2 - 4xy + 5y^2$; $4xy + x^2 - y^2$; $y^2 - 3x^2$
- (j) $1.4u - 3.2v + 6.1w$; $5.2u - 8v + 6.1w$; $0.7u - 0.2v + w$
- (k) $5.3xy - 4.2yz + 3.1xz$; $0.1yz - 2.3xy - 0.1xz$; $0.4xy - 3.2xz$

2. Subtract the first polynomial from the second.

- (a) $x + y + z$; $3x + 4y + 5z$
- (b) $3x - 4y + 2z$; $5x + 4y + 6z$
- (c) $4u - 3v + 2w$; $6u - 5v + w$
- (d) $7x + 3y - 4z$; $8x$
- (e) $5m - 3n$; $4n - 6mn + m$
- (f) $2x^2 - x - 4$; $x - 3x^2 + 6$
- (g) $4m - 3n$; $2x - y$
- (h) $5.2x - 3.1y$; $4.7x + 5.2y$
- (i) $7.1x - 3.4y + 5.7z$; $2.3x - 6.2y - 0.4z$
- (j) $6.1x^2 - 4.3x + 5$; $4.3x^2 + 5.9x - 8$

3. Subtract $4u - 3v + 2w$ from the sum of $4u - 3w + 2v$ and $6w - 3u - v$.

4. By how much does $4x^2 - 3x + 6$ exceed $3x - 2 + 3x^2$?

5. What must be added to $3x^2 - 2xy + y^2$ to give $7xy - 3y^2 + 4x^2$?

6. What polynomial decreased by $4xy - 3xz + 2yz$ equals $5yz - 3xy + 2xz$?

7. Subtract the sum of $3x^2 + 2x - 7$ and $4x^2 - x - 12$ from the sum of $5x + 2x^2 - 6$ and $3x^2 - 5$.

8. By how much does $3m - 4mn + 5n$ exceed $6m$?

EXERCISE 2

SUBSTITUTION

1. If $w = 2$, $x = 3$ and $y = 4$, evaluate

- (a) $4wx$
- (b) $7wxy$
- (c) $6wx + wxy$
- (d) $5wx - 3y$
- (e) $6w - 3x + 4y$
- (f) $w^2 + x^2 + y^2$
- (g) $w^2 + 3w - 7$
- (h) $w - x - y$
- (i) $2w^2 - 3w + x^2$
- (j) $2w^2xy - 3xy$
- (k) $3wx - 4xy - wy$
- (l) $2y^2 - w^2 - x^2$
- (m) $4(w + x)$
- (n) $-2(3w - 2x)$
- (o) $2x(4w - 3y)$
- (p) $-2wx(3y - 4)$
- (q) $0.5y(w^2 - x^2)$
- (r) $-3w^2(2x^2 - wx)$

2. If $w = 2$, $x = -2$ and $y = -1$, evaluate

- (a) $3x + 2y + 4w$
- (b) $6wxy$
- (c) $2xy + 3wx$
- (d) $2x - 3y - 2w$
- (e) $w - x - y$
- (f) $x^2 + y^2 + w^2$
- (g) $2w^2 - 3x^2 - 4y^2$
- (h) $x^2 - 2xy + y^2$
- (i) $2x^2y - y^2 - w^2$
- (j) $w^2 - x^2 - y^2$
- (k) $w^3 - y^3 - x^2$
- (l) $3y^5 - 13$
- (m) $2x(3w - 4y)$
- (n) $-(3x - 2wy)$
- (o) $0.5x(w^2 - y^2)$
- (p) $-1.5w(x^3 - y^3)$
- (q) $4y^4(5x - 7)$
- (r) $-2x^2(x^2y^2 - 3)$

EXERCISE 3

FUNCTION NOTATION

1. If $f(x) = x + 4$, evaluate

- | | |
|--------------|---------------|
| (a) $f(1)$ | (b) $f(2)$ |
| (c) $f(8)$ | (d) $f(20)$ |
| (e) $f(0)$ | (f) $f(-4)$ |
| (g) $f(-10)$ | (h) $f(-200)$ |

3. If $h(x) = x^2 + 6x - 24$, evaluate

- | | |
|--------------|--------------|
| (a) $h(2)$ | (b) $h(-3)$ |
| (c) $h(4)$ | (d) $h(0)$ |
| (e) $h(-6)$ | (f) $h(10)$ |
| (g) $h(-20)$ | (h) $h(-21)$ |

2. If $g(x) = 3x - 2$, evaluate

- | | |
|---------------|--------------|
| (a) $g(2)$ | (b) $g(7)$ |
| (c) $g(0)$ | (d) $g(-5)$ |
| (e) $g(14)$ | (f) $g(-15)$ |
| (g) $g(-100)$ | (h) $g(53)$ |

4. If $f(x) = x^3 - 2x^2 - 5x + 6$, evaluate

- | | |
|------------|-------------|
| (a) $f(1)$ | (b) $f(2)$ |
| (c) $f(0)$ | (d) $f(-2)$ |
| (e) $f(3)$ | (f) $f(-3)$ |
| (g) $f(5)$ | (h) $f(a)$ |

EXERCISE 4

THE DISTRIBUTIVE PROPERTY

1. Expand

$$a(b+c) \\ = ab+ac$$

- | | | |
|--------------------|---------------------|-----------------------|
| (a) $2(x+6)$ | (b) $3(x+4)$ | (c) $4(x-5)$ |
| (d) $7(5-m)$ | (e) $(4-3x)4$ | (f) $(x-y)5$ |
| (g) $-2(2x-7)$ | (h) $-7(3m-4n)$ | (i) $-(3m-7)$ |
| (j) $3(2x^2-4x+5)$ | (k) $-3(2t^2-5t+4)$ | (l) $-(3x^2-4xy+y^2)$ |
| (m) $2x(x-7)$ | (n) $5x(x^2-3x-4)$ | (o) $-m(1-3m-m^2)$ |

2. Expand

- | | | |
|--------------------------|-----------------------------|----------------------------|
| (a) $2xy(3x-4y)$ | (b) $-2xy(x^2-xy+y^2)$ | (c) $3t^3(t^2-2t-4)$ |
| (d) $2\pi r(w-x)$ | (e) $-\pi(3m-2n)$ | (f) $0.4(3x-5y)$ |
| (g) $-0.2(3x^2-4x)$ | (h) $-1.4(10x-y)$ | (i) $0.1x(3m-7n)$ |
| (j) $10t(0.3t-0.7)$ | (k) $-10x^2(4.8x-7.3y)$ | (l) $0.01(0.4m-0.5)$ |
| (m) $-0.3m(0.1m^2-0.2m)$ | (n) $1.5x^2y(20x^2y-5xy^2)$ | (o) $4mnt(3m^2-4n^2+5t^2)$ |

3. Expand and simplify.

- | | |
|---|-------------------------------------|
| (a) $2(x+4)+3(x+8)$ | (b) $3(2x+7)+4(5x+8)$ |
| (c) $7(3w-4)+5(5w-1)$ | (d) $6(m+4)+2(m-5)+4(3m-9)$ |
| (e) $2(4x-3)-3(2x-1)$ | (f) $5(3t-4)-2(t+7)-(t-6)$ |
| (g) $4(3x^2-2x+1)-5(x^2-x-1)$ | (h) $4(x^2-3x+4)-3(x^2-7x+6)$ |
| (i) $3(x-y)-2(2x+5y)-(x-3y)$ | (j) $2(x^2-4xy+y^2)+6(x^2-2xy-y^2)$ |
| (k) $5(x^2-8x-4)-3(2x-7)$ | (l) $(1-3t+4t^2)-(8+5t-6t^2)$ |
| (m) $4(3m-n)-(4m-7n+6)$ | (n) $-5(6m^2-n^2)-2(3n^2-2mn+m^2)$ |
| (o) $3x(2x-1)-4x(3x-5)$ | (p) $2w(1-3w)-w(2+5w)$ |
| (q) $3x(2x-4)-2x(1-3x)+x(2-x)$ | (r) $3x(x^2-2x+1)-x(2x^2-x-5)$ |
| (s) $2mn(m^2+n^2)-3mn(2m^2-n^2)$ | (t) $6x(1-4x+5x^2)-x(2x^2-5x+8)$ |
| (u) $10x(0.2x^2-0.3x+1.7)-20x(0.3x^2+0.2x-1.4)$ | (v) $4m^2n^2(2m-3n)-6m^2n^2(m-n)$ |
| (v) $0.1xy(10x-7y)-0.2xy(20x-9y)$ | |
| (x) $x(x-y)-y(y-x)-3(x^2-y^2)$ | |

4. If $w = 2$, $x = -1$ and $y = -2$, evaluate.

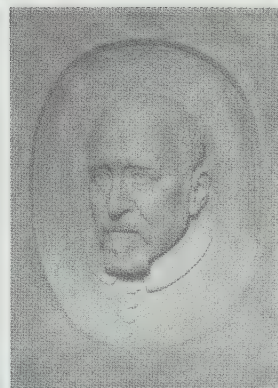
- | | |
|------------------------------------|-------------------------------------|
| (a) $3wx+2wy+4xy$ | (b) $w^2+x^2+y^2$ |
| (c) $2wx^2-w^2y^2-x$ | (d) $3w^2x^3-w^3x-2wxy$ |
| (e) $2(w-x)-(x-y)-2(x-w)$ | (f) $3(1-2x)-2(2w-1)-3(2y-3)$ |
| (g) $2w(w-x)-3x(w+x)+6wx$ | (h) $0.1(w-2x)+0.2(2x-y)-0.1(2y-w)$ |
| (i) $2x(2x-1)-x(1-3x)+4x(x-1)+x^2$ | |

Mathematics is the gate and key of the sciences.

Roger Bacon

ROBERT RECORDE (1510–1558)

Robert Recorde was a founder of the English mathematical school. He taught at Oxford and Cambridge and became physician to King Edward VI and Queen Mary. His most famous work was a book on algebra called *The Whetstone of Witte* and was published in 1557, a year before he died in prison. It was in this book that the equal sign occurred for the first time. In introducing this new symbol Recorde explained, “I will sette as I doe often in woorke use, a paire of paralleles, or twin lines of one length, thus: =, bicause noe 2 thynges, can be moare equalle.”

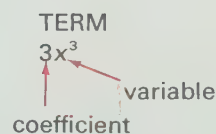


1.1 PRODUCT OF POLYNOMIALS

Various types of algebraic expressions and parts of algebraic expressions are given special names. When an expression is formed by multiplying numbers and/or variables it is called a term. Examples of terms are:

$$3x, \frac{xy}{3}, -6m^2, 14, 2(x-1).$$

The numerical part of a term is called the **numerical coefficient** or **coefficient**. A term may also have a variable or literal part. A variable is a symbol which may represent any member of a particular set. The degree of a term is the sum of the exponents of its variables.



$5x^3$ is a term of degree 3

$-4x^4y^2$ is a term of degree 6

Algebraic expressions are classified by the number of terms.

$3t$: one term : monomial

$4x^2 + 2y$: two terms : binomial

$3x^2 + 4x - 1$: three terms : trinomial

A polynomial is an expression in the form

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x^1 + a_0$$

where n is a non-negative integer, x is a variable and a_i ($i = n, n-1, \dots, 1, 0$) is a real number.

The degree of a polynomial is the greatest degree of its terms. A polynomial of degree 1, like $3x+4$, is called a *linear* polynomial. A second degree polynomial like $3x^2+7$ is called a *quadratic* polynomial.

$4t^3+2t-3$ is a polynomial of degree 3. It is called a *cubic* polynomial. Polynomials of higher degree are called *nth degree* polynomials.

Terms that have the same variable factors such as $3xy$ and $7xy$ are called *like terms*. We use the distributive property to *simplify* polynomials containing like terms. For example,

xy is a common factor.

$$3xy+7xy=(3+7)xy=10xy.$$

Polynomials are in simplest form when the terms contain no common factors.

When multiplying polynomials the distributive property is used:

$$(x+2)(2x^2+x-7)$$

$$\begin{aligned}(x+2)(2x^2+x-7) &= x(2x^2+x-7)+2(2x^2+x-7) \\ &= 2x^3+x^2-7x+4x^2+2x-14 \\ &= 2x^3+5x^2-5x-14.\end{aligned}$$

Stated simply,

To find the product of two polynomials, multiply each term of one of the polynomials by each term of the other and then add the products.

EXAMPLE 1. *Expand and simplify*

$$(2x^2-x-1)(x^2-2x+3)$$

Solution

$$\begin{aligned}(2x^2-x-1)(x^2-2x+3) \\ &= 2x^4-4x^3+6x^2-x^3+2x^2-3x-x^2+2x-3 \\ &= 2x^4-5x^3+7x^2-x-3\end{aligned}$$

A clock is stopped 1 min every 5 min. How long will it take the minute hand to complete one revolution?

Squaring a polynomial is simply finding the product of two polynomials.

$$\begin{aligned}(x^2-3x+1)^2 &= (x^2-3x+1)(x^2-3x+1) \\ &= x^4-3x^3+x^2-3x^3+9x^2-3x+x^2-3x+1 \\ &= x^4-6x^3+11x^2-6x+1\end{aligned}$$

Three special cases of binomial products are now illustrated.

$$\left. \begin{aligned}(x+3y)^2 &= (x+3y)(x+3y) \\ &= x^2+3xy+3xy+9y^2 \\ &= x^2+6xy+9y^2\end{aligned} \right\} \quad (a+b)^2 = a^2+2ab+b^2$$

$$\begin{aligned}
 (3x-4)^2 &= (3x-4)(3x-4) \\
 &= 9x^2 - 12x - 12x + 16 \\
 &= 9x^2 - 24x + 16
 \end{aligned}
 \left. \vphantom{\begin{aligned} (3x-4)^2 &= (3x-4)(3x-4) \\ &= 9x^2 - 12x - 12x + 16 \\ &= 9x^2 - 24x + 16 \end{aligned}} \right\} \boxed{(a-b)^2 = a^2 - 2ab + b^2} \quad /$$

$$\begin{aligned}
 (3x-2y)(3x+2y) &= 9x^2 - 6xy + 6xy - 4y^2 \\
 &= 9x^2 - 4y^2
 \end{aligned}
 \left. \vphantom{\begin{aligned} (3x-2y)(3x+2y) &= 9x^2 - 6xy + 6xy - 4y^2 \\ &= 9x^2 - 4y^2 \end{aligned}} \right\} \boxed{(a-b)(a+b) = a^2 - b^2} \quad /$$

Squaring a **trinomial** produces a pattern similar to the one resulting when a binomial is squared.

$$\begin{aligned}
 (2r-3s+t)^2 &= (2r-3s+t)(2r-3s+t) \\
 &= 4r^2 - 6rs + 2rt - 6rs + 9s^2 - 3st + 2rt - 3st + t^2 \\
 &= 4r^2 + 9s^2 + t^2 - 12rs + 4rt - 6st
 \end{aligned}$$

$$\boxed{(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc} \quad \rfloor$$

EXAMPLE 2. Expand and simplify

$$2(2x-3)(x+1) - 3(2x-1)^2$$

Solution

$$\begin{aligned}
 &2(2x-3)(x+1) - 3(2x-1)^2 \\
 &= 2(2x^2 + 2x - 3x - 3) - 3(4x^2 - 4x + 1) \\
 &= 2(2x^2 - x - 3) - 3(4x^2 - 4x + 1) \\
 &= 4x^2 - 2x - 6 - 12x^2 + 12x - 3 \\
 &= -8x^2 + 10x - 9
 \end{aligned}$$

EXERCISE 1-1

A 1. Classify each of the following as either monomial, binomial or trinomial and state the degree.

- | | | |
|-----------------------|------------------------|------------------------------|
| (a) $3x^4$ | (b) $-12x^3y^5$ | (c) $7rst$ |
| (d) $4x^2 - 7$ | (e) $3x^2 - 5x + 4$ | (f) $24x^6y^7 - 3$ |
| (g) $4r^3s^4 - 3rs^5$ | (h) $3x^7 - 2x^2y + 5$ | (i) $-16m^2n^3t$ |
| (j) $7xy^3 - 3x^4$ | (k) $2 - 4x^6y^3$ | (l) $1 - 3x + 5x^6$ |
| (m) $6xy - 5xyt$ | (n) $-11x^8y^9m^4$ | (o) $2x^6 - 3x^3y^5 + 5xy^4$ |

2. Arrange the terms from highest to lowest degree.

- | | |
|---------------------------------|---|
| (a) $16x^3 - 5x^5 + 4x^2 - 3x$ | (b) $17x^2 + 15x^4 - 3x^3 + 5x$ |
| (c) $5x^2y^3 - 4x^3y^3 + 5x^3y$ | (d) $8x^3y - 4x^4y^2 + 2m^2n - 3x^5y^2$ |
| (e) $3xy + 5x^3 - 4mn^3 + 5$ | (f) $4x^3y - 5m^6 + 11x^5y^2 - x^5$ |
| (g) $2m^4n^3 - 5m^6 + 2x^4$ | (h) $3xy - 5x^3 + y^4 - 3x + 2$ |

3. Each of the following expressions represents the perimeter of a geometric figure. Name the figure and state the degree of the expression.

- | | | |
|---------------|---------------------|-------------|
| (a) $2l + 2w$ | (b) $a + b + c$ | (c) $4x$ |
| (d) $2\pi r$ | (e) $a + b + c + d$ | (f) πd |

4. Each of the following expressions represents the area of a geometric figure. Name the figure and state the degree of the expression.

- | | | |
|----------|---------------|---------------------|
| (a) lw | (b) πr^2 | (c) $\frac{1}{2}bh$ |
|----------|---------------|---------------------|

(d) $\frac{h}{2}(a+b)$

(e) $\pi R^2 - \pi r^2$

5. Each of the following expressions represents the volume of a geometric figure. Name the figure and state the degree of the expression.

(a) lwh

(b) $\pi r^2 h$

(c) $\frac{4}{3}\pi r^3$

6. Expand.

(a) $(x+3)^2$

(b) $(x-2)(x+2)$

(c) $(t-2)^2$

(d) $(m-x)^2$

(e) $(r+7)(r-7)$

(f) $(y+4)^2$

(g) $(x-y)(x+y)$

(h) $(2m+1)^2$

(i) $(2x-3)^2$

(j) $(2x-3y)(2x+3y)$

(k) $(1-x)^2$

(l) $(2x+5)^2$

(m) $(3x-4y)^2$

(n) $(5s+3t)^2$

(o) $(x^2-1)(x^2+1)$

(p) $(2-3st)^2$

(q) $(3x^2-2y)(3x^2+2y)$

(r) $(3-7r)^2$

7. Expand.

(a) $(x+3)(x+2)$

(b) $(y+4)(y+5)$

(c) $(m-3)(m-2)$

(d) $(y-3)(y-7)$

(e) $(t+3)(t-4)$

(f) $(r+8)(r-5)$

(g) $(x-7)(x+3)$

(h) $(m-5)(m-7)$

(i) $(x+6)(x-4)$

(j) $(t+5)(t+11)$

(k) $(x+10)(x+11)$

(l) $(y+8)(y-6)$

(m) $(x^2-3)(x^2+6)$

(n) $(x^2+1)(x^2-2)$

(o) $(m^2-3)(m^2-5)$

(p) $(x^3-8)(x^3+6)$

(q) $(1-x)(5+x)$

(r) $(3-m)(6-m)$

(s) $(10-x)(8-x)$

(t) $(7-t)(8+t)$

(u) $(1+y)(8-y)$

1abcde when multiplied by 3 gives abcde1. Find a, b, c, d, and e.

B 8. Expand and simplify.

(a) $2(x-4)-3(x+2)$

(b) $2(x^2-7x+5)-3(x-4)$

(c) $5(3x-4y)-(2x-5y)+7$

(d) $3(r-2s-t)-3(4r+2s-6t)$

(e) $3(2x-4)-3-(2x+1)+5$

(f) $5(3x-1)-4(5y+2)-6$

(g) $2(2x^2-3x+1)-4(3x+5)$

(h) $2x(3x-5)-4(2x+7)+x^2$

(i) $2(1-3x+2x^2)-(1-4x+5x^2)$

(j) $2m(1-3m)-m(2m-3)+m$

(k) $3(x_1-2x_2+3x_3)-2(x_2-x_3)$

(l) $4(2x^2-3xy+4y^2)-2(x^2-3y^2)$

9. Expand and simplify.

(a) $(3x+4)(x+5)$

(b) $(2t+1)(3t+7)$

(c) $(4s+5)(s+7)$

(d) $(3x-4)(2x-1)$

(e) $(3m-8)(2m-3)$

(f) $(6t-5)(7t-8)$

(g) $(4x+3)(5x-4)$

(h) $(2r+7)(3r-1)$

(i) $(1-3x)(2+5x)$

(j) $(3-5y)(1-6y)$

(k) $(1-3m)(2m+5)$

(l) $(3x-10)(2-5x)$

(m) $(3x+y)(2x-3y)$

(n) $(4x-5y)(3x-10y)$

(o) $(5m+7n)(6m+n)$

(p) $(6w-11x)(w+3x)$

(q) $(7x+2y)(8x-7y)$

(r) $(3rs-4t)(5rs+7t)$

(s) $(5x^2-4x)(3x^2+2x)$

(t) $(2m-3m^2)(m^2+2m)$

(u) $(5x^2y-t)(7x^2y+4t)$

10. Expand and simplify.

(a) $(x+y+z)^2$

(b) $(w-x-y)^2$

(c) $(r-s+t)^2$

(d) $(2x+y+z)^2$

(e) $(2w-3x+y)^2$

(f) $(x^2-2x+3)^2$

(g) $(1-3x-4x^2)^2$

(h) $(5m-3n+4)^2$

(i) $(5x^4-2x^2-1)^2$

11. Find the following products.

(a) $(2x+3)(x^2+2x+1)$

(b) $(3w^2-4w-3)(2w-1)$

(c) $(2m^2+3m-1)(4m^2-2m+3)$

(d) $(2w-3x+2y)(4w-x+4y)$

(e) $(1-3x-x^2)(2+4x-5x^2)$

(f) $(3x-4y+2z)(x+3y-z)$

(g) $(x^3-x^2+x-1)(x^2-x-3)$

(h) $(x^3-x^2-2x-3)(x^3+2x^2+3x+1)$

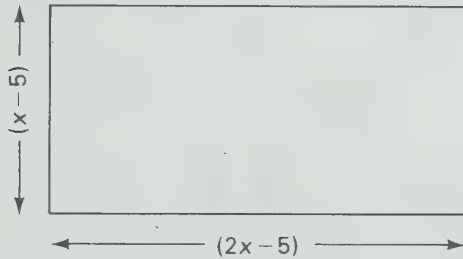
(i) $(m^3-2m^2-3m-1)(2m-5)$

(j) $(3x-4)(x^3-2x^2+5x-4)$

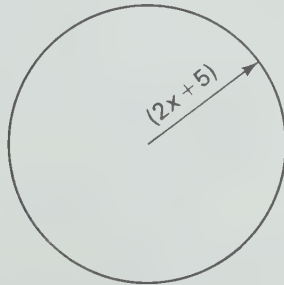
12. Expand and simplify.

- (a) $2(x-4)(x+3)+5(2x-1)(x+6)$
- (b) $3(2t-5)(t-4)-3(5t-3)(t+4)$
- (c) $2(m-3)(m-4)-3(m+5)^2-2(2m-1)(2m+1)$
- (d) $3(2m+3)^2-(m-5)^2-(2m-4)(m-5)$
- (e) $5(2x-5)(2x+5)-4(x-2)(x+3)-(2x+1)^2$
- (f) $(1-3x)(2+5x)-(x-4)(2x-5)-(2x+3)^2$
- (g) $5(2x-3)-2(x-4)(x-5)+3x^2-(x-6)$
- (h) $5x^2-(x-3)^2-2(x^2-5x)+2(2x-3)^2$
- (i) $1-(1-3x)-(x+5)^2-(3-4x)^2+6x^2$
- (j) $(x-y)(x+2y)-3(2x-3y)(x-4y)+3(x+y)^2$
- (k) $(2w+3x)(w-x)-4(w-2x)^2+5(w^2-x^2)$
- (l) $4(x^2-3xy)-(x+y)^2-2(x-y)(x+y)+5$
- (m) $2(x-1)(x^2-3x+2)-(2x^2-3x-4)(2x+3)$
- (n) $5(r-s+t)(r-2s-3t)-(r+s+t)^2-(r-s-3t)$

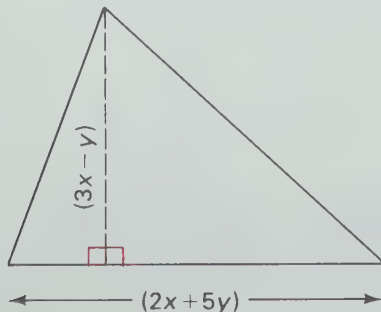
13. Express the area of the rectangle in terms of x .



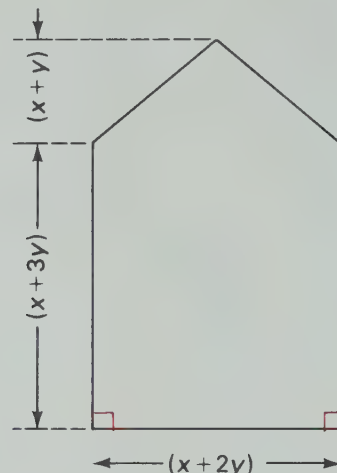
14. Express the circumference and area of the circle in terms of π and x .



15. Express the area of the triangle in terms of x and y .



16. Express the area of the figure in terms of x and y .



C 17. Expand and simplify.

(a) $(2x - 1)(x + 4)(3x - 5)$

(b) $(x - 2y)(x + 3y)(2x - 5y)$

(c) $(w + x + y + z)^2$

(d) $\left(x + \frac{1}{x}\right)\left(x - \frac{1}{x}\right)$

(e) $\left(m - \frac{2}{m}\right)\left(m + \frac{3}{m}\right)$

(f) $\left(1 - x + \frac{1}{x}\right)\left(2 + x - \frac{1}{x}\right)$

1.2 COMMON FACTOR

Since $4 \times 6 = 24$ we say that 4 and 6 are factors of 24. If we limit the factors to *prime numbers* we get

$$2 \times 2 \times 2 \times 3 = 24$$

Prime Numbers

2, 3, 5, 7, 11, 13, 17, ...

A prime number is an integer greater than 1 whose only integer factors are 1 and itself.

A positive integer is said to be completely factored when it is written as the product of prime numbers. A negative integer is factored completely when it is written as a product of -1 and prime numbers.

By the distributive property,

$$2x(x + 7) = 2x^2 + 14x.$$

Reversing the procedure, we get

$$2x^2 + 14x = 2x(x + 7).$$

This is called *factoring* the polynomial, where $2x$ is called a *common factor* since it is common to both terms of the polynomial.

—EXPANDING—→

$$2x(x + 7) = 2x^2 + 14x$$

←FACTORS—

To factor a polynomial is to express the polynomial as the product of polynomials. We shall agree that a polynomial is completely factored when no more variable factors can be removed and no more integer factors, other than 1 or -1 , can be removed.

This agreement is necessary since it is always possible to factor out something. For example

$$2x+3=2\left(x+\frac{3}{2}\right)$$

$$2x+3=x\left(2+\frac{3}{x}\right)$$

$$2x+3=3\left(\frac{2}{3}x+1\right)$$

In factoring polynomials with integer coefficients, unless otherwise stated, we will find factors with integer coefficients. This is called *factoring over the integers*

EXAMPLE 1. Factor completely.

(a) $4xy+6y$

(b) $12x^3-9x^2+3x$

Solution

(a) $4xy+6y$

$$=2y \cdot 2x+2y \cdot 3$$

$$=2y(2x+3)$$

(b) $12x^3-9x^2+3x$

$$=3x \cdot 4x^2-3x \cdot 3x+3x \cdot 1$$

$$=3x(4x^2-3x+1)$$

Addition

MOON
MEN
CAN
REACH

EXAMPLE 2. Factor completely.

$$x^2(x-1)+3x(x-1)-5(x-1)$$

Solution

Think of $(x-1)$ as one number.

$$x^2(x-1)+3x(x-1)-5(x-1)$$

$$=(x-1) \cdot x^2+(x-1) \cdot 3x-(x-1) \cdot 5$$

$$=(x-1)(x^2+3x-5)$$

EXAMPLE 3. Factor completely.

$$10x^2-5xy-6x+3y$$

Solution

The terms of the polynomial may be grouped to produce common factors.

$$10x^2-5xy-6x+3y$$

$$=(10x^2-5xy)-(6x-3y)$$

$$=5x(2x-y)-3(2x-y)$$

$$=(2x-y)(5x-3)$$

Factoring by grouping.

EXERCISE 1-2

A 1. Factor completely, where possible.

(a) $5x+10$

(b) $2x-6$

(c) $2x-4y$

(d) $5x+7$

(e) $3xy-4x$

(f) $10m-5$

(g) $3xy+4xw$

(h) $2t^2+4t-6$

(i) $5mn-6mnt$

(j) $8x^2-4x$

(k) $7r-14r^2$

(l) $2sw+4sx+6sy$

(m) $r(x-1)+t(x-1)$

(n) $7(w-z)-5x(w-z)$

(o) $3x(m-n)-(m+n)$

B 2. Factor completely.

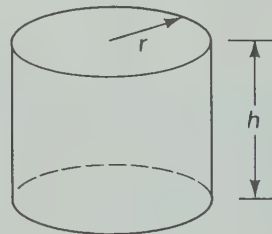
- | | |
|--|--|
| (a) $5x^2 + 25x^3 - 30x^4$ | (b) $x^5 - x^4 + x^3 - x^2$ |
| (c) $28x^3 + 24x^2$ | (d) $15x^5 - 7y^3 + 4z^2$ |
| (e) $6m^3 - 10m^2t + 6mt^2$ | (f) $14rst - 7r^2st + 21rs^2t - 28rst^2$ |
| (g) $x^3y^3 - x^2y + xy^2$ | (h) $13m^2n - 26m^2n^2$ |
| (i) $16p^4q^3 - 8p^3q^4 - 4p^3q^3$ | (j) $18x^2y^2 + 9xy - 3x^3y^4$ |
| (k) $11pqr - 15x + 23$ | (l) $x^6 - x^5 + x^4 - x^3 - x^2$ |
| (m) $36p^2x^4 - 16p^3x^5 + 24p^5x^4$ | (n) $m^{15}n^{11} - 3m^{10}n^{12} - 4m^9n^{15}$ |
| (o) $24x^9y^7 - 48x^{11}y^{13} + 36x^{10}y^{10}$ | (p) $2x^8y^4 - 6x^{17}y^{18} + 4x^8y^9 - 10x^{14}y^{21}$ |

3. Factor.

- | | |
|----------------------------|---|
| (a) $3(p+q) + 2x(p+q)$ | (b) $5(x-1) + 2m(x-1)$ |
| (c) $9m(x-y) - 14n(x-y)$ | (d) $8(m+4) - 9x(m+4)$ |
| (e) $5ty(m+n) - 6x(m+n)$ | (f) $(x-3) + y(x-3)$ |
| (g) $x(m+7) - (m+7)$ | (h) $2(x^2 - 2x - 1) - y(x^2 - 2x - 1)$ |
| (i) $14m^2(x-y) - 7m(x-y)$ | (j) $(x-y)(x+y) + 3(x+y)$ |
| (k) $2x(x-4) - 3(x-4)$ | (l) $5m(2m+1) - 4(2m+1)$ |
| (m) $(x-5)^2 - m(x-5)$ | (n) $(x-y)^2 - (x-y)$ |
| (o) $(y+2)x - (y+2)7$ | (p) $(m-n)^3 - 2(m-n)^2$ |

4. Factor.

- | | |
|-------------------------------|-----------------------------|
| (a) $mx + my + nx + ny$ | (b) $tx + 3t + wx + 3w$ |
| (c) $mx + my - 3x - 3y$ | (d) $x^3 + x^2 + x + 1$ |
| (e) $x^3 - x^2y + xy^2 - y^3$ | (f) $am + 3m + an + 3n$ |
| (g) $1 - x + x^2 - x^3$ | (h) $5x - 5y - x^2 + xy$ |
| (i) $3x^2 + y + 3xy + x$ | (j) $4mx + ny - 4nx - my$ |
| (k) $3ny + 2mx - 2my - 3nx$ | (l) $2mx - 3ny - 2my + 3nx$ |

C 5. Express the total area of the right circular cylinder in terms of r , h and π . Factor the expression.**6. Factor.**

- | | |
|--------------------------|------------------------|
| (a) $2m(x+y) - 3(y+x)$ | (b) $4x(m+n) - 2(n+m)$ |
| (c) $6n(x-y) + 5(y-x)$ | (d) $7m(x-4) - 3(4-x)$ |
| (e) $5m(3x-y) - 2(y-3x)$ | (f) $3x(m-n) - (n-m)$ |

1.3 FACTORING $ax^2 + bx + c$

Many polynomials appear in the form $ax^2 + bx + c$, such as $2x^2 + 7x + 3$, where $a = 2$, $b = 7$ and $c = 3$. Many of these quadratic polynomials can be written as the product of two linear polynomials.

————EXPANDING————→

$$(2x+1)(x+3) = 2x^2 + 7x + 3$$

←————FACTORING—————

First we will consider quadratics of the form $ax^2 + bx + c$, where $a = 1$. An analysis of a general expansion of two linear polynomials will simplify factoring.

$$\begin{aligned}(x+r)(x+s) &= x^2 + sx + rx + rs \\ &= x^2 + (s+r)x + rs \\ &= x^2 + bx + c\end{aligned}$$

Here, $b = (s+r)$ and $c = rs$.

If we can write $x^2 + bx + c$ as $(x+r)(x+s)$ where r and s are integers, we say we have factored $x^2 + bx + c$ over the integers.

To write $x^2 + 7x + 12$ in the form $(x+r)(x+s)$,

$$\begin{aligned}b = r + s &= 7 \\ \text{and } c = rs &= 12.\end{aligned}$$

The two integers that add to give 7 and multiply to give 12 are 4 and 3. Hence $r = 4$ and $s = 3$ and the factors are $(x + 4)$ and $(x + 3)$.

$$\therefore x^2 + 7x + 12 = (x+4)(x+3).$$

EXAMPLE 1. Factor $x^2 - 2x - 15$ over the integers.

Solution

For $x^2 - 2x - 15$

$$r + s = -2 \quad \text{and} \quad rs = -15$$

The two integers that add to give -2 and multiply to give -15 are -5 and 3 . Therefore $x^2 - 2x - 15 = (x-5)(x+3)$

Factoring trinomials in the form $ax^2 + bx + c$, where $a \neq 1$, is simplified if we factor by grouping. To do this we must break the middle term up into two parts.

$$\begin{aligned}12x^2 + 17x + 6 \\ &= 12x^2 + 9x + 8x + 6 \\ &= (12x^2 + 9x) + (8x + 6) \\ &= 3x(4x + 3) + 2(4x + 3) \\ &= (4x + 3)(3x + 2)\end{aligned}$$

A problem arises in trying to determine how to break up the middle term. For $17x$ in the example we might have used $10x + 7x$ or $2x + 15x$ or $-3x + 20x$ etc. The decision as to what two terms must be used will be clarified if we analyse a general expansion.

$$\begin{aligned}(px+r)(qx+s) \\ &= pqx^2 + psx + qrx + rs \\ &= pqx^2 + (ps + qr)x + rs \\ &= ax^2 + bx + c\end{aligned}$$

A perfect number is an integer which is equal to the sum of all its factors except itself.

Example: $6 = 1 + 2 + 3$. Find another perfect number.

If we break up the middle term bx into two terms, say mx and nx , then it is clear that

$$m + n = ps + qr = b$$

$$\text{and } m \times n = pqrs = ac$$

EXAMPLE 2. Factor $6x^2 + 5x - 6$ over the integers.

Solution

For $6x^2 + 5x - 6$, $a = 6$, $b = 5$ and $c = -6$. To factor, by grouping, we will replace $5x$ by $mx + nx$ where

Two integers that add to give 5 and multiply to give -36 .

$$\left\{ \begin{array}{l} m + n = b = 5 \\ \text{and } m \times n = ac = -36 \end{array} \right.$$

Therefore m and n are 9 and -4 .

$$\begin{aligned} 6x^2 + 5x - 6 &= 6x^2 + 9x - 4x - 6 \\ &= (6x^2 + 9x) - (4x + 6) \\ &= 3x(2x + 3) - 2(2x + 3) \\ &= (2x + 3)(3x - 2) \end{aligned}$$

EXAMPLE 3. Factor $24x^2 - 14xy - 20y^2$ over the integers.

Solution

Always check for common factors.

$$24x^2 - 14xy - 20y^2 = 2[12x^2 - 7xy - 10y^2]$$

We now factor $12x^2 - 7xy - 10y^2$ by breaking $-7xy$ into two parts. $m + n = -7$ and $mn = -120$. Therefore m and n are 8 and -15 .

$$\begin{aligned} 2[12x^2 - 7xy - 10y^2] &= 2[12x^2 + 8xy - 15xy - 10y^2] \\ &= 2[(12x^2 + 8xy) - (15xy + 10y^2)] \\ &= 2[4x(3x + 2y) - 5y(3x + 2y)] \\ &= 2[(3x + 2y)(4x - 5y)] \\ &= 2(3x + 2y)(4x - 5y) \end{aligned}$$

EXERCISE 1-3

A 1. Complete the factoring.

- | | |
|---|---|
| (a) $x^2 + 6x + 8 = (x + 4)(\quad)$ | (b) $x^2 + 10x + 16 = (x + 8)(\quad)$ |
| (c) $x^2 - 7x + 12 = (x - 4)(\quad)$ | (d) $m^2 - 8m + 12 = (m - 2)(\quad)$ |
| (e) $t^2 - 13t + 22 = (t - 2)(\quad)$ | (f) $n^2 - n - 2 = (\quad)(n + 1)$ |
| (g) $y^2 + 17y + 72 = (\quad)(y + 9)$ | (h) $x^2 + 3x - 18 = (\quad)(x + 6)$ |
| (i) $m^4 - 12m^2 + 20 = (m^2 - 2)(\quad)$ | (j) $x^6 - 20x^3 + 36 = (\quad)(x^3 - 2)$ |
| (k) $x^2 - 6x - 16 = (x + 2)(\quad)$ | (l) $x^2 - x - 20 = (x - 5)(\quad)$ |
| (m) $2 + 3x + x^2 = (2 + x)(\quad)$ | (n) $5 + 6s + s^2 = (5 + s)(\quad)$ |
| (o) $12 + x - x^2 = (\quad)(4 - x)$ | (p) $56 - 15t + t^2 = (7 - t)(\quad)$ |
| (q) $28 + 3m - m^2 = (4 + m)(\quad)$ | (r) $27 - 12x + x^2 = (\quad)(3 - x)$ |

2. If possible, determine integer values for m and n .

- | | |
|----------------------------|----------------------------|
| (a) $m + n = 7, mn = 12$ | (b) $m + n = -8, mn = 15$ |
| (c) $m + n = 20, mn = 36$ | (d) $m + n = 5, mn = 4$ |
| (e) $m + n = -3, mn = -18$ | (f) $m + n = 7, mn = -18$ |
| (g) $m + n = 6, mn = 7$ | (h) $m + n = -1, mn = -12$ |
| (i) $m + n = 3, mn = -40$ | (j) $m + n = -2, mn = -10$ |
| (k) $m + n = -8, mn = 16$ | (l) $m + n = 3, mn = -28$ |
| (m) $m + n = 20, mn = 75$ | (n) $m + n = -7, mn = -18$ |
| (o) $m + n = -6, mn = -27$ | (p) $m + n = 6, mn = -27$ |
| (q) $m + n = -5, mn = 4$ | (r) $m + n = 7, mn = 15$ |

B 3. Factor over the integers if possible.

- | | | |
|-----------------------|---------------------|-----------------------|
| (a) $x^2 + 7x + 10$ | (b) $y^2 - 8y + 15$ | (c) $m^2 + 10m + 21$ |
| (d) $w^2 - w - 56$ | (e) $x^2 + 5x + 7$ | (f) $s^2 - 4s - 96$ |
| (g) $t^2 + 36t + 320$ | (h) $x^2 + x + 1$ | (i) $x^2 + 26x + 165$ |
| (j) $w^2 - 13w + 42$ | (k) $z^2 + 3z - 40$ | (l) $m^2 - 7m - 18$ |
| (m) $x^2 - 8x + 16$ | (n) $9 + 8t + t^2$ | (o) $n^2 - 13n + 36$ |
| (p) $20 - 9x + x^2$ | (q) $42 - x - x^2$ | (r) $88 + 3y - y^2$ |

4. Factor over the integers.

- | | | |
|-----------------------|-------------------------|-----------------------|
| (a) $2x^2 + 4x + 2$ | (b) $3q^2 - 6q - 45$ | (c) $m^2 + 11m + 28$ |
| (d) $5p^2 - 45p + 70$ | (e) $2x^2 - 4x - 20$ | (f) $x^2 - 22x + 117$ |
| (g) $60 + 7m - m^2$ | (h) $10y^2 - 30y - 400$ | (i) $x^2 + 12x + 30$ |
| (j) $x^2 + 14x + 48$ | (k) $120 + 14x - 2x^2$ | (l) $s^2 - 6s + 9$ |
| (m) $3x^2 - 36x + 84$ | (n) $36 - 12m + m^2$ | (o) $w^2 + 4w - 77$ |

5. Factor over the integers.

- | | | |
|--------------------------|--------------------------|-------------------------|
| (a) $x^2 + 6xy - 160y^2$ | (b) $s^2 + 9st + 14t^2$ | (c) $x^2 + xy - 12y^2$ |
| (d) $56 - 15t + t^2$ | (e) $x^2 - 6xy - 55y^2$ | (f) $m^2 - 41m + 408$ |
| (g) $x^2 + 20x + 22$ | (h) $x^2 + 6xy - 91y^2$ | (i) $42 - 13x + x^2$ |
| (j) $x^4 - x^2 - 90$ | (k) $a^6 + 2a^3 - 35$ | (l) $s^2 - 9st + 20t^2$ |
| (m) $2y^2 - 20y - 48$ | (n) $112 + 22z + z^2$ | (o) $m^2 - 7m - 744$ |
| (p) $m^2 - 8mn - 240n^2$ | (q) $x^2 - 8xy - 660y^2$ | (r) $x^4 - 8x^2 - 48$ |

6. Factor over the integers where possible.

- | | | |
|------------------------|------------------------|-----------------------|
| (a) $6x^2 - 7x - 20$ | (b) $12x^2 + 23x + 5$ | (c) $22t^2 + 13t + 1$ |
| (d) $6r^2 + 11r + 6$ | (e) $6m^2 - 7m - 3$ | (f) $2y^2 - y - 28$ |
| (g) $12q^2 + 29q + 15$ | (h) $20x^2 - 3x - 9$ | (i) $12x^2 - 40x - 7$ |
| (j) $3m^2 - 19m - 20$ | (k) $10t^2 - 31t + 15$ | (l) $10x^2 - 16x + 3$ |
| (m) $6q^2 - 23q + 7$ | (n) $4m^2 + 23m - 35$ | (o) $3y^2 + 22y - 16$ |
| (p) $3x^2 + 25x + 42$ | (q) $4s^2 + 31s - 45$ | (r) $20x^2 + 11x - 4$ |
| (s) $20x^2 - 64x + 35$ | (t) $36m^2 - 7m - 15$ | (u) $16t^2 - 18t - 9$ |
| (v) $15x^2 + 27x + 8$ | (w) $4s^2 + 21s + 27$ | (x) $8x^2 - 2x - 3$ |

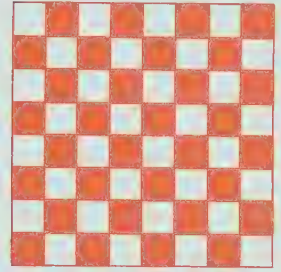
7. Factor over the integers.

- | | | |
|----------------------------|----------------------------|-----------------------------|
| (a) $24x^2 - 2x - 2$ | (b) $22y^2 + 13y + 1$ | (c) $6x^2 - 27x - 15$ |
| (d) $6x^2 + 17xy + 5y^2$ | (e) $12x^2 + 13xy - 35y^2$ | (f) $60m^2 + 370mn + 60n^2$ |
| (g) $24x^2 - 47xy + 20y^2$ | (h) $15t^2 + 22st + 8s^2$ | (i) $2 + 2x - 84x^2$ |
| (j) $5yx^2 + 18yx - 8y$ | (k) $4x^4 + 35x^2 + 49$ | (l) $2x^2 + 2xy + 2y^2$ |
| (m) $10 + 17x + 17x^2$ | (n) $15 - 44x - 20x^2$ | (o) $20 - 42x + 18x^2$ |

8. Factor over the integers.

- | | | |
|------------------------|------------------------|-------------------------|
| (a) $8x^2 + 38x + 45$ | (b) $20y^2 + 44y - 15$ | (c) $40t^2 - 47t + 12$ |
| (d) $48x^2 + 74x + 21$ | (e) $42m^2 - 51m + 15$ | (f) $40x^2 + 38x + 7$ |
| (g) $8m^2 + 46m + 63$ | (h) $48y^2 - 26y + 3$ | (i) $20s^2 - 29s - 33$ |
| (j) $15 - 53x + 42x^2$ | (k) $42 + t - 56t^2$ | (l) $28m^2 + 107m + 99$ |

Place 8 queens on a chess-board so that no queen attacks another queen.



watch for common factors

watch for common factors

C 9. Factor the following.

(a) $(x+3)^2 + 7(x+3) + 12$

(b) $(x+y)^2 - 9(x+y) + 18$

(c) $(x-y)^2 - 2(x-y) - 15$

(d) $(m+n)^2 - 10(m+n) + 21$

(e) $(x^2-x)^2 - 11(x^2-x) + 28$

(f) $x^2y^2 - 9xy + 20$

(g) $x^4y^2 + 8x^2y + 15$

(h) $6x^2y^2 - xyz - 12z^2$

(i) $(r+s)^2 - 2(r+s)(x+y) - 8(x+y)^2$

1.4 FACTORING SPECIAL QUADRATICS

We can use the following identities to factor special quadratics.

$$a^2 + 2ab + b^2 = (a+b)(a+b) = (a+b)^2$$

$$a^2 - 2ab + b^2 = (a-b)(a-b) = (a-b)^2$$

$$a^2 - b^2 = (a-b)(a+b)$$

EXAMPLE 1. Factor

(a) $16x^2 + 40x + 25$

(b) $9x^2 - 6xy + y^2$

Solution

(a) $16x^2 + 40x + 25$

$$= (4x)^2 + 2(4x)(5) + (5)^2$$

$$= (4x+5)(4x+5)$$

$$= (4x+5)^2$$

(b) $9x^2 - 6xy + y^2$

$$= (3x)^2 - 2(3x)(y) + (y)^2$$

$$= (3x-y)(3x-y)$$

$$= (3x-y)^2$$

EXAMPLE 2. Factor $25a^2 - 16b^2$

Solution

$$25a^2 - 16b^2 = (5a)^2 - (4b)^2$$

$$= (5a-4b)(5a+4b)$$

EXAMPLE 3. Factor $x^2 + 6x + 9 - y^2$

Solution

Grouping will give the difference of squares.

$$x^2 + 6x + 9 - y^2 = (x^2 + 6x + 9) - y^2$$

$$= (x+3)^2 - y^2$$

$$= [(x+3) - y][(x+3) + y]$$

$$= (x+3-y)(x+3+y)$$

EXERCISE 1-4

A 1. Expand.

(a) $(x-3)(x+3)$

(b) $(x+4)(x-4)$

(c) $(t-9)(t+9)$

(d) $(m-4)(m+4)$

(e) $(x+4)^2$

(f) $(x-3)^2$

(g) $(2x-3)(2x+3)$

(h) $(4y-1)(4y+1)$

(i) $(w+7)^2$

- (j) $(x-5)^2$ (k) $(2x-3y)(2x+3y)$ (l) $(4m+5n)(4m-5n)$
 (m) $(2m+3)^2$ (n) $(3x-4)^2$ (o) $(2x+5y)(2x-5y)$
 (p) $(7x-1)(7x+1)$ (q) $(1+5t)(1-5t)$ (r) $(m-n)(m+n)$

2. Factor over the integers if possible.

- (a) x^2-16 (b) m^2-25 (c) x^2-81
 (d) y^2-100 (e) x^2+6x+9 (f) t^2-4t+4
 (g) s^2+144 (h) $v^2+10v+25$ (i) $r^2-20r+100$
 (j) $25x^2-16$ (k) $1+4y^2$ (l) $16x^2-9y^2$
 (m) $x^2-22x+121$ (n) $s^2+16s+64$ (o) $36x^2-49y^2$
 (p) $64m^2-n^2$ (q) $n^2-12n+36$ (r) $100x^4-9y^2$

B 3. Factor.

- (a) $4x^2+12xy+9y^2$ (b) $16x^2-24xy+9y^2$
 (c) $25x^2+60xy+36y^2$ (d) $9r^2-12rs+4s^2$
 (e) $49x^2-56xy+16y^2$ (f) $36m^2-12mn+n^2$
 (g) $81m^2+36mn+4n^2$ (h) $64r^2-48rs+9s^2$

4. Factor.

- (a) $(x+y)^2-25$ (b) $(x+y)^2-1$
 (c) $(m-3)^2-n^2$ (d) $(s+5)^2-t^2$
 (e) $x^2-(m+n)^2$ (f) $m^2-(x-3)^2$
 (g) $16m^2-(n-2)^2$ (h) $9x^2-4(y-2)^2$
 (i) $25y^2-9(x+1)^2$ (j) $(x+2)^2-(y-1)^2$
 (k) $(m+3)^2-(n-4)^2$ (l) $4(x+2)^2-(y+7)^2$
 (m) $16(x-y)^2-49(w-3)^2$ (n) $25(m+5)^2-(n+1)^2$
 (o) t^4-16 (p) $x^{10}-25y^6$
 (q) $16m^8-9n^{10}$ (r) $36x^{14}-y^2$

5. Factor by grouping.

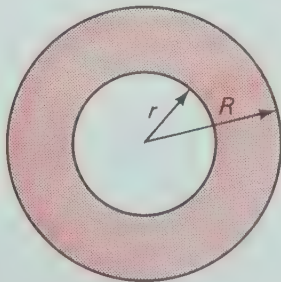
- (a) $x^2+2x+1-y^2$ (b) $x^2-6x+9-y^2$
 (c) $m^2-4m+4-n^2$ (d) $r^2-10r+25-s^2$
 (e) $r^2+14r+49-25s^2$ (f) $x^2-12x+36-9a^2$
 (g) $1-x^2-2xy-y^2$ (h) $9-x^2+4xy-4y^2$
 (i) $16x^2-y^2+6y-9$ (j) $25m^2-8n-n^2-16$
 (k) $9-36y^2+x^2-6x$ (l) $16y^2-4x^2-1-4x$
 (m) $12x-4+25y^2-9x^2$ (n) $24r-9-16r^2+9s^2$

6. Factor by grouping.

- (a) $x^2+2xy+y^2-m^2-2m-1$ (b) $m^2+6mn+9n^2-x^2+4x-4$
 (c) $8x-6mn-16-x^2+m^2+9n^2$ (d) $x^2-s^2-r^2+y^2-2rs+2xy$
 (e) $12pq-4p^2+12xy+9y^2-9q^2+4x^2$
 (f) $25-4m^2-20x-n^2+4mn+4x^2$

7. The area of the shaded region may be expressed as $\pi R^2 - \pi r^2$. Factor the expression and then determine the area (in terms of π) when

- (a) $R = 6 \text{ cm}, r = 2 \text{ cm}$
 (b) $R = 17 \text{ m}, r = 15 \text{ m}$
 (c) $R = 21 \text{ m}, r = 10 \text{ m}$



24 teams enter a basketball tournament. Each team plays until it loses. How many games are required to produce a winner?

1.5 DIVIDING A POLYNOMIAL BY A MONOMIAL

The rule for exponents in division can be illustrated as follows:

$$\frac{x^m}{x^n} = x^{m-n}$$

thus
$$\frac{x^5}{x^3} = x^{5-3} = x^2, \quad x \neq 0.$$

Since division by zero is not defined, replacements for variables which make the denominator zero will not be allowed.

EXAMPLE 1. Simplify

$$\frac{24x^5y^3}{6x^2y^2}$$

Solution

$$\begin{aligned} \frac{24x^5y^3}{6x^2y^2} &= \frac{24}{6} \cdot \frac{x^5}{x^2} \cdot \frac{y^3}{y^2} \\ &= 4x^3y, \quad x, y \neq 0. \end{aligned}$$

The distributive property applies to division as well as multiplication.

$$\frac{a+b}{c} = \frac{1}{c}(a+b) = \frac{a}{c} + \frac{b}{c}$$

To divide a polynomial by a monomial, each term of the polynomial is divided by the monomial.

EXAMPLE 2. Simplify

$$\frac{25x^5 - 15x^4 - 10x^3}{5x^2}$$

Solution

$$\begin{aligned} \frac{25x^5 - 15x^4 - 10x^3}{5x^2} &= \frac{25x^5}{5x^2} - \frac{15x^4}{5x^2} - \frac{10x^3}{5x^2} \\ &= 5x^3 - 3x^2 - 2x, \quad x \neq 0. \end{aligned}$$

EXAMPLE 3. Perform the indicated division.

$$\frac{36x^5 + 24x^3 - 4x^2 + 8}{2x^2}$$

Solution

$$\begin{aligned} \frac{36x^5 + 24x^3 - 4x^2 + 8}{2x^2} &= \frac{36x^5}{2x^2} + \frac{24x^3}{2x^2} - \frac{4x^2}{2x^2} + \frac{8}{2x^2} \\ &= 18x^3 + 12x - 2 + \frac{4}{x^2}, \quad x \neq 0. \end{aligned}$$

EXERCISE 1-5

A 1. Divide.

- | | | | |
|-----------------------------------|-----------------------------------|---------------------------------------|---|
| (a) $\frac{16x^2}{2x}$ | (b) $\frac{24t^5}{12t^2}$ | (c) $\frac{15xy}{3xy}$ | (d) $\frac{-36x^2y}{-4xy}$ |
| (e) $\frac{-48r^2s^2}{4rs}$ | (f) $\frac{54m^2n^4}{-6mn^2}$ | (g) $\frac{-100m^9n^{10}}{-20m^8n^9}$ | (h) $\frac{28m^2n}{-7n}$ |
| (i) $\frac{56x^2y^4z^6}{-7xy^3z}$ | (j) $\frac{-90m^{10}n^8}{9m^7n}$ | (k) $\frac{-75mn}{-25mn}$ | (l) $\frac{m^{10}n^{12}}{m^6n^6}$ |
| (m) $\frac{-30r^7s^9t}{-6r^2s^3}$ | (n) $\frac{-12x^6y^8m^3}{2xym^3}$ | (o) $\frac{-20xyz^5}{4xyz^3}$ | (p) $\frac{48r^{11}s^{15}t}{-6r^{10}s^{10}t}$ |

2. Reduce the following fractions to lowest terms.

- | | | | |
|------------------------|----------------------------|---------------------------|----------------------------|
| (a) $\frac{2}{2x}$ | (b) $\frac{x}{xy}$ | (c) $\frac{2t}{4t^3}$ | (d) $\frac{4x^2}{16x}$ |
| (e) $\frac{30y}{6y^6}$ | (f) $\frac{5xy}{20x^2y^2}$ | (g) $\frac{12mn^2}{24mn}$ | (h) $\frac{36x^6y}{12x^8}$ |

B 3. Perform the indicated divisions.

- | | | |
|--------------------------------|-----------------------------------|---------------------------------|
| (a) $\frac{2m+2n}{2}$ | (b) $\frac{4x-10y}{2}$ | (c) $\frac{tx+ty}{t}$ |
| (d) $\frac{15x^2-10x}{5x}$ | (e) $\frac{40s+8}{8}$ | (f) $\frac{18m^3-6m^2}{3m}$ |
| (g) $\frac{-7x^4+14x^3}{7x^3}$ | (h) $\frac{2m^4-4m^3-8m^2}{2m^2}$ | (i) $\frac{10x^5+15x^3-5x}{5x}$ |

4. Divide.

- | | |
|--|--|
| (a) $\frac{4x^2y+8xy^2}{4xy}$ | (b) $\frac{9x^3y^2-27x^2y-9xy}{3xy}$ |
| (c) $\frac{20m^4n^2-15m^2n^3-5mn}{5mn}$ | (d) $\frac{18r^3s^2t^4-9rs^2t-27r^4st}{9rst}$ |
| (e) $\frac{3x^4m^6-9x^3m^4-18x^2m^3}{3x^2m^2}$ | (f) $\frac{35m^6n^6-14m^4n^3+21m^2n^3}{7m^2n^2}$ |
| (g) $\frac{21x^4y^2-7x^3y^4+14x^5y^2}{7x^3}$ | (h) $\frac{-5x^2y^2-10x^3y^4-15x^5y^6}{5x^2y^2}$ |

5. Divide.

- | | | |
|--|---|-----------------------------------|
| (a) $\frac{2x-4y}{-2}$ | (b) $\frac{7t^2-14t^3}{-7t}$ | (c) $\frac{-20r^2s^2+15rs}{-5rs}$ |
| (d) $\frac{21rs^2t-15r^2st-6rst}{-3rst}$ | (e) $\frac{16m^4n^5-4m^3n^2-12m^3n^6}{-4m^3n^2}$ | |
| (f) $\frac{45x^7y^6z^5-9x^8y^5z^3}{-3x^5y^5z^3}$ | (g) $\frac{12x^3y^4z^3-8x^2y^2z^4-4x^2y^2z^3}{-4x^2y^2z^3}$ | |

C 6. Perform the indicated division.

- | | |
|---------------------------------|--------------------------------------|
| (a) $\frac{8x^3+4x^2+2x}{2x^2}$ | (b) $\frac{12x^4-8x^3+4x^2+2}{2x^2}$ |
|---------------------------------|--------------------------------------|

Augustus de Morgan, who lived in the 19th century, said "I was x years old in the year x^2 ." When was he born?

$$(c) \frac{m^5 - 6m^4 + 9m^3}{3m^3}$$

$$(d) \frac{21t^5 - 14t^4 - 7t^3 - t^2}{7t^3}$$

$$(e) \frac{10x^3y^4 - 5x^2y^5 + 15xy}{10x^2y^2}$$

$$(f) \frac{51w^4 - 34w^3 - 17w + 3}{17w^2}$$

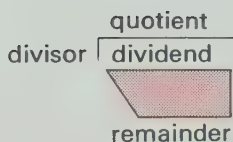
$$(g) \frac{20m^4n^3 - 8m^3n^4 - 4mn + 5}{-2m^2n^2}$$

$$(h) \frac{63r^3s^4 - 81r^4s^5 + 18r^3s^2 - 9r^2}{-3r^2s}$$

$$(i) \frac{x^5 + 24x^4y^5 - 8x^3y^6 - y^3}{-8x^3y^3}$$

1.6 DIVIDING A POLYNOMIAL BY A POLYNOMIAL

To illustrate the *division statement* two integers are divided.



$$\begin{array}{r} 2 \\ 23 \overline{)49} \\ \underline{46} \\ 3 \end{array}$$

To check the division we verify that

$$(2 \times 23) + 3 = 49$$

division statement } $\rightarrow$ or (quotient) $\times$ (divisor) + remainder = dividend

The method used to divide a polynomial by a polynomial is similar to the method used for long division in arithmetic.

EXAMPLE 1. Divide $(6x^2 + 16x - 1)$ by $(3x - 1)$ and write the division statement.

Solution

Divide the first term of the dividend by the first term of the divisor.

$$3x - 1 \overline{)6x^2 + 16x - 1} \quad \begin{array}{c} 2x \\ \hline \end{array}$$

$$3x - 1 \overline{)6x^2 + 16x - 1} \quad \begin{array}{c} 2x \\ \hline 6x^2 - 2x \\ \hline \end{array}$$

$$3x - 1 \overline{)6x^2 + 16x - 1} \quad \begin{array}{c} 2x \\ \hline 6x^2 - 2x \\ \hline 18x - 1 \end{array}$$

$$3x - 1 \overline{)6x^2 + 16x - 1} \quad \begin{array}{c} 2x + 6 \\ \hline 6x^2 - 2x \\ \hline 18x - 1 \end{array}$$

Multiply the first term of the quotient by the divisor.

Subtract the resulting product from the dividend.

Divide the first term of the remainder by the first term of the divisor.

Multiply the second term of the quotient by the divisor.

$$\begin{array}{r} 2x+6 \\ 3x-1 \overline{) 6x^2+16x-1} \\ \underline{6x^2-2x} \\ 18x-1 \\ \underline{18x-6} \end{array}$$

Subtract the resulting product from the previous remainder.

$$\begin{array}{r} 2x+6 \\ 3x-1 \overline{) 6x^2+16x-1} \\ \underline{6x^2-2x} \\ 18x-1 \\ \underline{18x-6} \\ 5 \end{array}$$

Stop when the remainder is zero or the degree of the remainder is less than the degree of the divisor.

Division statement:

$$(\text{quotient}) \times (\text{divisor}) + \text{remainder} = \text{dividend}$$

$$(2x+6)(3x-1)+5=6x^2+16x-1$$

EXAMPLE 2. Divide $(6x^2-6+x^3+7x)$ by $(x+3)$.

Solution

Arrange the terms of the dividend and divisor in descending (or ascending) powers of the variable.

$$\begin{array}{r} x^2+3x-2 \\ x+3 \overline{) x^3+6x^2+7x-6} \\ \underline{x^3+3x^2} \\ 3x^2+7x \\ \underline{3x^2+9x} \\ -2x-6 \\ \underline{-2x-6} \\ 0 \end{array}$$

What is the restriction on x ?

EXAMPLE 3. Divide $(x^2y^2+y^4-xy^3+x^4)$ by (x^2+y^2-xy) and write the division statement.

Solution

Arrange the terms in ascending or descending powers of *one* of the variables. When there are missing terms in the dividend represent them by using zero as the coefficient.

$$\begin{array}{r} x^2+xy+y^2 \\ x^2-xy+y^2 \overline{) x^4+0x^3y+x^2y^2-xy^3+y^4} \\ \underline{x^4-x^3y+x^2y^2} \\ x^3y+0x^2y^2-xy^3 \\ \underline{x^3y-x^2y^2+xy^3} \\ x^2y^2-2xy^3+y^4 \\ \underline{x^2y^2-xy^3+y^4} \\ -xy^3 \end{array}$$

Division statement:

$$(x^2 + xy + y^2)(x^2 - xy + y^2) - xy^3 = x^4 + x^2y^2 - xy^3 + y^4$$

EXERCISE 1-6

A 1. Arrange the following in descending or ascending powers of a variable.

- | | |
|-----------------------------------|-------------------------------------|
| (a) $x^3 + 3x^5 - 4x^2 + 7$ | (b) $1 - x^4 + 2x^2 - 5x$ |
| (c) $3m^2 + 2m^5 - 4m^3 + 5m$ | (d) $r^4 + rs^3 - s^4 + 2r^2s^2$ |
| (e) $3x - 5x^4 + 4x^2 + 2x^5$ | (f) $xy^3 + 2x^2y^2 + 4x^4 - 3x^3y$ |
| (g) $1 - t^5 + 4t^3 - 5t^2 + t^4$ | (h) $m - 3m^3 + 2m^2 + 5m^5$ |

2. State the division statement for the following.

- | | | | |
|-----------------|-----------------|------------------|------------------|
| (a) $19 \div 3$ | (b) $7 \div 2$ | (c) $11 \div 7$ | (d) $46 \div 20$ |
| (e) $21 \div 9$ | (f) $72 \div 7$ | (g) $65 \div 12$ | (h) $98 \div 9$ |

B 3. Divide, then write the division statement and state any restrictions on the variables.

- (a) $(2x^2 + 7x + 5) \div (x + 1)$
- (b) $(x^3 - 2x^2 - 11x + 13) \div (x - 4)$
- (c) $(2y^3 - 14y^2 + 31y - 30) \div (y - 4)$
- (d) $(6w^2 - 29w + 18) \div (2w - 7)$
- (e) $(9m^2 - 3m - 20) \div (3m + 4)$
- (f) $(2t^3 - t^2 - 5t + 20) \div (2t + 5)$
- (g) $(4m^3 - 4m^2 - 11m + 5) \div (2m + 1)$
- (h) $(12x^3 - 23x^2 + 38x - 35) \div (4x - 5)$
- (i) $(8y^3 + 12y^2 - 10y + 1) \div (2y + 3)$

4. Divide.

- (a) $(16x^3 + 40x^2 - 2x - 11) \div (2x + 5)$
- (b) $(x^2 + x^3 + 8 - 10x) \div (x^2 - 3x + 2)$
- (c) $(m^2 - 5m + m^3 + 7) \div (m + 3)$
- (d) $(4x^2 - 53) \div (2x - 7)$
- (e) $(7t - t^3 + t^4 - 5t^2 - 8) \div (2 + t^2 - 3t)$
- (f) $(8r^3 + 7) \div (2r - 1)$
- (g) $(2 + 15s^4 - 6s - 5s^3) \div (3s - 1)$
- (h) $(x^4 - 19) \div (x - 2)$
- (i) $(x^5 - 2) \div (x - 1)$

5. Divide.

- (a) $(x^2 + 2xy + 2y^2) \div (x + y)$
- (b) $(x^2 - 2xy + 4y^2) \div (x - y)$
- (c) $(x^2 - 7xy + 10y^2) \div (x - 4y)$
- (d) $(x^3 - y^3) \div (x - y)$
- (e) $(3x^2y + y^3 + 3xy^2 + 2x^3) \div (2x + y)$
- (f) $(2x^2y^2 + 2y^4 + xy^3 + x^4) \div (xy + x^2 + y^2)$
- (g) $(x^3y + x^4 - xy^3 - 8x^2y^2 + 3y^4) \div (x^2 + 3xy - y^2)$
- (h) $(10m^2n^2 - n^4 + 6m^4 - 15m^3n) \div (3m^2 - 3mn - n^2)$

6. The area of a rectangle is given by $6x^2 - 13x - 28$. If the width is expressed as $2x - 7$, find an expression for the length.

7. Divide $(x^7 - 128)$ by $(x - 2)$.

Divide the face of a watch with two straight lines so that the sums of the numbers in each part are equal.

1.7 THE REMAINDER THEOREM

Given the polynomial $P(x) = x^3 - x^2 - 7x + 9$, we shall divide $P(x)$ by $(x - 3)$ and then find $P(3)$.

$$\begin{array}{r}
 \overline{x^2 + 2x - 1} \\
 x-3 \overline{x^3 - x^2 - 7x + 9} \\
 \underline{x^3 - 3x^2} \\
 2x^2 - 7x \\
 \underline{2x^2 - 6x} \\
 -x + 9 \\
 \underline{-x + 3} \\
 6
 \end{array}
 \quad
 \begin{aligned}
 P(x) &= x^3 - x^2 - 7x + 9 \\
 P(3) &= (3)^3 - (3)^2 - 7(3) + 9 \\
 &= 27 - 9 - 21 + 9 \\
 &= 6
 \end{aligned}$$

When $P(x)$ is divided by $(x - 3)$ the remainder is $P(3)$. This leads to a statement of the Remainder Theorem.

Remainder Theorem

When a polynomial $P(x)$ is divided by $(x - b)$, and the remainder contains no term in x , then the remainder is $P(b)$.

Proof

The following division statement is true for all values of x .

$$P(x) = (x - b) \cdot Q(x) + R$$

$Q(x)$ represents a polynomial in x and R is a constant.

Substituting b for x we have

$$P(b) = (b - b) \cdot Q(b) + R$$

$$P(b) = 0 \cdot Q(b) + R$$

$$P(b) = R$$

In general,

When a polynomial $P(x)$ is divided by $(ax - b)$, and the remainder contains no term in x , then the remainder is $P\left(\frac{b}{a}\right)$.

$$\begin{array}{r}
 Q(x) \\
 x-b \overline{P(x)} \\
 \hline
 R
 \end{array}$$

$$\begin{aligned}
 &(ax - b) \\
 &= a\left(x - \frac{b}{a}\right)
 \end{aligned}$$

EXAMPLE 1. Use the Remainder Theorem to determine the remainder when $2x^3 - 4x^2 + 3x - 6$ is divided by $x + 2$.

Solution

$$P(x) = 2x^3 - 4x^2 + 3x - 6$$

$$\begin{aligned}
 P(-2) &= 2(-2)^3 - 4(-2)^2 + 3(-2) - 6 \\
 &= -16 - 16 - 6 - 6 \\
 &= -44
 \end{aligned}$$

The remainder is -44 .

EXAMPLE 2. Use the Remainder Theorem to determine the remainder when $2x^3 - 13x^2 + 19x + 7$ is divided by $2x - 3$.

Solution

$$P(x) = 2x^3 - 13x^2 + 19x + 7$$

$$\begin{aligned} P\left(\frac{3}{2}\right) &= 2\left(\frac{3}{2}\right)^3 - 13\left(\frac{3}{2}\right)^2 + 19\left(\frac{3}{2}\right) + 7 \\ &= \frac{27}{4} - \frac{117}{4} + \frac{57}{2} + 7 \\ &= \frac{27}{4} - \frac{117}{4} + \frac{114}{4} + \frac{28}{4} \\ &= \frac{52}{4} \\ &= 13 \end{aligned}$$

The remainder is 13.

EXERCISE 1-7

- B** 1. For $P(x) = x^3 + x^2 + x + 4$, find
 (a) $P(2)$ (b) $P(1)$ (c) $P(0)$ (d) $P(-2)$
2. For $f(x) = x^3 - 2x^2 - 2x - 3$, find
 (a) $f(3)$ (b) $f(-1)$ (c) $f(-3)$ (d) $f(5)$
3. For $g(x) = 2x^2 - 4x + 5$, find
 (a) $g(-3)$ (b) $g\left(\frac{1}{2}\right)$ (c) $g\left(-\frac{1}{2}\right)$ (d) $g\left(\frac{3}{2}\right)$
4. Use the Remainder Theorem to determine the remainder for each division.
 (a) $(x^3 + 2x^2 + 3x + 7) \div (x - 1)$
 (b) $(m^3 - m^2 + 7m - 4) \div (m + 1)$
 (c) $(x^3 - 8x^2 + 17x - 6) \div (x - 3)$
 (d) $(n^4 - 3n^2 + 7) \div (n + 3)$
 (e) $(1 - 2x - 3x^2 + x^3) \div (x - 1)$
 (f) $(x^3 - 27) \div (x - 3)$
 (g) $(2x^3 - 3x - 4) \div (x + 2)$
 (h) $(3m^3 - m^2 - m - 2) \div (m + 1)$
- C** 5. Find the remainder for each division.
 (a) $(2x^2 + 5x - 4) \div (2x - 1)$
 (b) $(2x^2 - 5x + 2) \div (2x + 1)$
 (c) $(5m^3 + m^2 - 5m - 3) \div (2m + 3)$
 (d) $(3x^3 - x^2 - 12x + 7) \div (3x - 1)$
 (e) $(2m^3 + 3m^2 - 7m - 5) \div (2m + 5)$

1.8 THE FACTOR THEOREM

We now use a corollary of the Remainder Theorem, the Factor Theorem, to factor polynomials of the third degree and higher. When the polynomial $P(x) = x^3 + 2x^2 - x - 2$ is divided by $(x - 1)$ the remainder is zero since $P(1) = 0$. Since division by $(x - 1)$ gives zero for a remainder, $(x - 1)$ is a *factor* of $x^3 + 2x^2 - x - 2$.

Factor Theorem

For a polynomial $P(x)$, if $P(b) = 0$, then $(x - b)$ is a factor of $P(x)$.

In this section we shall restrict ourselves to polynomials that have at least one factor of the form $(x - b)$, i.e. where the coefficient of x is 1.

EXAMPLE 1. Factor $x^3 - 6x^2 - x + 30$.

Solution

We are looking for a factor in the form $(x - b)$. From previous experience in multiplying and dividing polynomials, we can assume that b must divide 30 evenly. Therefore, if b is an integer, it could have values of $\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15, \pm 30$.

$$P(x) = x^3 - 6x^2 - x + 30$$

$$P(1) = 1 - 6 - 1 + 30 = 24$$

$$P(-1) = -1 - 6 + 1 + 30 = 24$$

$$P(2) = 8 - 24 - 2 + 30 = 12$$

$$P(-2) = -8 - 24 + 2 + 30 = 0$$

Since $P(-2) = 0$, $x + 2$ is a factor of $x^3 - 6x^2 - x + 30$. A polynomial of the third degree will have at most three linear factors. We could continue substituting to find other linear factors (if they exist). However, division yields another factor quickly.

$$\begin{array}{r} x^2 - 8x + 15 \\ x+2 \overline{) x^3 - 6x^2 - x + 30} \\ \underline{x^3 + 2x^2} \\ -8x^2 - x \\ \underline{-8x^2 - 16x} \\ 15x + 30 \\ \underline{15x + 30} \\ 0 \end{array}$$

$$\begin{aligned} x^3 - 6x^2 - x + 30 &= (x + 2)(x^2 - 8x + 15) \\ &= (x + 2)(x - 3)(x - 5) \end{aligned}$$

EXAMPLE 2. Find k so that $x^3 - 4x^2 - 2x + k$ has $x - 3$ as a factor.

Solution

Let $P(x) = x^3 - 4x^2 - 2x + k$. If $x - 3$ is a factor, $P(3) = 0$.

$$P(x) = x^3 - 4x^2 - 2x + k$$

$$P(3) = (3)^3 - 4(3)^2 - 2(3) + k = 0$$

$$27 - 36 - 6 + k = 0$$

$$-15 + k = 0$$

$$k = 15$$

38 teams are entered in a basketball tournament. Each team plays until it loses. How many games will be needed to determine a winner?

EXERCISE 1-8

A 1. Determine which of the polynomials have $x-1$ as a factor.

- (a) x^3+x^2-x-1 (b) $2x^3-x^2-3x-1$
 (c) $x^4-3x^3+2x^2-x+1$ (d) $3x^3-x-3$
 (e) $4x^4-2x^3+3x^2-2x+1$ (f) x^3-3x^2+4x-2
 (g) $2x^3+4x^2-5x-1$ (h) x^3-x^2-x-1

2. Is $(x+1)$ a factor of $x^{100}-1$? Is $(x-1)$? Explain.

3. Is $(x+1)$ a factor of $x^{99}+1$? Is $(x-1)$? Explain.

B 4. Use the factor theorem to show that

- (a) $(x+1)$ is a factor of x^3+2x^2+2x+1 .
 (b) $(x-2)$ is a factor of x^3-3x^2+4x-4 .
 (c) $(x-3)$ is a factor of x^3-3x^2+x-3 .
 (d) $(x+3)$ is a factor of $x^3+7x^2+17x+15$.
 (e) $(x+2)$ is a factor of $2x^3+4x^2-3x-6$.
 (f) $(x+5)$ is a factor of $x^4+5x^3+2x^2+7x-15$.
 (g) $(x-4)$ is a factor of $2x^4-11x^3+12x^2+x-4$.

5. Is $(x-3)$ a factor of x^3-5x^2+7x-1 ?

6. Factor the following over the integers.

- (a) $x^3-6x^2+11x-6$ (b) $x^3+8x^2+19x+12$
 (c) $t^3-2t^2-9t+18$ (d) m^3+4m^2+2m-3
 (e) $x^3+x^2-22x-40$ (f) $x^3+x^2-16x-16$
 (g) w^3-2w^2-6w-8 (h) $n^3+6n^2-7n-60$
 (i) x^3-27 (j) $x^3-27x+10$

7. Factor the following.

- (a) $2x^3-9x^2+10x-3$ (b) $3x^3-2x^2-12x+8$
 (c) $2x^3-3x^2+3x-10$ (d) $4m^3-7m-3$
 (e) $5t^3-23t^2+20t+12$ (f) $4y^3+13y^2+4y+3$
 (g) $6x^3-11x^2-26x+15$ (h) $2x^3-4x^2+11x-22$
 (i) $2x^4+x^3-26x^2-37x-12$ (j) $x^5-3x^4-10x^3+3x^2-9x-30$

8. (a) Show that $(x-2)$ is a factor of x^3-8 .

(b) Factor x^3-8 .

9. (a) Show that $(m+3)$ is a factor of m^3+27 .

(b) Factor m^3+27 .

FACTORIZING THE SUM OF CUBES

10. Use the distributive property to show that

(i) $a^3+b^3=(a+b)(a^2-ab+b^2)$

FACTORIZING THE DIFFERENCE OF CUBES

(ii) $a^3-b^3=(a-b)(a^2+ab+b^2)$

11. Using the results of question 10, factor the following.

- (a) x^3-27 (b) x^3+64 (c) m^3+8
 (d) y^3-125 (e) $8x^3-1$ (f) x^3-y^3
 (g) m^3+n^3 (h) $64m^3-1$ (i) $27x^3+8y^3$
 (j) x^6+y^9 (k) s^6-27t^9 (l) $1000x^{12}+y^{15}$

C 12. Find k so that x^3-2x^2+3x+k has $(x-1)$ as a factor.

13. Find k so that x^3+5x^2+kx+6 has $(x+2)$ as a factor.

14. Find k so that km^3-10m^2+2m+3 has $(m-3)$ as a factor.

15. Show that $(x-a)$ is a factor of x^n-a^n .

16. (a) If n is even, is $(x+a)$ a factor of $x^n + a^n$?
 (b) If n is odd, is $(x+a)$ a factor of $x^n + a^n$?
 17. Is $(x-a)$ a factor of $x^2(a-b) + a^2(b-x) + b^2(x-a)$?
 18. Use the Factor Theorem to show that $(x-c)$ is a factor of $(x-b)^2 - (b-c)^2 + (c-x)^2$.

1.9 SIMPLIFYING RATIONAL EXPRESSIONS

If x and y are two integers, then the quotient $\frac{x}{y}$, where $y \neq 0$, is a rational number. Examples of rational numbers are

$$\frac{3}{7}, \quad -8 = \frac{-8}{1}, \quad 0.23 = \frac{23}{100}$$

What is the last year which had no "Friday the 13ths"?

We define rational expressions in a similar manner. A rational expression is an expression that can be written as the quotient of two polynomials.

If P and Q are polynomials and $Q \neq 0$, then $\frac{P}{Q}$ is a rational expression.

Examples of rational expressions are

$$\frac{3x}{5+x}, \quad \frac{x^2+2x}{x^2}, \quad \frac{2x}{x-y}, \quad \frac{x^2+y^2}{x^2-y^2}$$

A rational number is said to be *reduced* or *simplified* if the numerator and denominator have no common factors other than 1.

To simplify rational numbers we (1) factor the numerator and denominator into prime factors, (2) divide both numerator and denominator by common factors.

$$\frac{30}{42} = \frac{2 \times 3 \times 5}{2 \times 3 \times 7} = \frac{5}{7}$$

Rational expressions are in *simplest form*, or *lowest terms*, when expressed as the quotient of two polynomials whose greatest common factor is 1.

To simplify rational expressions we (1) factor the numerator and denominator and (2) divide both numerator and denominator by every factor common to both.

Since division by zero is not defined, restrictions must be placed on the variables in rational expressions in order to prevent division by zero.

EXAMPLE 1. Reduce $\frac{x^2-9}{x^2-x-6}$ to lowest terms and state any restrictions on the variable.

Solution

3 and -2 are called the "zeros" of the denominator.

$$\begin{aligned}\frac{x^2-9}{x^2-x-6} &= \frac{(x-3)(x+3)}{(x-3)(x+2)} \\ &= \frac{x+3}{x+2} \quad \text{Restrictions: } x \neq 3, -2\end{aligned}$$

$x^2 + 2x - 3x - 6$

EXAMPLE 2. Simplify $\frac{3x^2-3x^3}{x^3-x}$ and state restrictions on the variable.

Solution

$$(1-x) = (-1)(x-1)$$

$$\begin{aligned}\frac{3x^2-3x^3}{x^3-x} &= \frac{3x^2(1-x)}{x(x^2-1)} \\ &= \frac{3x \cdot x(-1)(x-1)}{x(x+1)(x-1)} \\ &= \frac{-3x}{x+1} \quad \text{Restrictions: } x \neq 0, 1, -1\end{aligned}$$

EXAMPLE 3. Simplify

(a) $\frac{x^2-9}{3-x}$

(b) $\frac{x^2-x-20}{15+2x-x^2}$

Solution

$$\begin{aligned}\text{(a)} \quad \frac{x^2-9}{3-x} &= \frac{(x-3)(x+3)}{(3-x)} \\ &= \frac{(x-3)(x+3)}{(-1)(x-3)} \\ &= (-1)(x+3) \\ &= -x-3\end{aligned}$$

Restriction: $x \neq 3$

$$\begin{aligned}\text{(b)} \quad \frac{x^2-x-20}{15+2x-x^2} &= \frac{(x-5)(x+4)}{(5-x)(3+x)} \\ &= \frac{(x-5)(x+4)}{(-1)(x-5)(3+x)} \\ &= \frac{(-1)(x+4)}{(x+3)} \\ &= \frac{-x-4}{x+3}\end{aligned}$$

Restriction: $x \neq 5, -3$

EXERCISE 1-9

- A** 1. State the missing numerator or denominator to obtain an equivalent rational expression.

(a) $\frac{mx}{x^2} = \frac{\quad}{x}$

(b) $\frac{x}{y} = \frac{wx}{\quad}$

(c) $\frac{mx}{m} = \frac{x}{\quad}$

(d) $\frac{rst}{rstu} = \frac{\quad}{u}$

Multiplication distributes over addition

$$3(x+y) = 3x+3y$$

$$(e) \frac{x+y}{2} = \frac{\boxed{}}{6}$$

$$(f) \frac{1}{m-n} = \frac{7}{\boxed{}}$$

$$(g) \frac{(x+2)(x+3)}{(x+3)(x-1)} = \frac{x+2}{\boxed{}}$$

$$(h) \frac{3}{x-y} = \frac{-3}{\boxed{}}$$

$$(i) \frac{-2}{m-n} = \frac{\boxed{}}{n-m}$$

$$(j) \frac{2x(x+4)}{3x^2(x+4)} = \frac{\boxed{}}{3x}$$

2. State the restrictions on the variables in each of the following.

$$(a) \frac{5}{x}$$

$$(b) \frac{6}{x-4}$$

$$(c) \frac{x}{m-2}$$

$$(d) \frac{8}{t+3}$$

$$(e) \frac{1}{(x-1)(x+3)}$$

$$(f) \frac{x}{(x+2)(x+7)}$$

$$(g) \frac{r-4}{r+5}$$

$$(h) \frac{m+2}{m(m+8)}$$

$$(i) \frac{3x}{2x-1}$$

$$(j) \frac{16}{x-m}$$

$$(k) \frac{x-4}{(2x+1)(3x-2)}$$

$$(l) \frac{3x^2}{(1-x)(3-x)}$$

3. Simplify, where possible, the following rational expressions.

$$(a) \frac{12}{30}$$

$$(b) \frac{24x}{10}$$

$$(c) \frac{tx}{ty}$$

$$(d) \frac{-6rst}{3rs}$$

$$(e) \frac{4m}{3n}$$

$$(f) \frac{20xy}{10x}$$

$$(g) \frac{mx}{ny}$$

$$(h) \frac{-15xy}{-3w}$$

$$(i) \frac{16x^5}{4x^3}$$

$$(j) \frac{-21x^2y^4z}{3x^4y^2}$$

$$(k) \frac{x^5}{x^{10}}$$

$$(l) \frac{r^4s^6t^7}{xy}$$

$$(m) \frac{2xy}{8x^4y^3}$$

$$(n) \frac{-2m^3n}{6m^2n^7}$$

$$(o) \frac{15x^3y^4}{-3x^5y^4}$$

$$(p) \frac{(x+2)(x+5)}{(x+5)(x-2)}$$

$$(q) \frac{(m-3)^2}{(m-3)(m+3)}$$

$$(r) \frac{2(t-3)(t+2)}{3(t+2)(t-3)}$$

B 4. Simplify, where possible, and state any restrictions on the variables.

$$(a) \frac{x^2-3x-28}{x^2-4x-21}$$

$$(b) \frac{x^2+7x}{x^2+14x+49}$$

$$(c) \frac{2t-4}{t^2-3t+2}$$

$$(d) \frac{2m+6}{m^2-4}$$

$$(e) \frac{6x+6y}{3x-3y}$$

$$(f) \frac{2x-10}{x^2+2x-35}$$

$$(g) \frac{x^2-49}{x^2+6x-7}$$

$$(h) \frac{x^2-5x+6}{x^2-5x-6}$$

$$(i) \frac{x^2-4}{2-x}$$

$$(j) \frac{x-4}{4-x}$$

$$(k) \frac{2m^2-2}{m^2+2m+1}$$

$$(l) \frac{x^2+3x-28}{12+x-x^2}$$

Multiplication does not distribute over multiplication

$$3(x \cdot y) = 3 \cdot x \cdot y$$

Division distributes over addition

$$\frac{x+y}{3} = \frac{x}{3} + \frac{y}{3}$$

Division does not distribute over multiplication

$$\frac{x \cdot y}{3} = \frac{x}{3} \cdot y = x \cdot \frac{y}{3}$$

Knowing the above rules may eliminate errors such as

$$\frac{x+\cancel{3}}{\cancel{3}} = x+1$$

"ERROR"

Since division distributes over addition, then

$$\frac{x+3}{3} = \frac{x}{3} + \frac{3}{3} = \frac{x}{3} + 1$$

similarly

$$\frac{x+2}{x+3} \neq \frac{x+2}{x+3} \neq \frac{1+2}{1+3} = \frac{3}{4}$$

but

$$\frac{x+2}{x+3} = \frac{\frac{x}{x} + \frac{2}{x}}{\frac{x}{x} + \frac{3}{x}} = \frac{1 + \frac{2}{x}}{1 + \frac{3}{x}}$$

The 9 L jug is filled with wine. The 5 L, 4 L and 2 L jugs are empty. By decanting, divide the wine into three equal parts.



$$(m) \frac{3-3x}{x^2+3x-4}$$

$$(n) \frac{x^2-y^2}{x^2-2xy+y^2}$$

$$(o) \frac{5m}{5m+5n}$$

5. Simplify, where possible, and state restrictions on the variables.

$$(a) \frac{2x^2+3x-2}{x^2-4}$$

$$(b) \frac{6x^2+11x+3}{6x^2-7x-3}$$

$$(c) \frac{x^2-y^2}{y-x}$$

$$(d) \frac{x^2-5x}{x^2-x-20}$$

$$(e) \frac{2m^2+7m+3}{2m^2+5m+3}$$

$$(f) \frac{12-5m-2m^2}{2m^2-7m+6}$$

$$(g) \frac{6m^2-54}{2m^2+7m+3}$$

$$(h) \frac{x^3-8}{3x^2-x-10}$$

$$(i) \frac{-xm-ym}{x^2+2xy+y^2}$$

$$(j) \frac{3x+6}{x^4-16}$$

$$(k) \frac{4(x^2-y^2)}{(x-y)^2}$$

$$(l) \frac{6x^2+21x-12}{9x^2-3x-30}$$

C 6. Simplify the following.

$$(a) \frac{x^2-4}{x^3+4x^2-4x-16}$$

$$(b) \frac{x^3-7x-6}{x^2-2x-3}$$

$$(c) \frac{3x+4(x-2)}{4x+5(x-2)}$$

$$(d) \frac{(x-2)(x+3)+(x-2)}{(x-2)(x+3)(x-4)}$$

$$(e) \frac{1-(10-x^2)}{2x^2+9x+9}$$

$$(f) \frac{x^3+8}{3x^2-6x+12}$$

$$(g) \frac{x^2-y^2}{x^3+y^3}$$

$$(h) \frac{4-x^2}{2x^3-x^2-8x+4}$$

$$(i) \frac{2(x-3)-3(x-4)}{2x^2-6x-36}$$

$$(j) \frac{15x^2+14xy-8y^2}{6x^2+5xy-4y^2}$$

1.10 MULTIPLICATION AND DIVISION OF RATIONAL EXPRESSIONS

Rational expressions are multiplied in the same way that we multiply rational numbers. Recall that

$$\frac{3}{5} \cdot \frac{2}{7} = \frac{3 \cdot 2}{5 \cdot 7} = \frac{6}{35}$$

For rational expressions $\frac{P}{Q}$ and $\frac{R}{S}$,

$$\frac{P}{Q} \cdot \frac{R}{S} = \frac{PR}{QS}, \quad Q, S \neq 0$$

When multiplying rational expressions, first factor the numerators and denominators and divide by any common factors. Then express the product as a rational expression.

EXAMPLE 1. Multiply $\frac{x+1}{x^2-x-6}$ by $\frac{x^2-7x+12}{x^2-16}$

Solution

$$\begin{aligned}\frac{x+1}{x^2-x-6} \cdot \frac{x^2-7x+12}{x^2-16} &= \frac{(x+1)}{(x-3)(x+2)} \cdot \frac{(x-4)(x-3)}{(x-4)(x+4)} \\ &= \frac{(x+1)}{(x+2)(x+4)}, \quad x \neq 3, -2, 4, -4\end{aligned}$$

Rational expressions are divided in the same way that we divide rational numbers. Recall that

$$\frac{3}{4} \div \frac{11}{5} = \frac{3}{4} \cdot \frac{5}{11} = \frac{15}{44}$$

For rational expressions $\frac{P}{Q}$ and $\frac{R}{S}$,

$$\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \cdot \frac{S}{R} = \frac{PS}{QR}, \quad Q, R, S \neq 0$$

To divide one fraction by another, invert the divisor and multiply by the dividend.

EXAMPLE 2. Divide $\frac{2x-4}{x^2+9x+20}$ by $\frac{x^2+x-6}{x^2+7x+12}$

Solution

$$\begin{aligned}\frac{2x-4}{x^2+9x+20} \div \frac{x^2+x-6}{x^2+7x+12} &= \frac{2x-4}{x^2+9x+20} \cdot \frac{x^2+7x+12}{x^2+x-6} \\ &= \frac{2(x-2)}{(x+5)(x+4)} \cdot \frac{(x+4)(x+3)}{(x+3)(x-2)} \\ &= \frac{2}{x+5}, \quad x \neq -5, -4, -3, 2\end{aligned}$$

|| you switch the S and V in the word RESERVE, you get REVERSE. Switch two letters in LIMESTONE to make a new word.

EXAMPLE 3. Simplify

$$\frac{x^2-9y^2}{x^2+4xy} \cdot \frac{x^2+9xy+20y^2}{x^2+10xy+21y^2} \div x^2+2xy-15y^2$$

Solution

It is helpful to introduce 1 as a denominator.

$$\begin{aligned}\frac{x^2-9y^2}{x^2+4xy} \cdot \frac{x^2+9xy+20y^2}{x^2+10xy+21y^2} \div x^2+2xy-15y^2 \\ = \frac{x^2-9y^2}{x^2+4xy} \cdot \frac{x^2+9xy+20y^2}{x^2+10xy+21y^2} \div \frac{x^2+2xy-15y^2}{1}\end{aligned}$$

$$\begin{aligned}
&= \frac{x^2-9y^2}{x^2+4xy} \cdot \frac{x^2+9xy+20y^2}{x^2+10xy+21y^2} \cdot \frac{1}{x^2+2xy-15y^2} \\
&= \frac{(x-3y)(x+3y)}{x(x+4y)} \cdot \frac{(x+4y)(x+5y)}{(x+3y)(x+7y)} \cdot \frac{1}{(x+5y)(x-3y)} \\
&= \frac{1}{x(x+7y)}, \quad x \neq 0, -4y, -3y, -7y, -5y, 3y
\end{aligned}$$

EXERCISE 1-10

Express each of the following as rational expressions in lowest terms. Assume that all variables are restricted so that no denominator or divisor is equal to zero.

- A**
- (a) $\frac{x}{y} \cdot \frac{y}{w}$
 - (b) $\frac{2}{x} \cdot \frac{y}{2}$
 - (c) $\frac{1}{2} \cdot \frac{5}{6}$
 - (d) $2 \cdot \frac{x}{y}$
 - (e) $\frac{rs}{3} \cdot \frac{7}{rst}$
 - (f) $\frac{3x^2}{y} \cdot \frac{2y^2}{6x}$
 - (g) $\frac{3}{(x+2)} \cdot \frac{(x+2)}{5}$
 - (h) $\frac{1}{2} \div \frac{x}{y}$
 - (i) $x \div \frac{1}{x}$
 - (j) $\frac{(m-3)}{x} \div (m-3)$
 - (k) $\frac{m}{3n} \cdot \frac{3m}{n}$
 - (l) $\frac{x}{3y} \div \frac{3x}{y}$
 - (m) $\frac{x-y}{6} \cdot \frac{3}{x-y}$
 - (n) $\frac{x-4}{5} \div \frac{x-4}{15}$
 - (o) $\frac{1}{x+y} \div \frac{7}{x+y}$
 - (p) $\frac{x^3}{y^4} \cdot \frac{y^6}{x^5}$
 - (q) $\frac{1}{x+y} \div (x+y)$
 - (r) $\frac{(x+2)(x+3)}{(x-1)(x+2)} \cdot \frac{(x-1)}{(x+4)}$
 - (s) $\frac{(x-3)(x-2)}{(x-7)(x-3)} \cdot \frac{(x+5)(x-7)}{(x-2)(x+5)}$
 - (t) $\frac{x(x+2)}{y(x-3)} \div \frac{x}{y}$
 - (u) $\frac{x(x+y)}{x^2} \cdot \frac{x}{y}$
- B**
- (a) $\frac{14x^2}{21y^2} \cdot \frac{15y^4}{7x^3}$
 - (b) $\frac{6x^5y^3}{5xy} \cdot \frac{10x^2y}{4x^3y^4}$
 - (c) $\frac{21x^3y}{5xy} \div \frac{7x^2y^6}{10xy^4}$
 - (d) $\frac{56x^4y^6z}{7x^3yz} \div 2xyz$
 - (e) $\frac{20r^3s^4}{8x^2y} \div \frac{5rs^2}{2xy}$
 - (f) $\frac{6m^2n^2}{5x^2y^4} \cdot \frac{3xy}{12mn}$
 - (a) $\frac{x+3}{5x} \cdot \frac{10x}{x+3}$
 - (b) $\frac{m-2}{x} \cdot \frac{9x^2}{3m-6}$
 - (c) $\frac{7t-7}{4t} \div \frac{2t-2}{16t^4}$
 - (d) $\frac{x^2-9}{3x+9} \cdot \frac{2}{3-x}$
 - (e) $\frac{x^2+3x+2}{4-x^2} \cdot \frac{10x}{5x+5}$
 - (f) $\frac{x^2-y^2}{4} \div (x-y)$
 - (a) $\frac{x^2+7x+12}{x^2+3x+2} \cdot \frac{x^2+5x+6}{x^2+6x+9}$
 - (b) $\frac{x^2+2x-15}{x^2-6x+8} \cdot \frac{x^2+2x-8}{x^2-6x+9} \cdot \frac{x^2-7x+12}{x^2-x-30}$
 - (c) $\frac{x+7}{x^2-5x-36} \cdot \frac{x^2-15x+54}{x^2-36} \div \frac{x^2-2x-63}{x+4}$

$$(d) \frac{t^2-9}{t^2-2t-15} \cdot \frac{4+3t-t^2}{t^2-7t+12} \cdot \frac{t^2+2t-35}{6+5t-t^2}$$

$$(e) \frac{x^2-49}{x^2+3x-88} \cdot \frac{x^2+6x-55}{x^2-11x+28} \div \frac{x^2+2x-35}{x^2-12x+32}$$

$$(f) \frac{x^2-6x-27}{x^2+4x-5} \cdot \frac{x^2-1}{x^2+x-90} \div \frac{x^2+4x+3}{x^2+15x+50}$$

$$(g) \frac{5m^2-20}{m^2+14m+33} \cdot \frac{m^2+10m-11}{m^2-8m+12} \cdot \frac{m^2-3m-18}{2-m-m^2}$$

$$5. (a) \frac{2x^2+7x-4}{6x^2+x-2} \cdot \frac{15x^2+7x-4}{5x^2+19x-4}$$

$$(b) \frac{16-9x^2}{6x^2+13x-5} \cdot \frac{2x^2+7x+5}{6x^2-29x+28} \div \frac{3x^2+7x+4}{6x^2-19x-7}$$

$$(c) \frac{2x^2+5x-3}{4x^2-12x+5} \cdot \frac{6x^2-7x-20}{3x^2+13x+12}$$

$$(d) \frac{12x^2+13x-35}{5x^2-21x+18} \div \frac{3x^2+16x+21}{5x^2+9x-18}$$

$$(e) \frac{2m^2+5m-12}{m^2+9m+14} \cdot \frac{m^2+3m-28}{16-m^2} \div \frac{6m^2-7m-3}{3m^2+25m+8}$$

$$C \quad 6. (a) \frac{x^3+8}{2x^2+x-6} \cdot \frac{12x-18}{3x^2-6x+12} \quad (b) \frac{x^3+x^2-14x-24}{3x^2+17x-6} \cdot \frac{3x^2+14x-5}{x^2-x-12}$$

$$(c) \frac{x^2-9y^2}{6x^2+5xy-4y^2} \cdot \frac{10x-5y}{x^2+2xy-3y^2} \cdot \frac{3x^2+xy-4y^2}{20x-10y}$$

$$7. (a) \frac{\frac{x^2-4}{4x^2}}{\frac{x+2}{8x}} \quad (b) \frac{\frac{m^2+2m+1}{m+1}}{m} \quad (c) \frac{\frac{4-x}{x}}{x^2-16}$$

WORD LADDER

Start with the word "flour" and change one letter at a time to form a new word until you reach "bread". The best solution has the fewest steps.

f l o u r

b r e a d

1.11 LEAST COMMON MULTIPLE

In order to add or subtract rational expressions it is necessary to know how to determine the *least common multiple* (LCM) of two or more polynomials.

The LCM for 12 and 30 is the smallest number that has 12 and 30 as factors. In order to determine the LCM for 12 and 30 we first factor 12 and 30.

$$12 = 2 \cdot 2 \cdot 3$$

$$30 = 2 \cdot 3 \cdot 5$$

The LCM for 12 and 30 must include all the separate factors that make up 12 and 30.

$$\text{The LCM is } 2 \cdot \overbrace{2 \cdot 3}^{12} \cdot 5 \quad \text{or} \quad 60$$

30

EXAMPLE 1. Find the LCM for

$$x^2 + x - 6, x^2 + 7x + 12 \quad \text{and} \quad x^2 - 2x$$

Solution

Factor each polynomial

$$x^2 + x - 6 = (x + 3)(x - 2)$$

$$x^2 + 7x + 12 = (x + 3)(x + 4)$$

$$x^2 - 2x = x(x - 2)$$

$$\text{The LCM is } \underbrace{x}_{x^2-2x} \underbrace{(x-2)}_{x^2+x-6} \underbrace{(x+3)(x+4)}_{x^2+7x+12}$$

EXERCISE 1-11**B** Find the LCM for each of the following.

Answer may be left in factored form.

1. (a) 6, 14

(b) 8, 6, 3

(c) 15, 24, 45

(d) 20, 25, 30

(e) 7, 16, 18

(f) 4, 6, 9, 28

2. (a) x^2, xy, y^2

(b) $2x^2, 4xy, y^2$

(c) $6m^2, 2m^2n, 3mn^2$

(d) $8m^3n, 3m^2n^2, 2mn^3, 6n^4$

(e) $20xy, 15y, 30x^3$

(f) $18x^5y, 27txy, 15xy^4$

3. (a) $x^2 - x - 12, x^2 - 9$

(b) $mx - my, nx - ny$

(c) $2x + 6, x^2 + 6x + 9$

(d) $x^2 - xy, x^2 - y^2, x^2 + xy$

(e) $5x - 10, x^2 - 4, 6x + 12$

(f) $5x + 15, x^2 + 8x + 15, x^2 + 4x - 5$

(g) $t^2 - 1, t^2 + 7t + 6, t^2 + 6t - 7$

(h) $m^2 + m - 20, m^2 - 6m + 8, m^2 + 3m - 10$

4. (a) $x^2 - y^2, x^2 - 2xy + y^2$

(b) $m^2 + 2mn + n^2, m^2 - 2mn - 3n^2$

(c) $2x^2 + 5x - 3, 6x^2 + x - 2$

(d) $10m^2 + 9m - 9, 6m^2 + m - 12$

(e) $15r^2 - 17r - 4, 9r^2 - 16, 15r - 20$

(f) $6x^2 - 5x - 6, 6x^2 - 11x - 10, 4x^2 - 16x + 15$

1.12 ADDITION AND SUBTRACTION OF RATIONAL EXPRESSIONS

To write the sum of fractions with equal denominators as a single fraction we use the distributive property. Division, like multiplication, distributes over addition.

$$\frac{a}{b} + \frac{c}{b} = \frac{1}{b} \cdot a + \frac{1}{b} \cdot c = \frac{1}{b} (a + c) = \frac{a + c}{b}$$

For rational expressions $\frac{P}{Q}$ and $\frac{R}{Q}$ with the same denominator

$$\frac{P}{Q} + \frac{R}{Q} = \frac{P + R}{Q} \quad \text{and} \quad \frac{P}{Q} - \frac{R}{Q} = \frac{P - R}{Q}$$

EXAMPLE 1. Simplify

(a) $\frac{5}{13} + \frac{8}{13} - \frac{2}{13}$

(b) $\frac{3x}{x+2} + \frac{x+5}{x+2} - \frac{x}{x+2}$

Solution

$$\begin{aligned} \text{(a)} \quad \frac{5}{13} + \frac{8}{13} - \frac{2}{13} &= \frac{5+8-2}{13} \\ &= \frac{11}{13} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{3x}{x+2} + \frac{x+5}{x+2} - \frac{x}{x+2} &= \frac{3x+x+5-x}{x+2} \\ &= \frac{3x+5}{x+2} \end{aligned}$$

To add or subtract rational expressions with different denominators we

- (1) find the least common denominator (LCD) of the fractions. (The LCD is the LCM of the denominators)
- (2) express each fraction as an equivalent fraction having the LCD as the denominator.
- (3) perform the indicated operations and simplify.

EXAMPLE 2. Simplify $\frac{2w-3}{4} + \frac{3w-1}{5} - \frac{w-5}{2}$

Solution

The LCD is 20.

$$\begin{aligned} \frac{2w-3}{4} + \frac{3w-1}{5} - \frac{w-5}{2} &= \frac{5(2w-3)}{5 \cdot 4} + \frac{4(3w-1)}{4 \cdot 5} - \frac{10(w-5)}{10 \cdot 2} \\ &= \frac{5(2w-3)}{20} + \frac{4(3w-1)}{20} - \frac{10(w-5)}{20} \\ &= \frac{5(2w-3) + 4(3w-1) - 10(w-5)}{20} \\ &= \frac{10w - 15 + 12w - 4 - 10w + 50}{20} \\ &= \frac{12w + 31}{20} \end{aligned}$$

Find a four-digit number of the form aabb which is a perfect square.

EXAMPLE 3. Simplify $\frac{5}{3x^2} - \frac{1}{2x} + \frac{3}{5x^3}$

*Solution*The LCD is $30x^3$

$$\begin{aligned} \frac{5}{3x^2} - \frac{1}{2x} + \frac{3}{5x^3} &= \frac{10x \cdot 5}{10x \cdot 3x^2} - \frac{15x^2 \cdot 1}{15x^2 \cdot 2x} + \frac{6 \cdot 3}{6 \cdot 5x^3} \\ &= \frac{50x}{30x^3} - \frac{15x^2}{30x^3} + \frac{18}{30x^3} \\ &= \frac{50x - 15x^2 + 18}{30x^3}, \quad x \neq 0 \end{aligned}$$

EXAMPLE 4. Simplify

$$\frac{3x-12}{x^2-x-12} - \frac{2}{x^2+6x+9} - \frac{1}{x^2-4x-21}$$

Solution

$$\begin{aligned} & \frac{3x-12}{x^2-x-12} - \frac{2}{x^2+6x+9} - \frac{1}{x^2-4x-21} \\ &= \frac{3(x-4)}{(x-4)(x+3)} - \frac{2}{(x+3)(x+3)} - \frac{1}{(x-7)(x+3)} \\ &= \frac{3}{(x+3)} - \frac{2}{(x+3)(x+3)} - \frac{1}{(x-7)(x+3)} \\ &= \frac{3(x+3)(x-7)}{(x+3)(x+3)(x-7)} - \frac{2(x-7)}{(x+3)(x+3)(x-7)} - \frac{1(x+3)}{(x-7)(x+3)(x+3)} \\ &= \frac{3(x+3)(x-7) - 2(x-7) - (x+3)}{(x+3)(x+3)(x-7)} \\ &= \frac{3(x^2-4x-21) - 2(x-7) - (x+3)}{(x+3)(x+3)(x-7)} \\ &= \frac{3x^2-12x-63-2x+14-x-3}{(x+3)(x+3)(x-7)} \\ &= \frac{3x^2-15x-52}{(x+3)(x+3)(x-7)}, \quad x \neq 4, -3, 7 \end{aligned}$$

simplify expressions where possible

LCD is
 $(x+3)(x+3)(x-7)$ **EXERCISE 1-12****A 1.** State the LCD for each of the following.

(a) $\frac{3}{7} - \frac{5}{6}$

(b) $\frac{3}{4} - \frac{5}{6} + \frac{1}{3}$

(c) $\frac{5}{8} - \frac{1}{3} + \frac{3}{4} - \frac{5}{6}$

(d) $\frac{3}{x} - \frac{4}{y} + \frac{5}{xy}$

(e) $\frac{2}{3x^2} - \frac{3}{2x^3} + \frac{4}{x}$

(f) $\frac{a}{x^2y} - \frac{b}{xy^2} + \frac{c}{3x^3y^2}$

(g) $\frac{1}{2} + \frac{3}{m+2}$

(h) $\frac{3}{x-1} + \frac{4}{x+2} + \frac{3}{4}$

(i) $\frac{5}{x} - \frac{4}{x-3}$

(j) $\frac{2}{m-1} + \frac{3}{m+2} - \frac{4}{m+3}$

(k) $\frac{5}{(x+1)(x+2)} - \frac{7}{(x+2)(x-4)}$

(l) $\frac{1}{t(t+5)} - \frac{2}{(t+5)(t+7)} - \frac{3}{t^2(t+7)}$

2. Simplify.

(a) $\frac{5}{x} + \frac{3}{x}$

(b) $\frac{7}{y} - \frac{3}{y}$

(c) $\frac{7}{m} - \frac{3}{m} + \frac{4}{m}$

(d) $\frac{3}{x+1} + \frac{2}{x+1}$

$$(e) \frac{3x}{x-4} - \frac{5}{x-4}$$

$$(f) \frac{4x+3y}{8} + \frac{x+2y}{8}$$

$$(g) \frac{5x^2}{(x+5)(x+3)} - \frac{3x^2}{(x+5)(x+3)}$$

$$(h) \frac{2m^2}{m-5} - \frac{3m}{m-5} + \frac{4}{m-5}$$

B Perform the indicated operations and simplify. Assume that all variables are restricted so that no denominator is equal to zero.

$$3. (a) \frac{3}{4} - \frac{1}{3} + \frac{5}{6}$$

$$(b) \frac{7}{8} + \frac{1}{12} - \frac{2}{3}$$

$$(c) \frac{3x}{2} - \frac{x}{3} + \frac{5x}{6}$$

$$(d) \frac{2m+3n}{2} + \frac{m-n}{3}$$

$$(e) \frac{x-1}{4} - \frac{x+1}{5} + \frac{x+3}{10}$$

$$(f) \frac{4x^2}{5} - \frac{2x^2+1}{6} - \frac{3x^2-5}{15}$$

$$(g) 5 + \frac{m}{7} + \frac{n}{2}$$

$$(h) \frac{2t-3}{8} - 2 + \frac{1-3t}{3}$$

$$4. (a) \frac{5}{2x} + \frac{3}{3x} - \frac{4}{6x}$$

$$(b) \frac{7}{5x} - \frac{2}{3x} + \frac{9}{10x}$$

$$(c) \frac{2}{x} - \frac{3}{x^2} + \frac{4}{x^3}$$

$$(d) \frac{2}{3y^2} - \frac{1}{2y} + \frac{5}{9y^3}$$

$$(e) \frac{2}{s^2} - \frac{s-3}{st} + \frac{t-2}{t^2}$$

$$(f) \frac{2x-1}{xy} - \frac{3}{y} + \frac{4}{x}$$

$$(g) m - \frac{x}{y} + y$$

$$(h) w - \frac{x}{w} + x$$

$$(i) 3x + \frac{1}{x}$$

$$(j) 4 - \frac{1}{x} + x$$

$$5. (a) \frac{3}{x+1} + \frac{4}{x-3}$$

$$(b) \frac{6}{w+4} + \frac{5}{w}$$

$$(c) \frac{6}{m-4} - \frac{3}{m-2}$$

$$(d) \frac{3}{t-2} - \frac{4}{2-t}$$

$$(e) \frac{5x}{3} - \frac{2}{x+5} + \frac{x}{4}$$

$$(f) \frac{2}{x-y} - \frac{1}{x+y}$$

$$(g) \frac{7}{2r-1} - \frac{3}{1-2r}$$

$$(h) \frac{2t}{t-4} - \frac{3t}{t+5}$$

$$(i) \frac{x+3}{x+2} + \frac{x+2}{x+3}$$

$$(j) \frac{m-5}{m-1} - \frac{m-1}{m-3}$$

$$(k) 7 + \frac{3x}{x-4} - \frac{x}{x+2}$$

$$(l) \frac{2x}{x-y} - \frac{3y}{x+y} + 1$$

$$6. (a) \frac{2}{(x-4)(x+3)} + \frac{5}{(x+3)(x+5)}$$

$$(b) \frac{4}{x(x+1)} - \frac{2}{(x+1)(x+2)}$$

$$(c) \frac{2w}{(w-1)(w-5)} - \frac{3w}{(w-5)(w-6)}$$

$$(d) \frac{1}{x+4} + \frac{3}{x^2+7x+12}$$

$$(e) \frac{1}{t^2+5t+4} - \frac{2}{t^2+9t+20}$$

$$(f) \frac{5}{x^2-5x} + \frac{2}{x^2-4x-5}$$

$$(g) \frac{3x-3}{x^2-5x+4} + \frac{4}{x^2-16}$$

$$(h) \frac{2}{m-3} + \frac{4m-12}{9-m^2}$$

$$(i) \frac{3}{x^2-xy} - \frac{4}{y^2-xy}$$

$$(j) \frac{w}{w-x} - \frac{x}{w+x} + \frac{wx}{x^2-w^2}$$

$$7. (a) \frac{x+6}{x^2+9x+18} + \frac{x-3}{x^2-2x-3}$$

$$(b) \frac{x^2+x-6}{x^2-3x+2} - \frac{x^2+3x+2}{x^2-x-2}$$

$$(c) \frac{x+2}{x^2-x-6} - \frac{x-4}{x^2-x-12} + \frac{x+3}{x^2+2x-3}$$

$$(d) \frac{1}{(w-x)(x-y)} - \frac{2}{(x-y)(y-w)} + \frac{3}{(w-x)(w-y)}$$

$$(e) \frac{x-3}{x+3} - \frac{2}{x-3} + \frac{3}{x^2-9}$$

$$(f) \frac{x}{x+1} + \frac{x+1}{2} - \frac{x}{x^2-1} + \frac{2}{1-x}$$

$$C \quad 8. (a) \frac{2}{2x^2+3x-2} - \frac{3}{3x^2+4x-4}$$

$$(b) \frac{3t+1}{6t^2-7t-3} - \frac{2t-1}{2t^2+9t-5} - \frac{5}{2t^2+7t-15}$$

$$(c) \frac{2}{x^2-3x-4} \cdot \frac{x-4}{x-2} + \frac{3}{x+1}$$

$$(d) \frac{x+1}{x+2} \div \frac{x^2-2x-15}{x^2+5x+6} - \frac{4}{x^2-25}$$

$$(e) \frac{1 - \frac{2}{x} + \frac{3}{x^2}}{2 - \frac{1}{x} - \frac{1}{x^2}}$$

$$(g) x - \frac{x}{x + \frac{1}{3}}$$

$$(f) \frac{\frac{r}{st} + \frac{s}{rt} + \frac{t}{rs}}{\frac{1}{r^2s^2} - \frac{1}{s^2t^2} - \frac{1}{r^2t^2}}$$

$$(h) \frac{\frac{x-1}{x+1} + \frac{x+1}{x-1}}{\frac{x-1}{x+1} - \frac{x+1}{x-1}}$$

REVIEW EXERCISE

A 1. Expand.

$$(a) (x+7)(x+11)$$

$$(b) (2x+5)^2$$

$$(c) (3x-1)(3x+1)$$

$$(d) (t-3)(t+7)$$

$$(e) (m-8)(m+9)$$

$$(f) (m^2-8)(m^2+2)$$

2. Factor.

$$(a) 5x+30$$

$$(b) 8m^2-4m$$

$$(c) 3m(x-y)+2n(x-y)$$

$$(d) x^2+9x+18$$

$$(e) m^2-14m+49$$

$$(f) n^2-2n-24$$

$$(g) x^2-16$$

$$(h) x^2-11x+28$$

$$(i) y^2-y-20$$

$$(j) 4x^2-25$$

$$(k) t^2+8t+16$$

$$(l) 1-25w^2$$

3. State the restrictions on the variables in each of the following.

(a) $\frac{7}{m}$ (b) $\frac{5}{x-5}$ (c) $\frac{8}{n+7}$ (d) $\frac{x}{(x-5)(x+2)}$

4. Simplify.

(a) $\frac{5}{x} \cdot \frac{3}{x}$ (b) $\frac{5}{x} + \frac{3}{x}$ (c) $\frac{5}{x} \div \frac{3}{x}$ (d) $\frac{5}{x} - \frac{3}{x}$
 (e) $\frac{3}{x+2} - \frac{4}{x+2}$ (f) $\frac{3}{x+2} \div \frac{4}{x+2}$ (g) $\frac{3}{x+2} \cdot \frac{4}{x+2}$
 (h) $x \div \frac{1}{x}$ (i) $\frac{1}{x} \div x$ (j) $\frac{(x+2)(x+1)}{(x+1)(x-3)} \cdot \frac{(x-3)}{(x+2)}$

B 5. Expand and simplify.

(a) $2(x-7)+3(x+5)$ (b) $4(3m-2n)-5(m+3n)$
 (c) $2x(3x-5)-x(x-11)$ (d) $(4x-3)(3x+8)$
 (e) $(1-7t)(3t-4)$ (f) $(3x^2-5x)(4x^2-7x)$
 (g) $(x^2-3x+7)^2$ (h) $(3m+2)(2m^2-m-4)$
 (i) $(3s^2-4s+5)(1-2s)$ (j) $(3w^2-4w+1)(w^2-w-5)$
 (k) $2(x-3)(x+5)-4(3x-1)(2x+5)$
 (l) $3(x-3y)(x-4y)-(x-2y)^2+16x^2$
 (m) $2(x-3)(x^2-6x+5)-3(2x-1)(3x+7)$

6. Factor over the integers.

(a) $m^2+8m+15$ (b) $2x^2-2x-24$
 (c) $2x^2-17x+21$ (d) $6x^2+41x+63$
 (e) $15x^2-17x+4$ (f) $12m^2-7m-10$
 (g) $12t^2-43t+35$ (h) $8n^2-38n-60$
 (i) $12r^2+28r-5$ (j) $mx+nx+my+ny$
 (k) $4y+mx-4x-my$ (l) $sw+5s+tw+5t$
 (m) $8m^2+14mn-15n^2$ (n) $20p^2+11pq-3q^2$
 (o) $6x^2+23xy+20y^2$ (p) $18s^2+15st-75t^2$

7. Factor.

(a) $25x^2-36y^2$ (b) $1-100m^2$
 (c) $t^4-36x^2y^2$ (d) $9x^2-30x+25$
 (e) $16m^2-24mn+9n^2$ (f) $4r^2+20rs+25s^2$
 (g) $(x+y)^2-36$ (h) $25m^2-(x-y)^2$
 (i) $(m+2n)^2-4(r+s)^2$ (j) $x^2+4xy+4y^2-m^2-6mn-9n^2$
 (k) $x^2-36m^2-8xy+16y^2$ (l) $9t^2-9y^2-x^2+6xy$

8. Divide.

(a) $\frac{30x+5}{5}$ (b) $\frac{12x^4-8x^3+4x^2}{4x^2}$
 (c) $\frac{25x^4y^5-30x^3y^7+35x^4y^6}{-5x^2y}$ (d) $\frac{3m^2n^4-9m^3n^5+12m^5n^6}{3m^2n^4}$
 (e) $\frac{40r^2t^4-30r^3t^5+50r^2t^7}{-10r^2t^4}$ (f) $\frac{-7m^3n^4-14m^5n^2+21m^6n^7}{-7m^3n^2}$

9. Divide.

(a) $(x^3+2x^2-4x-3) \div (x+3)$
 (b) $(2x^3-7x^2-7x+5) \div (2x-1)$
 (c) $(15m-4m^3-9m^2+3m^4-4) \div (3m-4)$
 (d) $(14t^2-7t^3+t^4-7t+1) \div (t^2-4t+1)$
 (e) $(6x^3-13x^2y+8xy^2-3y^3) \div (2x-3y)$



Find the missing digits in this division.

10. Use the Remainder Theorem to determine the remainder for each division.

(a) $(x^2 - 4x + 11) \div (x - 1)$

(b) $(m^2 - 3m - 5) \div (m + 2)$

(c) $(x^3 - 4x^2 + 3x - 5) \div (x + 1)$

(d) $(2x^4 - 3x^2 + 5x - 7) \div (x - 2)$

11. Factor.

(a) $x^3 - 4x^2 - 11x + 30$

(b) $x^3 + 3x^2 - 16x - 48$

(c) $x^3 + 10x^2 + 31x + 30$

(d) $3x^3 + 5x^2 - 16x - 12$

(e) $3x^3 - 11x^2 - 9x + 20$

(f) $2x^3 - 9x^2 - 8x + 15$

(g) $x^3 - 7x + 6$

(h) $x^3 + 3x^2 - 2$

12. Factor.

(a) $x^3 + 27$

(b) $m^3 - 8$

(c) $125t^3 - 1$

(d) $x^9 + 27y^{12}$

13. Simplify the following.

(a) $\frac{25x^4}{7y^3} \cdot \frac{21y^4}{5x^2}$

(b) $\frac{35x^4y^7}{7x^3y^4} \div \frac{5x^6y}{2xy^2}$

(c) $\frac{x^2 + 11x + 28}{x^2 + 4x - 21} \cdot \frac{x^2 - 2x - 3}{x^2 + 6x + 5}$

(d) $\frac{t^2 + 9t + 20}{t^2 + 7t + 12} \div \frac{t^2 + 3t - 10}{t^2 + t - 6}$

(e) $\frac{x^2 - 4x}{x^2 + 4x + 4} \cdot \frac{x^2 - 6x - 16}{x^2 - 12x + 32}$

(f) $\frac{x^2 + 5x + 4}{x^2 - 5x - 14} \cdot \frac{x^2 - 4x - 21}{x^2 + 7x + 12} \cdot \frac{x^2 - 4x - 12}{x^2 - 6x - 7}$

(g) $\frac{x^2 - 25}{x^2 - 6x} \cdot \frac{x^2 - 12x + 36}{x^2 + 2x - 15} \div \frac{x^2 - 11x + 30}{x^2 + 4x - 21}$

14. Simplify.

(a) $\frac{1}{x+3} + \frac{1}{x+4}$

(b) $\frac{2}{t-3} - \frac{3}{t-4}$

(c) $\frac{5}{2x+3} - \frac{6}{2x-3}$

(d) $\frac{2}{x^2+3x+2} + \frac{4}{x^2+5x+6}$

(e) $\frac{2m}{m^2+2m-3} - \frac{3m}{m^2+m-6}$

(f) $\frac{3}{x^2-x-6} - \frac{4}{x^2-2x-8}$

(g) $\frac{6}{x^2-4} + \frac{4}{x^2+4x+4}$

(h) $\frac{5}{2x^2-7x-4} - \frac{2}{3x^2-10x-8}$

(i) $\frac{3x-3}{x^2+x-2} + \frac{4x+20}{x^2+6x+5}$

(j) $\frac{5m-10}{m^2-4} - \frac{2m-6}{m^2-7m+12}$

(k) $\frac{x^2+3x+2}{x^2-3x-4} + \frac{x^2+2x-15}{x^2+3x-10}$

15. Simplify.

(a) $\frac{6x^2-13x-5}{8x^2-22x+5} \cdot \frac{4x^2-x-14}{3x^2-5x-2}$

(b) $\frac{49x^2-1}{21x^2+11x-2} \div \frac{14x^2+9x+1}{6x^2+7x+2}$

(c) $\frac{15x^2-14x-8}{9x^2+26x-3} \cdot \frac{x^2+9x+18}{3x^2+14x-24} \div \frac{5x^2+32x+12}{9x^2-46x+5}$

(d) $\frac{x^3+2x^2-5x-6}{x^3+6x^2+5x-12} \cdot \frac{x^3-8x^2+19x-12}{x^3-4x^2+x+6}$

(e) $\frac{x}{6x^2+x-2} - \frac{3}{12x^2+23x+10}$

(f) $\frac{2}{6x^2+7x+2} - \frac{6x+21}{6x^2+25x+14}$

Any natural number can be written as the sum of at most 4 perfect squares.

$$126 = 10^2 + 5^2 + 1^2$$

$$211 = 12^2 + 7^2 + 3^2 + 3^2$$

Write 366 as the sum of at most 4 perfect squares.

REVIEW AND PREVIEW TO CHAPTER 2

EXERCISE 1

PERCENTAGES

Complete the following calculations.

- | | | |
|--------------------|---------------------|----------------------|
| 1. 7% of \$55.60 | 2. 12% of \$125.00 | 3. 12.5% of \$800 |
| 4. 0.5% of \$6500 | 5. 3.5% of \$15.80 | 6. 1.5% of \$29.95 |
| 7. 22% of \$122.50 | 8. 28% of \$65.70 | 9. 120% of \$85.00 |
| 10. 65% of \$38.25 | 11. 150% of \$28.00 | 12. 325% of \$257.00 |

Express the first quantity as a percentage of the second.

- | | | |
|------------------------|----------------------|------------------------|
| 13. \$2.50, \$25.00 | 14. \$7.50, \$150.00 | 15. \$12.00, \$48.00 |
| 16. \$67.50, \$1350.00 | 17. \$21.50, \$10.00 | 18. \$125.00, \$75.00 |
| 19. \$55.20, \$326.50 | 20. \$6.82, \$75.81 | 21. \$65.20, \$4260.00 |
| 22. \$12.50, \$75.00 | 23. \$5.20, \$36.80 | 24. \$25.00, \$3500.00 |

25. Trudy Kuchev works in sales for Park Motors Ltd. She receives \$400/month plus 2% of sales. In the first quarter she had the following sales.

Jan.	\$36 800	Feb.	\$89 000	Mar.	\$31 200
	\$27 900				\$68 700

- (a) How much did she earn each month?
(b) How much should she contribute to a pension plan for the first quarter at 3% of gross income?

26. An equipment salesman receives a commission of 5% on sales in a month with a bonus of an additional 1.5% on sales over \$20 000. What commission is received on monthly sales of

- (a) \$18 000
(b) \$30 000
(c) \$2450
(d) \$42 800

27. A student working part-time in a clothing store receives \$2.00/h plus a commission of 2% on sales. Find the student's gross pay for each of the following weeks.

- (a) 18 h with sales of \$680.20
(b) 21 h with sales of \$1757.18
(c) 11 h with sales of \$478.72
(d) 16 h with sales of \$825.78

EXERCISE 2

OPERATIONS

Perform the following operations.

- | | |
|--------------------------|---|
| 1. $3 - 2[5 + (3 - 7)2]$ | 2. $(2 - 5)(3 + 7) \div (5 - 4)(2 - 7)$ |
| 3. $3[7 - 5(2 - 4)]$ | 4. $7(3 - 5) - 2[5 - 7(2 - 4)]$ |

$$5. 3 - \frac{1}{1 - \frac{1}{2}}$$

$$7. \frac{2}{3} \left[\frac{1}{2} - \left(\frac{1}{4} - \frac{1}{2} \right) \right]$$

$$9. 25^{\frac{1}{2}} + 25^{\frac{3}{2}}$$

$$11. -5^2 + (-5)^2$$

Simplify to 3 significant figures.

$$13. \sqrt{38.7}$$

$$14. \sqrt{4.97}$$

$$6. \frac{1}{2 - \frac{1}{2 - 1}}$$

$$8. \frac{5}{8} \left[\frac{4}{3} - \left(\frac{2}{3} - \frac{1}{2} \right) \right]$$

$$10. \sqrt{64} + \sqrt[3]{64} + \sqrt[6]{64}$$

$$12. \sqrt[3]{8^2} - \sqrt{4^3}$$

$$17. \sqrt{0.205}$$

$$18. \sqrt{0.0473}$$

$$19. \sqrt{12.7}$$

$$16. \sqrt{125\,000}$$

$$20. \sqrt{350\,000}$$

$$21. 5.86(2.75 + 7.63) \quad 22. 1.35 \div (7.83 + 8.51)$$

$$23. (6.87 - 5.24)(3.12 + 4.16)$$

$$24. 0.0357(6.12 - 8.35)$$

$$25. \sqrt{0.752 \div 0.143}$$

The Real Numbers

I often say that when you can measure what you are speaking about, and express it in numbers, you know something about it, but when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind; it may be the beginning of knowledge, but you have scarcely, in your thoughts, advanced to the stage of Science, whatever the matter may be.

Lord Kelvin

2.1 THE NUMBER SYSTEM

We have established the following number systems.

$$N = \{1, 2, 3, 4, \dots\}$$

$$W = \{0, 1, 2, 3, \dots\}$$

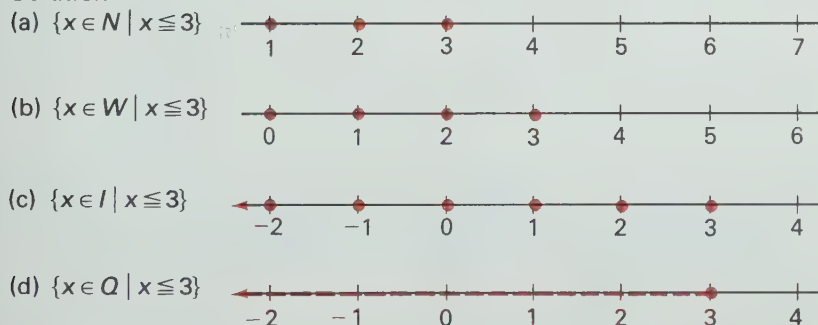
$$I = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

$$Q = \left\{ \frac{a}{b} \mid a, b \in I, b \neq 0 \right\}$$

EXAMPLE 1. Graph $\{x \mid x \leq 3\}$ where

- (a) $x \in N$ (b) $x \in W$ (c) $x \in I$ (d) $x \in Q$

Solution



Note that we use a dotted line when graphing on the Q -line.

The rational numbers, Q , can be expressed as common fractions or as decimals. Decimal equivalents such as $\frac{1}{2} = 0.5$ and $\frac{3}{4} = 0.75$ are called **terminating** decimals, while $\frac{1}{3} = 0.\dot{3}$ and $\frac{2}{7} = 0.428\,57\dot{1}$ are **periodic** decimals. The periodic decimal $\frac{3}{13} = 0.230\,769\,230\,7\ldots$ has the period 230 769 with period length 6.

EXAMPLE 2. Express (a) $0.\dot{5}4$ and (b) $2.1\dot{4}5\dot{3}$ as common fractions.

Solution

$$\begin{aligned} \text{(a) Let } x &= 0.5454 \dots \\ 100x &= 54.54 \dots \\ x &= 0.54 \dots \\ \hline 99x &= 54 \\ x &= \frac{54}{99} \\ &= \frac{6}{11} \end{aligned}$$

$$\begin{aligned} \text{(b) Let } x &= 2.145\,345\,3 \dots \\ 10\,000x &= 21\,453.453 \dots \\ 10x &= 21.453 \dots \\ \hline 9990x &= 21\,432 \\ x &= \frac{21\,432}{9990} \\ &= 2\frac{242}{1665} \end{aligned}$$

Using the method of Example 2, we can express all periodic decimals as common fractions. Numbers such as

$$0.121\,122\,111\,222 \dots$$

$$0.122\,112\,221\,112\,222 \dots$$

$$0.123\,124\,125\,126\,127 \dots$$

have patterns that are easily recognized, however they are not periodic and hence do not represent rational numbers.

$\bar{Q}$ is the set of non-periodic decimals.

Other familiar examples of non-periodic decimals are $\sqrt{2}$, $\sqrt{3}$, and π . In Figure 2.1, the number π has been located by rolling a circle with diameter 1. It turns out that the numbers in $\bar{Q}$ are irrational.

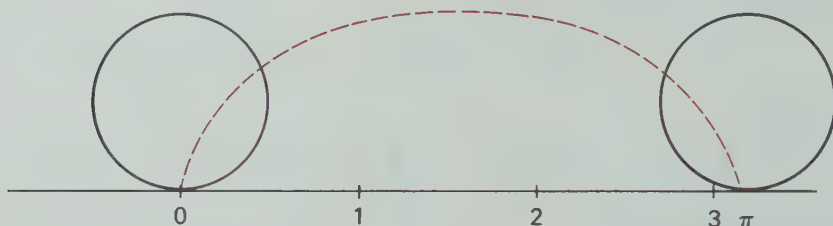


Figure 2-1

The set of real numbers, R , is the set of all periodic and non-periodic decimals. We can also define R as the union of the set of rational numbers, Q , and the set of irrational numbers, $\bar{Q}$,

$$R = Q \cup \bar{Q}$$

EXAMPLE 3. Graph each set on a number line.

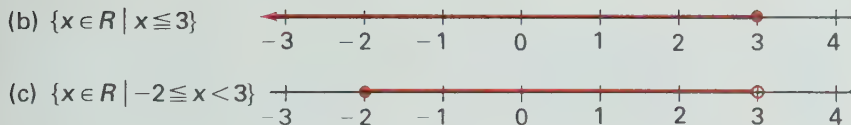
(a) $\{x \in R \mid x > -3\}$

(b) $\{x \in R \mid x \leq 3\}$

(c) $\{x \in R \mid -2 \leq x < 3\}$

Solution





○ point not included

● point included

EXAMPLE 4.

Given $A = \{x \in \mathbb{R} \mid x > -2\}$

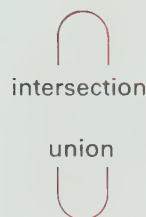
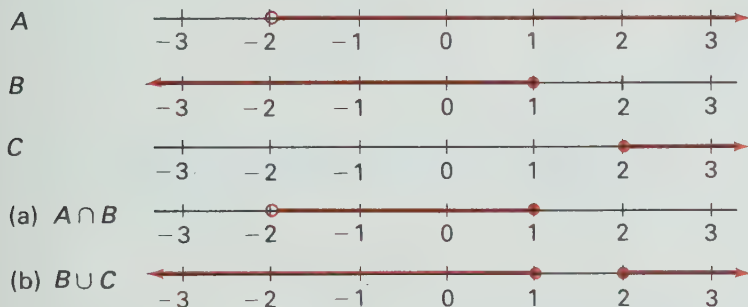
$B = \{x \in \mathbb{R} \mid x \leq 1\}$

$C = \{x \in \mathbb{R} \mid x \geq 2\}$

graph each of the following on a number line.

(a) $A \cap B$ (b) $B \cup C$

Solution



In set-builder notation, $A \cap B$ can be written

$$\{x \in \mathbb{R} \mid -2 < x \leq 1\}$$

EXERCISE 2-1

B 1. Express in decimal form.

- | | | | | |
|---------------------|-------------------|--------------------|--------------------|---------------------|
| (a) $\frac{1}{8}$ | (b) $\frac{1}{4}$ | (c) $\frac{3}{8}$ | (d) $\frac{5}{8}$ | (e) $\frac{7}{8}$ |
| (f) $\frac{21}{50}$ | (g) $\frac{1}{9}$ | (h) $\frac{5}{11}$ | (i) $\frac{7}{13}$ | (j) $\frac{15}{11}$ |

2. Express in the form $\frac{a}{b}$.

- | | | | |
|-----------|---------------|-----------|-----------|
| (a) 0.375 | (b) 0.215 | (c) 0.618 | (d) 0.035 |
| (e) 0.36 | (f) 0.5 | (g) 0.235 | (h) 4.754 |
| (i) 0.49 | (j) 0.142 857 | (k) 4.773 | (l) 0.018 |

3. Graph the following sets on an appropriate number line.

- | | |
|---|--|
| (a) $\{x \in \mathbb{N} \mid x \leq 4\}$ | (b) $\{x \in \mathbb{W} \mid x < 5\}$ |
| (c) $\{x \in \mathbb{W} \mid x > 3\}$ | (d) $\{x \in \mathbb{I} \mid x \leq 2\}$ |
| (e) $\{x \in \mathbb{I} \mid x \geq -3\}$ | (f) $\{x \in \mathbb{Q} \mid x < 5\}$ |
| (g) $\{x \in \mathbb{R} \mid x \geq -2\}$ | (h) $\{x \in \mathbb{R} \mid x > -3\}$ |

4. Graph the following.

- | | |
|--|--|
| (a) $\{x \in \mathbb{W} \mid 2 \leq x \leq 5\}$ | (b) $\{x \in \mathbb{I} \mid -2 \leq x < 3\}$ |
| (c) $\{x \in \mathbb{Q} \mid -3 < x < 3\}$ | (d) $\{x \in \mathbb{Q} \mid 0 \leq x \leq 4\}$ |
| (e) $\{x \in \mathbb{R} \mid -1 < x \leq 4\}$ | (f) $\{x \in \mathbb{R} \mid -3 \leq x \leq 3\}$ |
| (g) $\{x \in \mathbb{I} \mid -3 \leq x \leq 5\}$ | (h) $\{x \in \mathbb{R} \mid -2 < x < 3\}$ |

The day before yesterday I was 19 a old. Next year I will turn 22. When is my birthday?

5. Graph the following for $x \in R$.

- (a) $\{x \mid x > -2\} \cap \{x \mid x \leq 3\}$
- (b) $\{x \mid x \leq 0\} \cup \{x \mid x \geq 4\}$
- (c) $\{x \mid x \leq -3\} \cup \{x \mid x > 0\}$
- (d) $\{x \mid x \leq 3\} \cap \{x \mid x \geq 3\}$

6. Graph the following for $x \in R$.

- (a) $\{x \mid -1 \leq x \leq 2\} \cap \{x \mid 1 < x \leq 4\}$
- (b) $\{x \mid -2 \leq x < 1\} \cup \{x \mid 1 \leq x \leq 3\}$
- (c) $\{x \mid -3 \leq x \leq 2\} \cap \{x \mid x \neq 2\}$
- (d) $\{x \mid -2 \leq x < 3\} \cup \{x \mid x - 3 = 0\}$

C 7. Express in the form $\frac{a}{b}$.

- (a) $(0.5)(0.\dot{6})$
- (b) $(0.25)(0.3)$
- (c) $(0.1\dot{8})(0.5)$
- (d) $(0.3)^3$
- (e) $(0.\dot{4})(0.5)$
- (f) $(0.2)(0.\dot{3}\dot{6})$

8. Express in the form $\frac{a}{b}$.

- (a) $0.4\dot{9}$
- (b) $0.3\dot{9}$
- (c) $0.6\dot{9}$
- (d) $0.\dot{9}$

9. Express in the form $\frac{a}{b}$.

- (a) $0.1 + 0.01 + 0.001 + \dots$
- (b) $0.3 + 0.03 + 0.003 + \dots$
- (c) $0.36 + 0.0036 + 0.000\,036 + \dots$
- (d) $0.156 + 0.000\,156 + 0.000\,000\,156 + \dots$

10. Graph on a number line.

- (a) $\{x \in R \mid x^2 > 9\}$
- (b) $\{x \in R \mid x^2 < 4\}$
- (c) $\{x \in R \mid x \leq 3 \text{ and } x \geq -3\}$
- (d) $\{x \in R \mid x \leq -3 \text{ or } x \geq 2\}$

What happens to the area of a rectangle if you triple the lengths of the sides?

2.2 PROPERTIES OF REAL NUMBERS

The operations $+$, $-$, $\times$, $\div$ and $\sqrt{}$ are called algebraic operations. Because $-$, $\div$, and $\sqrt{}$ can be defined in terms of $+$ and $\times$ we can summarize the properties of the real numbers in terms of addition and multiplication.

For $a, b, c \in R$	$+$	$\times$
Closure	$(a + b) \in R$	$ab \in R$
Associative	$(a + b) + c = a + (b + c)$	$(ab)c = a(bc)$
Neutral Element	$a + 0 = a$	$a \times 1 = a$
Inverse Element	$a + (-a) = 0$	$a \times \frac{1}{a} = 1, a \neq 0$
Commutative	$a + b = b + a$	$ab = ba$
Distributive	$a(b + c) = ab + ac$	

There is a one-to-one correspondence between the real numbers, R , and the points on the number line. We determine whether one number is greater than, equal to, or less than another number according to their positions on the number line. Given any two real numbers a and b , one and only one of

- (i) $a < b$
- (ii) $a = b$
- (iii) $a > b$

is true. This is called the *trichotomy* property of order.

If $a - b > 0$, then $a > b$.

If $a - b = 0$, then $a = b$.

If $a - b < 0$, then $a < b$.

We can now state some properties of inequality in R .

Transitive Property of Order $a, b, c \in R$
If $a < b$, and $b < c$, then $a < c$.
If $a > b$, and $b > c$, then $a > c$.

If we add the same positive or negative quantity to both sides of an inequality, the sense of the inequality is not changed.

For $a, b, c \in R$
If $a < b$, then $a + c < b + c$.
If $a > b$, then $a + c > b + c$.

If we multiply both sides of an inequality by the same quantity, then the sense of the inequality is

- (i) unchanged if the quantity is positive.
- (ii) reversed if the quantity is negative.

$$\begin{aligned} -5 &< 2 \\ -5(-3) &? 2(-3) \\ 15 &> -6 \end{aligned}$$

For $a, b, c \in R$
If $a < b$, and $c > 0$, then $ac < bc$. If $a > b$, and $c > 0$, then $ac > bc$.
If $a < b$, and $c < 0$, then $ac > bc$. If $a > b$, and $c < 0$, then $ac < bc$.

An important property of real numbers is the *axiom of equality* which we state in three parts.

Axiom of Equality	For $a, b, c \in R$
1. Reflexive Property	$a = a$
2. Symmetric Property	If $a = b$, then $b = a$.
3. Transitive Property	If $a = b$, and $b = c$, then $a = c$.

R has the property that between any two real numbers we can find a third real number. If we assume that $a < b$, then we can prove

$$a < \frac{a+b}{2} < b$$

as follows.

$$\begin{aligned} \text{(i)} \quad b - \frac{a+b}{2} &= \frac{2b-a-b}{2} \\ &= \frac{b-a}{2} \\ &> 0 \end{aligned}$$

because $a < b$

$$\therefore \frac{a+b}{2} < b$$

$$\begin{aligned} \text{(ii)} \quad \frac{a+b}{2} - a &= \frac{a+b-2a}{2} \\ &= \frac{b-a}{2} \\ &> 0 \end{aligned}$$

because $a < b$

$$\therefore a < \frac{a+b}{2}$$

$$\text{and } a < \frac{a+b}{2} < b.$$

This shows that between any two real numbers we can find another real number.

EXERCISE 2-2

A 1. State the property that justifies each statement.

- | | | |
|-------------------------|-------------------------|----------------------|
| (a) $5 + x \in R$ | (b) $t \times 1 = t$ | (c) $m + 0 = m$ |
| (d) $5(x+3) = (x+3)5$ | (e) $(2+x)+4 = 2+(x+4)$ | (f) $ax = xa$ |
| (g) $4+(2-a) = (2-a)+4$ | (h) $x+(-x) = 0$ | (i) $5(x-y) = 5x-5y$ |
| (j) $3a \in R$ | (k) $3x-6 = 3(x-2)$ | (l) $5x \in R$ |

2. State the property that justifies each statement.

- (a) $1 < \sqrt{3}$ or $1 = \sqrt{3}$ or $1 > \sqrt{3}$ (b) $x = x$
 (c) Every point on the number line corresponds to a real number.
 (d) If $3 < 5$, and $5 < 7$, then $3 < 7$.
 (e) If $3 < x$, and $x < 8$, then $3 < 8$.
 (f) If $x = 8$, and $8 = y$, then $x = y$.
 (g) If $a = 5 + b$, then $5 + b = a$.
 (h) If $(a+b)x = ax + bx$, then $ax + bx = (a+b)x$.

3. State the greater number in each pair, and subtract a number to obtain the lesser.

- | | | | |
|--------------|--------------|--------------------------------|---------------------------------|
| (a) 3, 9 | (b) 0, -5 | (c) -3, 3 | (d) -4, 3 |
| (e) -3, -5 | (f) -3, 0 | (g) $\frac{2}{5}, \frac{3}{5}$ | (h) $-\frac{3}{4}, \frac{1}{4}$ |
| (i) 3.5, 2.7 | (j) $a, a-3$ | (k) $b, b+2$ | (l) $\frac{1}{a}, a > 1$ |

4. Make a conclusion from each of the following using the axiom of equality and the trichotomy property of order.

- | | |
|------------------------------|---|
| (a) $a < 5$ and $5 < b$ | (b) $a = ?$ |
| (c) $a \neq b$, $a < b$ | (d) $3(a+b) = 3a+3b$,
hence $3a+3b = ?$ |
| (e) $x+3 = y$ and $2x+1 = y$ | (f) $a+b > c$ and $c > a-b$ |
| (g) $-1 < a$ and $a < 3c$ | (h) $0.25 < \frac{1}{4}$ and $0.25 > \frac{1}{4}$ |

5. State the condition ($y < 0$, $y = 0$, $y > 0$) for each of the following to be a true statement.

- | | | |
|-----------------|--------------|---------------|
| (a) $x = x + y$ | (b) $2y < y$ | (c) $3y > y$ |
| (d) $y = -y$ | (e) $-y < 0$ | (f) $y^2 = y$ |

6. Find simpler inequalities.

- | | | |
|---------------|---------------|---------------|
| (a) $x+5 > 7$ | (b) $x-3 < 2$ | (c) $x+5 < 5$ |
| (d) $x+4 < 0$ | (e) $3x > 12$ | (f) $4x < 16$ |

(g) $-x > 3$

(h) $-x < 3$

(i) $-3x > 6$

(j) $\frac{1}{3}x < 2$

(k) $-\frac{1}{2}x > 3$

(l) $-\frac{x}{5} < 1$

7. Find a real number between each of the following pairs of real numbers.

(a) 2, 3

(b) -1, 2

(c) -3, -4

(d) 33, 34

(e) 1.4, 1.5

(f) -2.5, -2.6

(g) $\frac{1}{2}, \frac{1}{3}$

(h) $-\frac{2}{3}, -\frac{3}{4}$

8. Use each pair of equations and the axiom of equality (AXEQ) to solve for x .

(a) $x + 5 = c$
 $2x + 3 = c$

(b) $3x - 1 = b$
 $2x + 5 = b$

(c) $4x - 3 = a$
 $2x + 5 = a$

(d) $y = 5$
 $2x + y = 11$

(e) $y = 3 - x$
 $x + 4 = y$

(f) $x = a + 5$
 $x = 2a + 3$

2.3 LINEAR EQUATIONS

The polynomial $ax + b$ is of the first degree and is called a linear expression in x ; if we write $ax + b = 0$ then we have a linear equation in x . An equation is solved when we have the set of values of the variable that make the sentence true. We solve equations using the rules as given below.

If $a = b$ and $a, b, c \in R$	
I $a + c = b + c$	II $a \times c = b \times c$

EXAMPLE 1. Solve

(a) $3x + 5 = 19$

(b) $3(x - 5) = 2x + 3$

(c) $\frac{1}{3}(2x - 5) - \frac{1}{2}(x + 3) = 2$

Solution

(a) $3x + 5 = 19$

$$3x + 5 - 5 = 19 - 5 \quad \text{Rule I}$$

$$3x = 14$$

$$\frac{3x}{3} = \frac{14}{3} \quad \text{Rule II}$$

$$x = \frac{14}{3}$$

$\therefore$ the solution set is $\{\frac{14}{3}\}$

(b) $3(x - 5) = 2x + 3$

$$3x - 15 = 2x + 3$$

$$3x - 2x - 15 + 15 = 2x - 2x + 3 + 15 \quad \text{Rule I}$$

$$x = 18$$

$\therefore$ The root is 18.

Do these steps mentally

You can clear fractions by multiplying both sides by the L.C.D.

$$(c) \quad \frac{1}{3}(2x-5) - \frac{1}{2}(x+3) = 2$$

$$6 \left[\frac{1}{3}(2x-5) - \frac{1}{2}(x+3) \right] = (2)6 \quad \text{Rule II}$$

$$2(2x-5) - 3(x+3) = 12$$

$$4x - 10 - 3x - 9 = 12$$

$$x - 19 = 12$$

$$x - 19 + 19 = 12 + 19 \quad \text{Rule I}$$

$$x = 31$$

$$\text{Check: L.S.} = \frac{1}{3}(2(31) - 5) - \frac{1}{2}(31 + 3) \quad \text{R.S.} = 2$$

$$= \frac{1}{3}(57) - \frac{1}{2}(34)$$

$$= 19 - 17$$

$$= 2$$

$\therefore$ the root is 31.

EXERCISE 2-3

A 1. Determine whether the given value is a solution of the equation.

(a) $2a - 5 = 11$, 6

(b) $3b - 5 = 2b$, -5

(c) $2(c - 3) = 0$, 3

(d) $5(d + 3) = 0$, 0

(e) $\frac{-2c}{3} = 12$, -6

(f) $3f + 2 = 5$, 1

(g) $3g - 2 = g + 4$, 3

(h) $5 - h = 7$, -2

(i) $3(i - 2) = 2 - i$, -2

(j) $\frac{1}{2}j + \frac{3}{4} = 1\frac{1}{2}$, $\frac{2}{3}$

2. Solve each of the following.

(a) $4a - 2 = 10$

(b) $2b + 5 = 13$

(c) $\frac{2}{3}c = -14$

(d) $\frac{3}{4}d = 6$

(e) $-3e + 5 = -7$

(f) $2(f - 5) - f = 4$

(g) $3(g - 2) = 2(g + 1)$

(h) $3 - 2(h + 1) = 0$

(i) $\frac{1}{2}(i - 3) = 2$

(j) $\frac{2}{3}(j - 5) = 4$

B 3. Solve each of the following.

(a) $7x - 2 = 19$

(b) $5x - 7 = 23$

(c) $4x - 3 = 2x + 5$

(d) $5 - 2x = 3x + 10$

(e) $3(x - 5) = 2x + 3$

(f) $3 - 5x = 2(3 - 2x)$

(g) $\frac{1}{2}x + \frac{3}{4} = \frac{1}{4}x - \frac{1}{4}$

(h) $\frac{1}{2}(x - 3) = 4$

(i) $\frac{2}{3}x + \frac{1}{6} = \frac{1}{3}(x - 2)$

(j) $\frac{3}{4}(x - \frac{1}{2}) = \frac{1}{2}(x + 3)$

4. Solve and check:

(a) $2(x - 3) + 3(x - 5) = 4$

(b) $3(x + 4) - 2(x + 3) = 1$

(c) $5(x + 2) + 2(x - 4) = 3x$

(d) $2(x + 4) + 3(x + 2) = -2x$

(e) $4(x + 7) - 2(x - 5) = 3(x - 2)$

(f) $3(x - 7) + 2(x + 3) = 4(x + 1)$

(g) $\frac{1}{2}(x - 5) + \frac{2}{3}(x + 3) = 2$

(h) $\frac{3}{4}(x - 1) - \frac{1}{2}(x + 3) = \frac{1}{4}$

(i) $0.5(x + 0.7) - 0.2(x - 0.3) = 0.4$

(j) $0.3(x - 0.2) + 0.7(x + 0.4) = 0.25$

C 5. Solve for x

(a) $ax - b = c$

(b) $\frac{ax}{b} = b + c$

Addition

ZERO
ONE
TWO
THREE

$$(c) \frac{a}{x} - b = c$$

$$(d) b - \frac{a}{x} = \frac{1}{c}$$

$$(e) a(x - b) = c$$

$$(f) 3a - 2x = 5(b - c)$$

$$(g) 2a(x - b) + c = b$$

$$(h) 3x(a - b) - b = c - a$$

$$(i) 3(a - bx) = c$$

$$(j) 2(ax - b) - 3c = 5a + 2b$$

2.4 LINEAR INEQUALITIES

In section 2.2, we saw the rules for transforming an inequality. In this section we will solve inequalities using the following rules.

Addition and Multiplication Axioms of Order
Let $a, b, c \in \mathbb{R}$

I If $a < b$, then $a + c < b + c$
 $a > b$, then $a + c > b + c$

II If $a < b$, and $c > 0$, then $ac < bc$
 $a > b$, and $c > 0$, then $ac > bc$

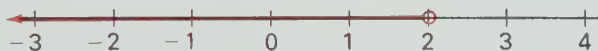
III If $a < b$ and $c < 0$, then $ac > bc$
 $a > b$ and $c < 0$, then $ac < bc$

EXAMPLE 1. Solve $5x - 2 < 2x + 4$, $x \in \mathbb{R}$ and graph the solution set.

Solution

$$\begin{aligned} 5x - 2 &< 2x + 4 \\ 5x - 2 + 2 &< 2x + 4 + 2 && \text{Rule I} \\ 5x &< 2x + 6 \\ 5x - 2x &< 2x + 6 - 2x && \text{Rule I} \\ 3x &< 6 \\ \frac{3x}{3} &< \frac{6}{3} && \text{Rule II} \\ x &< 2 \end{aligned}$$

$\therefore$ the solution set is $\{x \in \mathbb{R} \mid x < 2\}$



EXAMPLE 2. Solve $3 - 2(x + 2) \geq 3x - (2x - 5)$, $x \in \mathbb{R}$

Solution

$$\begin{aligned} 3 - 2(x + 2) &\geq 3x - (2x - 5) \\ 3 - 2x - 4 &\geq 3x - 2x + 5 \\ -2x - 1 &\geq x + 5 \\ -2x - 1 - x + 1 &\geq x + 5 - x + 1 && \text{Rule I} \\ -3x &\geq 6 \\ \frac{-3x}{-3} &\leq \frac{6}{-3} && \text{Rule III} \\ x &\leq -2 \end{aligned}$$

$\therefore$ the solution set is $\{x \mid x \leq -2\}$

EXERCISE 2-4

A 1. Solve the following inequalities.

- | | | |
|-------------------|-----------------------|---------------------|
| (a) $2x - 3 > 1$ | (b) $3x - 5 < 4$ | (c) $2x + 1 \leq 7$ |
| (d) $-x \geq 5$ | (e) $-3x + 1 \geq 7$ | (f) $2 - x > 3$ |
| (g) $-5x + 2 > 2$ | (h) $2(3 - x) \leq 6$ | (i) $-3(x - 2) < 9$ |

2. Determine whether the given value is a solution of the inequality.

- | | |
|-------------------------|--------------------------|
| (a) $3x - 5 \leq 7, -2$ | (b) $4 - 3x \geq 3, 0$ |
| (c) $2x + 5 \geq 8, -1$ | (d) $3(2 - x) \leq 5, 0$ |
| (e) $3 - 2x < 5, -4$ | (f) $2x > x, -2$ |
| (g) $3 - 5x \geq 2, -1$ | (h) $4 \leq 3 - 2x, -2$ |

3. Solve the following inequalities.

- | | | |
|--|--|-----------------------------|
| (a) $\frac{1}{2}x \leq 3$ | (b) $-\frac{x}{3} > 2$ | (c) $\frac{x-3}{2} \geq 2$ |
| (d) $\frac{2}{3}x + 1 \leq 5$ | (e) $\frac{1}{6}x - \frac{1}{3} > \frac{1}{2}$ | (f) $-\frac{3}{4}x + 1 < 6$ |
| (g) $\frac{1}{2} - x \leq \frac{3}{4}$ | (h) $3 - \frac{x}{3} > \frac{1}{3}$ | (i) $4 - \frac{2}{3}x > 4$ |

B 4. Solve the following inequalities and graph the solution set.

- | | |
|--|--|
| (a) $2(x - 3) + x < 3$ | (b) $2(3 - x) + 1 \geq -3$ |
| (c) $3 - 2(x + 3) \geq 5$ | (d) $3x - 5 < 7 - 2(x - 4)$ |
| (e) $\frac{1}{2}(x - 3) + \frac{2}{3} < \frac{3}{2}$ | (f) $-\frac{3}{4}x - \frac{1}{2} \geq \frac{3}{4}$ |

5. Solve the following inequalities.

- | | |
|---|--|
| (a) $5(3 - 2x) \leq 2x + 7$ | (b) $-3x + 2(3 - x) > 2(x - 5)$ |
| (c) $2[x + 3(x - 1)] \geq 3(x + 3)$ | (d) $2[x - 3(x - 2)] > 8 - x$ |
| (e) $\frac{5-x}{2} + \frac{x}{4} < \frac{x+2}{4}$ | (f) $\frac{1}{2}(5 - x) - \frac{2}{3}(5 - 2x) \leq \frac{1}{2}(x - 1)$ |

C 6. Solve for x , if $a, b, c > 0$.

- | | |
|--|---|
| (a) $ax + b < c$ | (b) $a(bx - c) \geq bc$ |
| (c) $a(b - cx) \leq bc$ | (d) $(a - b)x + c > ab$ |
| (e) $\frac{a-x}{b} \geq \frac{a+b}{c}$ | (f) $\frac{x}{a} - \frac{x-a}{b} < \frac{1}{c}$ |

7. Solve for x , if $a, b, c < 0$.

- | | |
|-------------------------|-------------------------|
| (a) $ax + b < c$ | (b) $a(bx - c) \geq bc$ |
| (c) $a(b - cx) \leq bc$ | (d) $(a - b)x + c > ab$ |

2.5 ABSOLUTE VALUE

The Absolute Value of $n, n \in R$.

$$\begin{aligned} |n| &= n, & \text{if } n \geq 0 \\ |n| &= -n, & \text{if } n < 0 \end{aligned}$$

For example

$$\begin{aligned} |3| &= 3, & \text{since } 3 > 0 \\ |-3| &= -(-3), & \text{since } -3 < 0 \\ &= 3 \end{aligned}$$

This definition enables us to solve more complicated equations and inequalities. $|n|$ is often called the "magnitude of n ."

If we relate absolute value to a number line then it can be used to indicate the measure of the distance between a point and the origin without reference to direction.

EXAMPLE 1. Draw the graph for $x \in R$

- (a) $|x|=3$ (b) $|x|<3$ (c) $|x|>3$

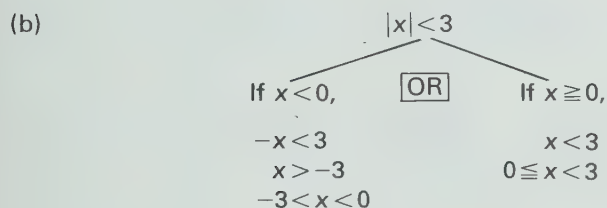
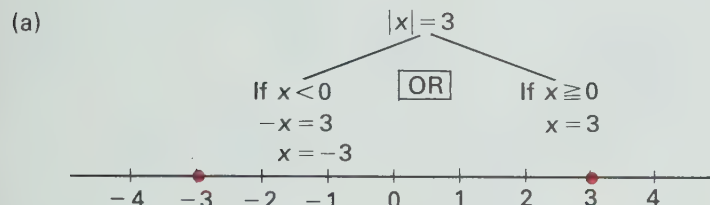
Solution

From the definition of absolute value, we have

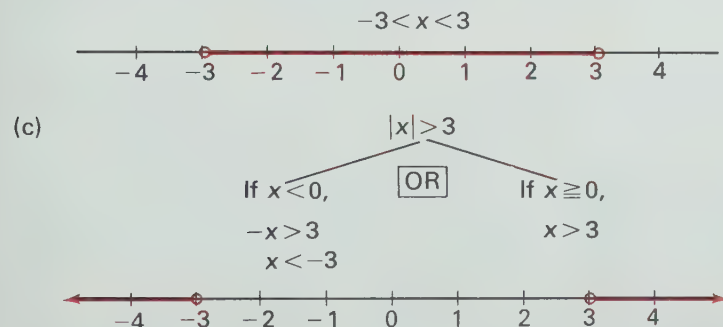
$$|x| = x, \quad x \geq 0$$

or

$$|x| = -x, \quad x < 0$$



These results are combined as we write



We can generalize the results of example 1 in the following statements.

I	$ x = a$	$\Leftrightarrow$	$x = -a$ or $x = a$
II	$ x < a$	$\Leftrightarrow$	$-a < x < a$
III	$ x > a$	$\Leftrightarrow$	$x < -a$ or $x > a$

EXAMPLE 2. Solve $|2x - 1| = 7$ for $x \in R$.

Solution

$$|2x - 1| = 7$$

Using statement I

$$2x - 1 = 7 \quad \text{OR} \quad 2x - 1 = -7$$

$$2x = 8 \qquad 2x = -6$$

$$x = 4 \qquad x = -3$$

$\therefore$ the solution set is $\{-3, 4\}$

EXAMPLE 3. Solve $|x - 4| \leq 2$, and graph for $x \in R$.

Solution

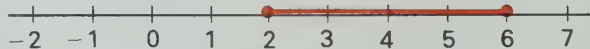
$$|x - 4| \leq 2$$

Using the definition,

$$\begin{array}{ll} \text{If } (x - 4) < 0, -(x - 4) \leq 2 & \text{OR} \quad \text{If } (x - 4) \geq 0, (x - 4) \leq 2 \\ x < 4, -x + 4 \leq 2 & x \geq 4, x - 4 \leq 2 \\ -x \leq -2 & x \leq 6 \\ x \geq 2 & \end{array}$$

These can now be combined in one statement:

$$2 \leq x \leq 6$$



We may also use statement II:

$$\begin{array}{l} |x - 4| \leq 2 \\ -2 \leq x - 4 \leq 2 \\ -2 + 4 \leq x - 4 + 4 \leq 2 + 4 \\ 2 \leq x \leq 6 \end{array}$$

which gives the same result.



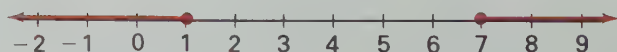
EXAMPLE 4. Solve $|x - 4| \geq 3$, and graph for $x \in R$.

Solution

$$|x - 4| \geq 3$$

By statement III

$$\begin{array}{ll} -(x - 4) \geq 3 & \text{OR} \quad (x - 4) \geq 3 \\ -x + 4 \geq 3 & x - 4 \geq 3 \\ -x \geq -1 & x \geq 7 \\ x \leq 1 & \\ x \leq 1 & \text{OR} \quad x \geq 7 \end{array}$$



EXERCISE 2-5

A 1. Evaluate.

- (a) $|-5|$ (b) $|-7+3|$ (c) $|-7|+|3|$ (d) $|5-9|$
 (e) $|5|-|9|$ (f) $|5|+|-9|$ (g) $|-5-7|$ (h) $|-5|-|7|$

2. Solve for the variable, $x \in R$.

- (a) $|x|=5$ (b) $|x|-2=5$ (c) $|x|+5=3$ (d) $|x+1|=3$
 (e) $|x|\leq 6$ (f) $|x|>4$ (g) $|x|+3<5$ (h) $|x|-2>3$

B 3. Solve for $x \in R$.

- (a) $|3x+2|=5$ (b) $|3x-2|=5$ (c) $|1-x|=3$
 (d) $|1-2x|=3$ (e) $|4x-3|=5$ (f) $|2x-5|=3$
 (g) $|3x-5|=x$ (h) $|2x|=x+3$ (i) $|3x+2|=-3x$

4. Solve and draw a graph for $x \in R$.

- (a) $|x+2|<3$ (b) $|x-3|\geq 2$ (c) $|3x|\leq 6$
 (d) $|-3x|\leq 6$ (e) $|2x-1|>5$ (f) $|3x-2|\geq 1$
 (g) $|2x-5|\geq 0$ (h) $|2x|-3\leq 5$ (i) $|3-2x|\leq 7$

$$\begin{aligned}
 |n| = |x| &\Leftrightarrow n = |x| \text{ or } -n = |x| \\
 &\Leftrightarrow [n = x \text{ or } n = -x] \text{ or } [-n = x \text{ or } n = x] \\
 &\Leftrightarrow n = x \text{ or } n = -x
 \end{aligned}$$

To solve equations or inequalities with absolute value, we use the definition to eliminate the absolute value signs.

5. Solve for $x \in R$.

- (a) $|2x|=|x+5|$ (b) $|3x-2|=|2x|$ (c) $|5x-3|=|x+1|$
 (d) $|2x-5|=|x-1|$ (e) $|x+3|=|x-3|$ (f) $|3x-7|=|2-x|$
 (g) $|3x-2|-|4x|=0$ (h) $|2-3x|=|5+x|$ (i) $|2x+4|=|-3x|$

C 6. Solve for $x \in R$.

- (a) $|x|\leq a, a>0$ (b) $|x|\geq a, a<0$ (c) $|x+a|=a+x$
 (d) $|x+a|=b$ (e) $|xa-b|=c$ (f) $|x+a|<b$

2.6 QUADRATIC EQUATIONS SOLVED BY FACTORING

Equations of degree 2, such as $x^2-x-12=0$ or $3x^2-7x+4=0$, are called quadratic equations. Some quadratic equations can be solved by factoring.

If two quantities a and b are such that $a \cdot b = 0$ then either $a = 0$ or $b = 0$.

The examples which follow show how this property is used to solve quadratic equations.

EXAMPLE 1. Solve $x^2-x-12=0$

Solution

$$\begin{aligned}
 x^2-x-12 &= 0 \\
 (x-4)(x+3) &= 0
 \end{aligned}$$

$$\begin{array}{ll} \text{Either } x-4=0 & \text{or } x+3=0 \\ x=4 & x=-3 \end{array}$$

The roots are 4 and -3

$$\begin{array}{ll} \text{Check: } x=4 & x=-3 \\ \text{L.S.} = x^2 - x - 12 & \text{R.S.} = 0 \quad \text{L.S.} = x^2 - x - 12 \\ = 4^2 - 4 - 12 & = (-3)^2 - (-3) - 12 \\ = 0 & = 0 \end{array}$$

Note that both roots, 4 and -3, are checked in the original equation.

EXAMPLE 2. Solve

- (a) $25x^2 - 36 = 0$
 (b) $3x^2 - 13x + 7 = 3$

Solution

$$\begin{array}{l} \text{(a)} \quad 25x^2 - 36 = 0 \\ (5x+6)(5x-6) = 0 \end{array}$$

$$\begin{array}{l} \text{Either } 5x+6=0 \quad \text{or } 5x-6=0 \\ 5x=-6 \quad \quad \quad 5x=6 \\ x=-\frac{6}{5} \quad \quad \quad x=\frac{6}{5} \end{array}$$

$\therefore$ the solution set is $\{-\frac{6}{5}, \frac{6}{5}\}$

$$\text{(b)} \quad 3x^2 - 13x + 7 = 3$$

We first rewrite the equation

$$\begin{array}{l} 3x^2 - 13x + 4 = 0 \\ (3x-1)(x-4) = 0 \end{array}$$

$$\begin{array}{l} \text{Either } (3x-1)=0 \quad \text{or } (x-4)=0 \\ 3x=1 \quad \quad \quad x=4 \\ x=\frac{1}{3} \end{array}$$

$\therefore$ the solution set is $\{\frac{1}{3}, 4\}$

EXAMPLE 3. Solve

- (a) $x^2 = 3x$
 (b) $(x+2)(x-3) = 4(x-3)$

Solution

$$\begin{array}{l} \text{(a)} \quad x^2 = 3x \\ x^2 - 3x = 0 \\ x(x-3) = 0 \end{array}$$

$$\begin{array}{l} \text{Either } x=0 \quad \text{or } x-3=0 \\ x=3 \end{array}$$

Check:

$$\begin{array}{l} x=0 \\ \text{L.S.} = 0^2 = 0 \\ \text{R.S.} = 3(0) = 0 \\ x=3 \\ \text{L.S.} = 3^2 = 9 \\ \text{R.S.} = 3(3) = 9 \end{array}$$

$\therefore$ the solution set is $\{0, 3\}$

$$\begin{array}{l} \text{(b)} \quad (x+2)(x-3) = 4(x-3) \\ x^2 - x - 6 = 4x - 12 \\ x^2 - 5x + 6 = 0 \end{array}$$

$$(x-2)(x-3) = 0$$

$$\begin{array}{l} \text{Either } (x-2)=0 \quad \text{or } (x-3)=0 \\ x=2 \quad \quad \quad x=3 \end{array}$$

Check:

$$\begin{array}{l} x=2 \\ \text{L.S.} = (2+2)(2-3) = -4 \\ \text{R.S.} = 4(2-3) = -4 \\ x=3 \\ \text{L.S.} = (3+2)(3-3) = 0 \\ \text{R.S.} = 4(3-3) = 0 \end{array}$$

$\therefore$ the solution set is $\{2, 3\}$

In example 3, we must take care not to divide both sides of the equation in (a) by x and in (b) by $(x-3)$, because we would be dividing both sides by zero. Care must be taken not to divide by zero when the expression contains a variable.

EXAMPLE 4. Solve

- (a) $(x-1)(x+4)(x-5)=0$
 (b) $x^3+2x^2-5x-6=0$

Solution

Equations of higher order can often be solved by factoring as in the previous examples.

(a) $(x-1)(x+4)(x-5)=0$

Either $x-1=0$ or $x+4=0$ or $x-5=0$
 $x=1$ $x=-4$ $x=5$

$\therefore$ the solution set is $\{-4, 1, 5\}$.

(b) $x^3+2x^2-5x-6=0$

We factor the left side using the factor theorem.

$$(x+1)(x^2+x-6)=0$$

$$(x+1)(x-2)(x+3)=0$$

Either $x+1=0$ or $x-2=0$ or $x+3=0$
 $x=-1$ $x=2$ $x=-3$

$\therefore$ the solution set is $\{-3, -1, 2\}$

$$\begin{array}{r}
 x^2 + x - 6 \\
 x + 1 \overline{) x^3 + 2x^2 - 5x - 6} \\
 \underline{x^3 + x^2} \\
 x^2 - 5x - 6 \\
 \underline{x^2 + x} \\
 -6x - 6 \\
 \underline{-6x - 6} \\
 0
 \end{array}$$

EXERCISE 2-6**A 1. Solve the following for x.**

- (a) $(x-2)(x+5)=0$ (b) $(x-5)(x+2)=0$ (c) $x(x+3)=0$
 (d) $(x-3)(x+7)=0$ (e) $(x-10)(x+7)=0$ (f) $(x-7)x=0$
 (g) $(x-a)(x+b)=0$ (h) $(x-5)(x-3)(x-1)=0$

B 2. Solve for x.

- (a) $x^2-5x+6=0$ (b) $x^2-x-6=0$ (c) $x^2+x-6=0$
 (d) $x^2-7x+6=0$ (e) $x^2-7x+12=0$ (f) $x^2-7x+10=0$
 (g) $x^2-x-12=0$ (h) $x^2-2x-15=0$ (i) $x^2+8x-20=0$

3. Solve for x and check:

- (a) $2x^2+11x+12=0$ (b) $3x^2-11x+10=0$ (c) $4x^2+8x+3=0$
 (d) $3x^2+13x+10=0$ (e) $3x^2+2x-8=0$ (f) $15x^2-14x-8=0$
 (g) $4x^2-7x-2=0$ (h) $6x^2+11x+3=0$ (i) $4x^2-15x-4=0$

4. Solve for x.

- (a) $x^2-5x=6$ (b) $x^2-3x=0$ (c) $x^2+12=7x$
 (d) $x-1-\frac{12}{x}=0$ (e) $x^2=5x$ (f) $(x+2)^2=9$
 (g) $x^2-10x+25=0$ (h) $2x^2-2x-84=0$ (i) $(2x-3)^2=4$

5. Solve for x.

- (a) $(x+3)(x^2-5x+6)=0$
 (b) $(2x-5)(x^2-7x+12)=0$
 (c) $x^3+2x^2-5x-6=0$
 (d) $x^3-4x^2+5x-2=0$
 (e) $x^3+2x^2-x-2=0$
 (f) $x^3-8x^2+19x-12=0$
 (g) $2x^3-x^2-5x-2=0$

Every day at noon a ship leaves Lisbon for New York and at the same instant a ship leaves New York for Lisbon. Each trip takes exactly 192 h. How many ships from New York will each ship from Lisbon meet?

C 6. Solve for x .

(a) $\frac{x+3}{x-2} = \frac{2x}{2x-5}$

(b) $\frac{2x+1}{x+2} = \frac{3x}{2x+1}$

(c) $\frac{x+3}{x} = \frac{2x-1}{2x+2}$

(d) $\frac{2x+2}{x+3} = \frac{3x+3}{2x+3}$

7. Find an equation whose roots are

- (a) 1 and 2 (b) 3 and -2 (c) 4 and -4 (d) 0 and 5
(e) 1, 3, and 4 (f) -1, 0, and 4 (g) -2, 2, and 3 (h) -1, -2, and -3

8. Solve for x and verify

$$\frac{x^2}{x-1} = \frac{7}{x-1} - 3$$

9. Solve for x and verify.

- (a) $x(x+3) = 5(x+3)$
(b) $(2x-3)(x+1) = 4(x+1)$

Put the numbers from 1 to 9 in the spaces to make the statements true.

$$\square - \square + \square = 10$$

$$\square + \square - \square = 10$$

$$\square \div \square + \square = 10$$

2.7 QUADRATIC EQUATIONS SOLVED BY FORMULA

We have solved the quadratic equation $ax^2+bx+c=0$ by factoring the polynomial ax^2+bx+c . In this section, we shall develop a formula to solve quadratic equations where the polynomial cannot be factored.

$$\begin{array}{ll} \text{If} & x^2 = 25 \\ \text{then} & x = 5 \text{ or } x = -5 \end{array}$$

$$\begin{array}{ll} \text{If} & (x-2)^2 = 25 \\ \text{Then} & x-2 = 5 \text{ or } x-2 = -5 \\ & x = 7 \qquad \qquad x = -3 \end{array}$$

These equations were solved by taking the square root of both sides. Note that when taking the square root, we account for both positive and negative roots. We now use the method of taking the square root of both sides in the following examples.

EXAMPLE 1. Solve $x^2 - 2x - 15 = 0$

Solution

$$x^2 - 2x - 15 = 0$$

$$x^2 - 2x = 15 \quad \text{Add 15 to both sides}$$

$$x^2 - 2x + 1 = 15 + 1 \quad \text{Add 1 to both sides}$$

$$(x-1)^2 = 16$$

$$\begin{array}{ll} x-1 = 4 \text{ or } x-1 = -4 & \text{Take the square root of} \\ x = 5 \qquad \qquad x = -3 & \text{both sides.} \end{array}$$

EXAMPLE 2. Solve $3x^2 - 8x + 5 = 0$

Solution

$$3x^2 - 8x + 5 = 0$$

$$x^2 - \frac{8}{3}x + \frac{5}{3} = 0$$

$$x^2 - \frac{8}{3}x = -\frac{5}{3}$$

$$x^2 - \frac{8}{3}x + \frac{16}{9} = -\frac{5}{3} + \frac{16}{9}$$

$$(x - \frac{4}{3})^2 = \frac{1}{9}$$

$$x - \frac{4}{3} = \frac{1}{3} \quad \text{or} \quad x - \frac{4}{3} = -\frac{1}{3}$$

$$x = \frac{1}{3} + \frac{4}{3} \quad x = -\frac{1}{3} + \frac{4}{3}$$

$$= \frac{5}{3} \quad = 1$$

$\therefore$ the roots are $\frac{5}{3}$ and 1.

We now use the method of example 2 to develop the general quadratic formula.

$$ax^2 + bx + c = 0, \quad a \neq 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Dividing by a , the coefficient of x^2

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Subtracting $\frac{c}{a}$ from both sides.

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

Adding $\frac{b^2}{4a^2}$.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Rewriting as the square of a binomial.

$$x + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x + \frac{b}{2a} = -\frac{\sqrt{b^2 - 4ac}}{2a}$$

Taking the square root of both sides.

$$x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \quad x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

The above formulas are stated as follows.

The solution of $ax^2 + bx + c = 0$, $a \neq 0$, is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

EXAMPLE 3. Solve $3x^2 - 2x - 2 = 0$.

Solution

By comparison with $ax^2 + bx + c = 0$,

$$a = 3, \quad b = -2, \quad c = -2, \quad \text{and}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-2)}}{2(3)}$$

$$= \frac{2 \pm \sqrt{4+24}}{6}$$

$$= \frac{2 \pm \sqrt{28}}{6}$$

$$= \frac{2 \pm 2\sqrt{7}}{6}$$

$$= \frac{1 \pm \sqrt{7}}{3}$$

$\therefore$ the roots are $\frac{1+\sqrt{7}}{3}$ and $\frac{1-\sqrt{7}}{3}$

EXAMPLE 4. Find the roots of $3x^2 + 5x + 1 = 0$ correct to two decimal places.

Solution

By comparison with $ax^2 + bx + c = 0$,

$$a = 3 \quad b = 5 \quad c = 1$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-5 \pm \sqrt{25 - 12}}{6}$$

$$= \frac{-5 \pm \sqrt{13}}{6}$$

$$\doteq \frac{-5 \pm 3.606}{6}$$

$$x \doteq \frac{-5+3.606}{6} \quad \text{or} \quad x \doteq \frac{-5-3.606}{6}$$

$$\doteq \frac{-1.39}{4}$$

$$\doteq -0.232$$

$$\doteq \frac{-8.606}{6}$$

$$\doteq -1.434$$

$\therefore$ the roots are -0.23 and -1.43 correct to two decimal places.

EXERCISE 2-7

B Solve the following equations by any method:

1. $3x^2 - 5x + 2 = 0$

2. $6x^2 - 5x - 7 = 0$

3. $4x^2 - 2x - 5 = 0$

4. $3x^2 - 7x + 5 = 0$

5. $2x^2 - 8x + 3 = 0$
6. $12x^2 - 17x + 6 = 0$
7. $4x^2 - 9x + 3 = 0$
8. $8x^2 - 11x + 3 = 0$
9. $7x^2 - 5x - 12 = 0$
10. $9x^2 - 11x - 3 = 0$
11. $2x^2 - 7x - 15 = 0$
12. $3x^2 + 10x + 8 = 0$
13. $4x^2 + 7x - 6 = 0$
14. $11x^2 - 8x - 3 = 0$
15. $x^2 + x + 1 = 0$

C Solve the following equations to three significant digits.

16. $0.25x^2 - 1.5x + 1 = 0$
17. $2.7x^2 + 1.6x - 1.5 = 0$
18. $4.7x^2 - 2.8x - 2 = 0$
19. $3.6x^2 - 5.2x + 1.2 = 0$
20. $50x^2 - 2.5x - 8.2 = 0$
21. $5.8x^2 + 6.3x - 2.5 = 0$
22. $x^2 - 7.5x + 3.2 = 0$
23. $2.7x^2 - 5.8x + 0.25 = 0$
24. $6.3x^2 - 2.7 = 0$
25. $6.5x^2 + 42x + 1.6 = 0$

2.8 QUADRATIC INEQUALITIES

If $a \cdot b > 0$, then there are two possibilities.

Case I $a > 0$ and $b > 0$

Case II $a < 0$ and $b < 0$

If $a \cdot b < 0$ then the two possibilities are

Case I $a < 0$ and $b > 0$

Case II $a > 0$ and $b < 0$

We use these facts to solve certain quadratic inequalities as in the examples which follow.

What common English word ends with the two letters MT?

EXAMPLE 1. Graph the solution set of $x^2 - 2x - 8 > 0$

Solution

$$x^2 - 2x - 8 > 0$$

$$(x+2)(x-4) > 0$$

This inequality is now solved by taking cases where the factors on the left side are either both negative or both positive.

Case I	OR	Case II
$x+2 < 0$ and $x-4 < 0$		$x+2 > 0$ and $x-4 > 0$
$\therefore x < -2$ and $x < 4$		$\therefore x > -2$ and $x > 4$

The intersection of these sets is $x < -2$

The intersection of these sets is $x > 4$.

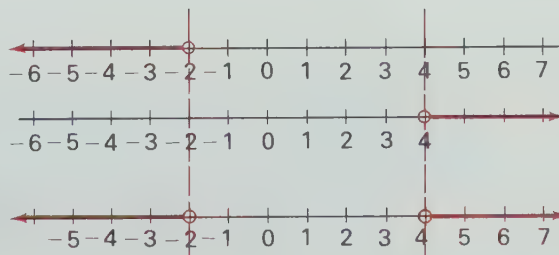
Graphing we have

$$\{x \in \mathbb{R} \mid x < -2\}$$

$$\{x \in \mathbb{R} \mid x > 4\}$$

The solution set is

$$\{x \in \mathbb{R} \mid x < -2 \text{ or } x > 4\}$$



EXAMPLE 2. Graph the solution set of

$$2x^2 + 7x - 4 \leq 0$$

Solution

$$2x^2 + 7x - 4 \leq 0$$

$$(2x - 1)(x + 4) \leq 0$$

This inequality is now solved by taking cases where the factors on the left side are neither both negative, nor both positive.

Case I	OR	Case II
$2x - 1 \leq 0$ and $x + 4 \geq 0$		$2x - 1 \geq 0$ and $x + 4 \leq 0$
$2x \leq 1$ and $x \geq -4$		$2x \geq 1$ and $x \leq -4$
$x \leq \frac{1}{2}$ and $-4 \leq x$		$x \geq \frac{1}{2}$ and $x \leq -4$
$-4 \leq x \leq \frac{1}{2}$		There is no value for x here.

$\therefore$ the solution set is $\{x \in R \mid -4 \leq x \leq \frac{1}{2}\}$

and the graph follows.

**EXERCISE 2-8**

- B** 1. Graph the solution set of each of the following.
- (a) $(x - 2)(x + 3) < 0$ (b) $(x - 3)(x + 2) > 0$
 (c) $x^2 - 4 \geq 0$ (d) $x^2 - 4 \leq 0$
 (e) $x^2 - x - 12 > 0$ (f) $x^2 + 2x - 3 < 0$
2. Graph the solution set:
- (a) $x^2 + x \leq 6$ (b) $x^2 - 2x > 4$
 (c) $x^2 \geq x + 20$ (d) $2x + 3 \leq x^2$
 (e) $x^2 - 10 < 3x$ (f) $0 \leq x^2 - 9$
3. Graph the solution set:
- (a) $(2x - 3)(x + 5) < 0$ (b) $(x - 2)(2x - 5) \geq 0$
 (c) $3x^2 + 7x - 6 < 0$ (d) $9 + 3x - 2x^2 \leq 0$
 (e) $2x^2 + 7x + 5 > 0$ (f) $4x^2 - 4x - 15 \leq 0$
- C** 4. Solve the following and graph.
- (a) $x(x - 1)(x + 3) < 0$
 (b) $(x - 2)(x + 3)^2 \geq 0$
 (c) $x^3 - 4x > 0$
 (d) $x^3 - x^2 - 20x \leq 0$

2.9 RADICAL EXPRESSIONS

Expressions of the form $\sqrt{3}$, $\sqrt[5]{7}$ and $\sqrt[3]{-16}$ are called radicals where the 5 and 3 are called *indices* (in $\sqrt{3}$, the index 2 is understood) and 3, 7, and -16 are the radicands.

$$x = \sqrt{a} \text{ means } x^2 = a, \text{ and } x > 0$$

$$x = \sqrt[3]{a} \text{ means } x^3 = a$$

In general,

$x = \sqrt[n]{a}$ means $x^n = a$

If n is even, the convention is that $a > 0$ and $x > 0$.

We now state two general rules for n th roots.

I	$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$	If n is even then $a, b \geq 0$
II	$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}, \quad b \neq 0$	

Proof of Rules I and II

Let $x = \sqrt[n]{a}, \quad y = \sqrt[n]{b}.$

Then $x^n = a \quad \text{and} \quad y^n = b$

Rule I

$$(xy)^n = x^n y^n = ab$$

$$xy = \sqrt[n]{ab}$$

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

Rule II

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n} = \frac{a}{b}$$

$$\frac{x}{y} = \sqrt[n]{\frac{a}{b}}$$

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

In working with second order radicals, the radicand is never negative so that radicals such as $\sqrt{x-2}$ have the restriction $x \geq 2$.

EXAMPLE 1. Express as entire radicals.

- (a) $3\sqrt{2}$ (b) $-5\sqrt{7}$ (c) $a\sqrt{b}$ (d) $2x^2y\sqrt{z}$
 (e) $2x^{23}\sqrt[3]{3x}$ (f) $-3x^2y\sqrt[3]{2xy^2}$

Solution

(a) $3\sqrt{2} = \sqrt{9} \cdot \sqrt{2} = \sqrt{18}$

(b) $-5\sqrt{7} = -\sqrt{25} \cdot \sqrt{7} = -\sqrt{175}$

(c) $a\sqrt{b} = \sqrt{a^2} \cdot \sqrt{b} = \sqrt{a^2b}$

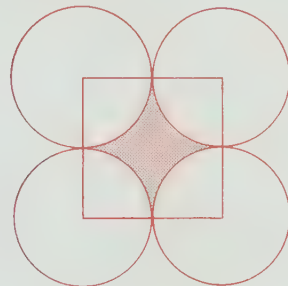
(d) $3x^2y\sqrt{z} = \sqrt{9x^4y^2} \cdot \sqrt{z} = \sqrt{9x^4y^2z}$

(e) $2x^2\sqrt[3]{3x} = \sqrt[3]{8x^6} \cdot \sqrt[3]{3x} = \sqrt[3]{24x^7}$

(f) $-3x^2y\sqrt[3]{2xy^2} = \sqrt[3]{-27x^6y^3} \cdot \sqrt[3]{2xy^2} = \sqrt[3]{-54x^7y^5}$

In part (f) of example 1, we find that $-3x^2y\sqrt[3]{2xy^2} = \sqrt[3]{-54x^7y^5}$ where the radicand on the right side is negative. This is correct, because $(-2)^3 = -8$ and $\sqrt[3]{-8} = -2$, and the index 3 is odd. However where the index is even, both the radical and the radicand are positive.

The square formed by joining the centres has a side of 4. Find the area of the shaded region.



EXAMPLE 2. Express as mixed radicals.

- (a) $\sqrt{18}$ (b) $\sqrt{a^2b}$ (c) $\sqrt{25a^4b}$ (d) $\sqrt{27a^3b^2c}$
 (e) $\sqrt[3]{-8x^4}$ (f) $\sqrt[5]{243x^5y^7}$

Solution

- (a) $\sqrt{18} = \sqrt{9 \cdot 2} = 3\sqrt{2}$
 (b) $\sqrt{a^2b} = \sqrt{a^2} \cdot \sqrt{b} = |a|\sqrt{b}$
 (c) $\sqrt{25a^4b} = \sqrt{25a^4} \cdot \sqrt{b} = 5a^2\sqrt{b}$
 (d) $\sqrt{27a^3b^2c} = \sqrt{9a^2b^2} \cdot \sqrt{3ac} = 3|ab|\sqrt{3ac}$
 (e) $\sqrt[3]{-8x^4} = \sqrt[3]{-8x^3} \cdot \sqrt[3]{x} = -2x\sqrt[3]{x}$
 (f) $\sqrt[5]{243x^5y^7} = \sqrt[5]{243x^5y^5} \cdot \sqrt[5]{y^2} = 3xy\sqrt[5]{y^2}$

Note in example 2 that we use absolute value bars to have a positive solution because, by definition, the symbol $\sqrt{\quad}$ means “the principal square root of”.

EXAMPLE 3. Simplify.

- (a) $\frac{\sqrt{48}}{\sqrt{3}}$ (b) $\frac{\sqrt{75x^4}}{\sqrt{3x^2}}$ (c) $\frac{\sqrt{80a^4b}}{\sqrt{5b}}$

Solution

- (a) $\frac{\sqrt{48}}{\sqrt{3}} = \sqrt{\frac{48}{3}} = \sqrt{16} = 4$
 (b) $\frac{\sqrt{75x^4}}{\sqrt{3x^2}} = \sqrt{\frac{75x^4}{3x^2}} = \sqrt{25x^2} = 5|x|$
 (c) $\frac{\sqrt{80a^4b}}{\sqrt{5b}} = \sqrt{\frac{80a^4b}{5b}} = \sqrt{16a^4} = 4a^2$

Radicals which are of the same order and have the same radicand are called *like* radicals. We can add like radicals by combining their coefficients using the rules of algebra as in $2x + 3x = 5x$.

EXAMPLE 4. Simplify

- (a) $3\sqrt{12} + \sqrt{18} - \sqrt{27} + 3\sqrt{8}$ (b) $3\sqrt{2}(2\sqrt{3} - 5)$ (c) $(3\sqrt{2a} - 2)^2$

Solution

- (a) We first simplify all radicals then collect like radicals.

$$\begin{aligned} 3\sqrt{12} + \sqrt{18} - \sqrt{27} + 3\sqrt{8} &= 3(2\sqrt{3}) + 3\sqrt{2} - 3\sqrt{3} + 3(2\sqrt{2}) \\ &= 6\sqrt{3} - 3\sqrt{3} + 3\sqrt{2} + 6\sqrt{2} \\ &= 3\sqrt{3} + 9\sqrt{2} \end{aligned}$$

$$(b) 3\sqrt{2}(2\sqrt{3}-5) = 6\sqrt{6} - 15\sqrt{2}$$

$$(c) (3\sqrt{2a}-2)^2 = 9\sqrt{4a^2} - 12\sqrt{2a} + 4$$

$$= 18|a| - 12\sqrt{2a} + 4$$

EXERCISE 2-9

A 1. Express as entire radicals.

$$(a) 5\sqrt{5} \quad (b) 2\sqrt{7} \quad (c) -3\sqrt{11} \quad (d) 4\sqrt{2} \quad (e) 3\sqrt{7}$$

$$(f) 3\sqrt[3]{2} \quad (g) -2\sqrt[3]{5} \quad (h) 2\sqrt[4]{3}$$

2. Express as mixed radicals.

$$(a) \sqrt{27} \quad (b) \sqrt{98} \quad (c) \sqrt{288} \quad (d) \sqrt{75} \quad (e) \sqrt{200}$$

$$(f) \sqrt[3]{-27} \quad (g) \sqrt[3]{-128} \quad (h) \sqrt[4]{48}$$

3. Simplify.

$$(a) \frac{\sqrt{75}}{\sqrt{3}} \quad (b) \frac{\sqrt{16}}{\sqrt{2}} \quad (c) \frac{\sqrt{20}}{\sqrt{5}} \quad (d) \frac{\sqrt{44}}{\sqrt{11}} \quad (e) \frac{\sqrt{2}}{\sqrt{128}}$$

$$(f) \frac{\sqrt[3]{-54}}{\sqrt[3]{-2}} \quad (g) \frac{\sqrt[4]{80}}{\sqrt[4]{5}} \quad (h) \frac{\sqrt[5]{64}}{\sqrt[5]{2}}$$

4. Collect like radicals.

$$(a) 5\sqrt{3} + 2\sqrt{3} \quad (b) 3\sqrt{5} - 2\sqrt{5} \quad (c) 6\sqrt{3} + 2\sqrt{3} - 4\sqrt{3}$$

$$(d) 4\sqrt{6} - 3\sqrt{5} + 5\sqrt{6} \quad (e) 4\sqrt{2} - 5\sqrt{2} + \frac{1}{2}\sqrt{2} \quad (f) 5\sqrt{2} - 7\sqrt{3} + 3\sqrt{2}$$

5. Expand and simplify.

$$(a) \sqrt{2}(\sqrt{6} - \sqrt{3}) \quad (b) 2\sqrt{3}(\sqrt{5} - 2) \quad (c) 3\sqrt{5}(2 - \sqrt{2})$$

$$(d) 3\sqrt{2}(\sqrt{2} + 1) \quad (e) \sqrt{a}(\sqrt{b} - 2) \quad (f) (\sqrt{a} + \sqrt{b})\sqrt{a}$$

$$(g) (\sqrt{3} - 1)^2 \quad (h) (\sqrt{2} - 3)^2 \quad (i) (\sqrt{2} - \sqrt{5})2\sqrt{3}$$

6. State the restrictions on the variables so that the following radicals represent real numbers.

$$(a) \sqrt{x-1} \quad (b) \sqrt{x+3} \quad (c) \sqrt{x} \quad (d) \sqrt{x^2+1}$$

$$(e) \sqrt{1-x^2} \quad (f) \sqrt{x(x+1)} \quad (g) \sqrt{2x-3} \quad (h) \sqrt{3-2x}$$

B 7. Express as mixed radicals.

$$(a) \sqrt{25x^2y^4} \quad (b) \sqrt{36x^2(x+y)} \quad (c) \sqrt{16a^3(a+b)}$$

$$(d) \sqrt{49a^2b(a-b)}$$

8. Express as entire radicals.

$$(a) \sqrt{5x} \times \sqrt{3a} \quad (b) -3x\sqrt{2x}$$

$$(c) 3xy^2\sqrt{xy} \quad (d) 3x^3\sqrt{x+4}$$

9. Simplify and collect like radicals.

$$(a) 3\sqrt{2} + 5\sqrt{8} - 2\sqrt{18} + 2\sqrt{32} \quad (b) 2\sqrt{20} - \sqrt{125} + 2\sqrt{45} + 6\sqrt{5}$$

$$(c) 3\sqrt{3} + 2\sqrt{20} - \sqrt{12} + \sqrt{80} \quad (d) 3\sqrt{72} - 5\sqrt{27} + 4\sqrt{8} + 2\sqrt{48}$$

$$(e) 4\sqrt{99} - 7\sqrt{12} + 3\sqrt{44} - 2\sqrt{22} \quad (f) 4\sqrt{50} + 7\sqrt{32} - 5\sqrt{7} - 2\sqrt{98}$$

10. Expand and simplify.

$$(a) 3\sqrt{2}(2 - \sqrt{6}) \quad (b) 5\sqrt{3}(2 - \sqrt{6}) \quad (c) 2\sqrt{5}(\sqrt{10} - 2)$$

WORD LADDER

Start with the word "one" and change one letter at a time to form a new word until you reach "two". The best solution has the fewest steps.

o n e

t w o

- (d) $(3\sqrt{2}-1)^2$ (e) $(\sqrt{3}-\sqrt{2})^2$ (f) $(2\sqrt{3}-5\sqrt{2})^2$
 (g) $(3\sqrt{2}-1)(3\sqrt{2}+1)$ (h) $(4-\sqrt{5})(4+\sqrt{5})$ (i) $(3\sqrt{7}-1)(3\sqrt{7}+1)$
 (j) $(3\sqrt{2}+7)(3\sqrt{2}+5)$ (k) $(\sqrt{2}+5)(2\sqrt{2}-3)$ (l) $(2\sqrt{5}-3\sqrt{2})(\sqrt{5}+2\sqrt{2})$

C 11. Express as mixed radicals.

- (a) $\sqrt{(x+3)^3}$ (b) $\sqrt{x^2-6x+9}$
 (c) $\sqrt{x^3-10x^2+25x}$ (d) $\sqrt{x^5+8x^4+16x^3}$

12. Express as entire radicals.

- (a) $|3x-2|\sqrt{5x}$ (b) $3(x-1)\sqrt{x-1}$
 (c) $(x-5)\sqrt{x+3}$, $x > 5$

13. Simplify.

- (a) $\sqrt{4x}-3\sqrt{16x}+\sqrt{81x}$, $x \geq 0$ (b) $\sqrt{8x^4}-\sqrt{2x^4}+3\sqrt{18x^4}$
 (c) $2\sqrt{x^3}-3\sqrt{4x^3}+|x|\sqrt{x}$ (d) $3\sqrt{x}-4\sqrt{y}+\sqrt{49x}-3\sqrt{9y}$
 (e) $\sqrt{8x^3}-2\sqrt{3x}+\sqrt{27x}+2\sqrt{2x}$ (f) $2\sqrt{16x^5y}+3x^2\sqrt{xy}-\sqrt{25x^5y}$

14. Expand and simplify.

- (a) $\sqrt{a}(\sqrt{a}-5)$ (b) $2\sqrt{a}(3-5\sqrt{a})$ (c) $(2\sqrt{a}-3\sqrt{b})^2$
 (d) $(\sqrt{a}+2\sqrt{b})(\sqrt{a}-2\sqrt{b})$ (e) $(3\sqrt{2a}-\sqrt{2})^2$ (f) $(3\sqrt{a}-5\sqrt{x})(3\sqrt{a}+7\sqrt{x})$

15. Expand and simplify.

- (a) $(\sqrt{x}-3)^2$ (b) $(\sqrt{x-1}+2)^2$ (c) $(4-\sqrt{x+1})^2$
 (d) $(\sqrt{x-1}+\sqrt{x+1})^2$ (e) $(\sqrt{x}-\sqrt{x-1})^2$ (f) $(\sqrt{x-3}+\sqrt{x})^2$

2.10 RATIONALIZING THE DENOMINATOR

If we wish to find the value of $\frac{1}{\sqrt{2}}$ to three decimal places, then we

can substitute $\sqrt{2} \doteq 1.414$ immediately and divide, or we can rationalize the denominator.

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} \doteq \frac{1.414}{2} = 0.707$$

$$\therefore \frac{1}{\sqrt{2}} \doteq 0.707$$

Note that the value of the original fraction is not changed by multiplying by 1.

In order to rationalize a **binomial** denominator, we multiply numerator and denominator by the **conjugate** radical of the denominator. If we multiply conjugate radicals, such as

$$(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})=5-3=2$$

then the product of the two irrational numbers is a rational number.

EXAMPLE 1. Simplify by first rationalizing the denominator.

- (a) $\frac{2\sqrt{5}}{3\sqrt{2}}$ (b) $\frac{2}{\sqrt{7}-\sqrt{3}}$ (c) $\frac{3+2\sqrt{5}}{4+3\sqrt{5}}$

Solution

$$\begin{aligned} \text{(a)} \quad \frac{2\sqrt{5}}{3\sqrt{2}} &= \frac{2\sqrt{5}}{3\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{2\sqrt{10}}{3(2)} \\ &= \frac{\sqrt{10}}{3} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{2}{\sqrt{7}-\sqrt{3}} &= \frac{2}{\sqrt{7}-\sqrt{3}} \cdot \frac{\sqrt{7}+\sqrt{3}}{\sqrt{7}+\sqrt{3}} \\ &= \frac{2(\sqrt{7}+\sqrt{3})}{7-3} \\ &= \frac{2(\sqrt{7}+\sqrt{3})}{4} \\ &= \frac{\sqrt{7}+\sqrt{3}}{2} \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \frac{3+2\sqrt{5}}{4+3\sqrt{5}} &= \frac{3+2\sqrt{5}}{4+3\sqrt{5}} \cdot \frac{4-3\sqrt{5}}{4-3\sqrt{5}} \\ &= \frac{12-\sqrt{5}-30}{16-45} \\ &= \frac{-18-\sqrt{5}}{-29} \\ &= \frac{18+\sqrt{5}}{29} \end{aligned}$$

EXERCISE 2-10

A 1. Multiply the following and state which are conjugate radicals.

- | | |
|--|--|
| (a) $(3\sqrt{5}+1)(3\sqrt{5}-1)$ | (b) $(2\sqrt{3}+1)^2$ |
| (c) $(\sqrt{7}-\sqrt{5})(\sqrt{7}+\sqrt{5})$ | (d) $(\sqrt{5}+\sqrt{7})(\sqrt{7}-\sqrt{5})$ |
| (e) $(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})$ | (f) $(3\sqrt{2}+1)(3\sqrt{2}+1)$ |

B 2. Rationalize the denominator.

- | | | | |
|----------------------------------|--|-----------------------------------|-----------------------------------|
| (a) $\frac{3\sqrt{2}}{\sqrt{5}}$ | (b) $\frac{\sqrt{3}+\sqrt{5}}{\sqrt{2}}$ | (c) $\frac{2\sqrt{3}}{3\sqrt{6}}$ | (d) $\frac{2\sqrt{5}}{3\sqrt{6}}$ |
|----------------------------------|--|-----------------------------------|-----------------------------------|

3. Rationalize the denominator.

- | | | |
|---|---|---|
| (a) $\frac{3}{2-\sqrt{2}}$ | (b) $\frac{3+\sqrt{3}}{2-\sqrt{3}}$ | (c) $\frac{\sqrt{2}+1}{\sqrt{2}-1}$ |
| (d) $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ | (e) $\frac{3\sqrt{5}}{\sqrt{2}-3}$ | (f) $\frac{2\sqrt{5}-3\sqrt{2}}{\sqrt{5}+\sqrt{2}}$ |
| (g) $\frac{12}{2\sqrt{6}-\sqrt{3}}$ | (h) $\frac{2\sqrt{5}}{\sqrt{5}+\sqrt{3}}$ | (i) $\frac{3\sqrt{7}}{2\sqrt{2}-\sqrt{5}}$ |

4. Calculate each of the following to two decimal places by first rationalizing the denominator.

- | | | |
|-----------------------------------|----------------------------|-------------------------------------|
| (a) $\frac{1}{\sqrt{3}-\sqrt{2}}$ | (b) $\frac{1}{3-\sqrt{5}}$ | (c) $\frac{\sqrt{6}+1}{\sqrt{6}-1}$ |
|-----------------------------------|----------------------------|-------------------------------------|

$$\begin{aligned} \sqrt{2} &\doteq 1.414 \\ \sqrt{3} &\doteq 1.732 \\ \sqrt{5} &\doteq 2.236 \\ \sqrt{6} &\doteq 2.449 \end{aligned}$$

(d) $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$

(e) $\frac{2\sqrt{5}}{\sqrt{5}-1}$

(f) $\frac{3\sqrt{2}}{2\sqrt{3}-\sqrt{2}}$

(g) $\frac{3\sqrt{3}-2}{2\sqrt{3}-3\sqrt{2}}$

(h) $\frac{2\sqrt{5}-1}{\sqrt{5}+2}$

(i) $\frac{5\sqrt{3}-\sqrt{2}}{2\sqrt{3}+\sqrt{2}}$

5. Rationalize the denominator.

(a) $\frac{1}{\sqrt{a}-\sqrt{b}}$

(b) $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$

(c) $\frac{a+b+2\sqrt{ab}}{\sqrt{a}+\sqrt{b}}$

(d) $\frac{3\sqrt{a}}{\sqrt{a}-2}$

(e) $\frac{3\sqrt{a}-5}{2\sqrt{a}+3}$

(f) $\frac{a+b-2\sqrt{ab}}{\sqrt{a}-\sqrt{b}}$

(g) $\frac{1}{\sqrt{x-1}+1}$

(h) $\frac{3}{\sqrt{x+3}+\sqrt{x}}$

(i) $\frac{5x}{\sqrt{x}-\sqrt{2x-5}}$

2.11 RADICAL EQUATIONS

Equations such as $\sqrt{x+3}-2=5$, where the variable appears in the radicand, are called **radical equations**. The solution of these equations is based on the fact that if two real numbers are equal then their squares are equal. The converse of this is not necessarily true because although $(-5)^2 = (5)^2$, $(-5) \neq (5)$. Because the radicand contains a variable, we must ensure that the radicand is non-negative so that the domain of the variable is limited to those values for which the equation has meaning.

EXAMPLE 1. State the domain of x for each of the following expressions.

(a) $\sqrt{x-3}$

(b) $\sqrt{x+5}$

(c) $\sqrt{x^2-9}$

(d) $\frac{1}{\sqrt{x-3}}$

Solution

(a) $\sqrt{x-3}$ is defined for $x-3 \geq 0$
 $x \geq 3$

(b) $\sqrt{x+5}$ is defined for $x+5 \geq 0$
 $x \geq -5$

(c) $\sqrt{x^2-9}$ is defined for $x^2-9 \geq 0$
 $x \geq 3$ or $x \leq -3$

(d) $\frac{1}{\sqrt{x-3}}$ is defined for $x > 3$

EXAMPLE 2. Solve $\sqrt{x-2}+3=5$.

Solution

$$\sqrt{x-2}+3=5, \quad x-2 \geq 0 \quad \text{and} \quad x \geq 2$$

We first *isolate the radical*.

$$\sqrt{x-2} = 2$$

Squaring both sides,

$$[\sqrt{x-2}]^2 = [2]^2$$

$$x-2 = 4$$

$$x = 6$$

$\therefore$ the root is 6 (because $6 \geq 2$)

EXAMPLE 3. Solve $\sqrt{2x-3} + \sqrt{x+2} = 3$.

Solution

$$\sqrt{2x-3} + \sqrt{x+2} = 3, \quad \begin{cases} 2x-3 \geq 0, & \text{and } x \geq \frac{3}{2} \\ x+2 \geq 0, & \text{and } x \geq -2 \end{cases}$$

We first isolate the radical we wish to eliminate.

$$\sqrt{2x-3} = 3 - \sqrt{x+2}$$

Squaring both sides,

$$[\sqrt{2x-3}]^2 = [3 - \sqrt{x+2}]^2$$

$$2x-3 = 9 - 6\sqrt{x+2} + x+2$$

$$x-14 = -6\sqrt{x+2}$$

Squaring both sides again,

$$[x-14]^2 = [-6\sqrt{x+2}]^2$$

$$x^2 - 28x + 196 = 36x + 72$$

$$x^2 - 64x + 124 = 0$$

$$(x-2)(x-62) = 0$$

$$(x-2) = 0 \quad \text{or} \quad (x-62) = 0$$

$$x = 2$$

$$x = 62$$

Check: for $x = 2$

$$\begin{aligned} \text{L.S.} &= \sqrt{2(2)-3} + \sqrt{2+2} & \text{R.S.} &= 3 \\ &= \sqrt{1} + \sqrt{4} \\ &= 3 \end{aligned}$$

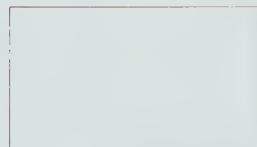
for $x = 62$

$$\begin{aligned} \text{L.S.} &= \sqrt{2(62)-3} + \sqrt{62+2} & \text{R.S.} &= 3 \\ &= \sqrt{121} + \sqrt{64} \\ &= 19 \end{aligned}$$

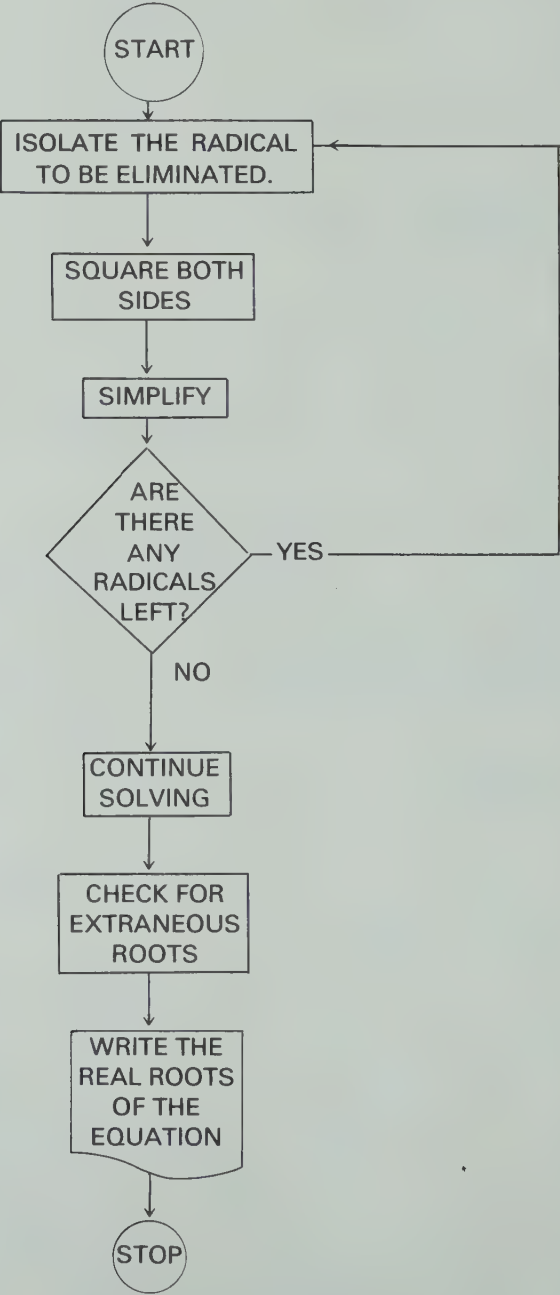
The solution set is $\{2\}$.

Since $x = 62$ did not verify in the original equation, we discard this value and say that 62 is an *extraneous* root.

Cut the 9 cm $\times$ 16 cm shape and form a square.



The method of solving radical equations can be summarized in the following flow chart.



EXERCISE 2-11

A 1. State the domain of x for each of the following expressions.

- (a) $\sqrt{x}$ (b) $\sqrt{x-2}$ (c) $\sqrt{x+3}$ (d) $\sqrt{x^2+1}$
 (e) $\sqrt{x^2-1}$ (f) $\sqrt{x^2-4}$ (g) $\frac{1}{\sqrt{x+2}}$ (h) $\frac{-3}{\sqrt{x-3}}$

2. Determine which of the given values are roots of the following equations.

- (a) $\sqrt{x+3}=5$; 2, 4 (b) $3-\sqrt{x}=2$; 1, 4
 (c) $\sqrt{x+1}=0$; 1, -1 (d) $\sqrt{x-1}-2=0$; -3, 5
 (e) $\sqrt{2x+1}=x-1$; -3, 4 (f) $\sqrt{2x+1}=\sqrt{3x-3}$; 0, 4

B 3. Solve the following.

- (a) $\sqrt{x+1}-5=2$ (b) $\sqrt{2x-1}+3=7$
 (c) $3-\sqrt{x+2}=2$ (d) $\sqrt{3x-27}=3$

4. Solve.

- (a) $\sqrt{2x-5}=x-4$ (b) $\sqrt{3x-2}=x$
 (c) $\sqrt{2x-1}=x-2$ (d) $\sqrt{x^2-5}=x-1$

5. Solve.

- (a) $\sqrt{x-3}+\sqrt{x}=3$ (b) $\sqrt{2x+1}-\sqrt{x}=1$
 (c) $\sqrt{2x+1}+\sqrt{x}=1$ (d) $\sqrt{2x+3}+\sqrt{3-x}=3$

C 6. Solve.

- (a) $\frac{\sqrt{x-7}}{\sqrt{x-2}}=\sqrt{2}$ (b) $\sqrt{x+1}=\frac{3}{\sqrt{x-1}}-\sqrt{x-1}$
 (c) $\sqrt{x}=\frac{1}{\sqrt{x-1}}-\sqrt{x-1}$ (d) $\frac{\sqrt{x+3}}{\sqrt{x}}=\sqrt{x-1}$

2.12 APPLICATIONS OF REAL NUMBERS

EXAMPLE 1. The sum of the squares of two consecutive positive integers is 85. Find the numbers.

Solution

Let the smaller integer be x . Then the larger integer is $x+1$ and

$$\begin{aligned}x^2 + (x+1)^2 &= 85, & x > 0 \\x^2 + (x^2 + 2x + 1) &= 85 \\2x^2 + 2x - 84 &= 0 \\x^2 + x - 42 &= 0 \\(x+7)(x-6) &= 0 \\x+7=0 &\quad \text{or} \quad x-6=0 \\x=-7 &\quad x=6 \\ \text{is inadmissible} &\quad x+1=7 \\ \text{since } x > 0\end{aligned}$$

$\therefore$ the numbers are 6 and 7.

Check: 1. 6 and 7 are consecutive integers.

2. $6^2 + 7^2 = 36 + 49 = 85$

EXAMPLE 2. A tourist travels 200 km by car and 240 km by bus. He travelled 10 km/h faster by bus than by car. What was the speed of the car if the total trip took 8 h?

Solution

We introduce the variable and sort the data in a DRT table.

	D(km)	R(km/h)	T(h)
Car	200	x	$\frac{200}{x}$
Bus	240	$x + 10$	$\frac{240}{x + 10}$

$$\frac{200}{x} + \frac{240}{x + 10} = 8, \quad x \geq 0$$

$$x(x + 10) \left[\frac{200}{x} + \frac{240}{x + 10} \right] = 8x(x + 10)$$

$$200(x + 10) + 240x = 8x^2 + 80x$$

$$200x + 2000 + 240x = 8x^2 + 80x$$

$$-8x^2 + 360x + 2000 = 0$$

$$x^2 - 45x - 250 = 0$$

$$(x - 50)(x + 5) = 0$$

$$x = 50 \quad \text{or} \quad x = -5$$

$\therefore$ the speed of the car was 50 km/h.

Check: car $\rightarrow$ 200 km at 50 km/h takes $\frac{200}{50} = 4$ h

Bus $\rightarrow$ 240 km at 60 km/h takes $\frac{240}{60} = 4$ h

Total time = 8 h

Find a four-digit perfect square whose first two digits and last two digits each represent squares.

EXAMPLE 3. The altitude d of an object t seconds after it is launched from an altitude of h metres with an initial velocity of v metres per second is given by the formula

$$d = h + vt - 4.9t^2$$

During what period of time will a missile be above 400 m from the ground if it is fired from a tower 91.3 m high with an initial velocity of 117.6 m/s.

Solution

We rewrite the formula

$$-4.9t^2 + vt + h > d$$

and substitute the values $v = 117.6$, $h = 91.3$ and $d = 400$.

$$\begin{aligned}
 -4.9t^2 + 117.6t + 91.3 &> 400, & t &\geq 0 \\
 -4.9t^2 + 117.6t - 308.7 &> 0 \\
 t^2 - 24t + 63 &< 0 \\
 (t-3)(t-21) &< 0
 \end{aligned}$$

Case I: $t-3 < 0$ and $t-21 > 0$ or Case II: $t-3 > 0$ and $t-21 < 0$

$$t < 3 \text{ and } t > 21$$

$$t > 3 \text{ and } t < 21$$

There is no value for t .

$$3 < t < 21$$

The missile will be above 400 m between the third and the twenty-first second after launch.

EXERCISE 2-12

- A**
- State algebraic expressions using x as a variable.
 - Mike scored 3 goals more than Dave.
 - Ed is 3 times as old as his son.
 - It takes 0.7 s to react and $3x$ seconds to stop a car.
 - Sam drives x hours at 80 km/h.
 - State whether you would subtract or divide.
 - To find the length AB if AC and BC are given.
 - To find the average mass of five samples of steel if the total mass is x kilograms.
 - To find the cost of 50 items that sold for $\$x$ making a profit of $\$30$.
 - To find the year Mr. Jonas was born if he is now x years old.
 - State the distance travelled by
 - A car travelling 80 km/h for 8 h.
 - A sound wave travelling 410 m/s for 8 s.
 - A car travelling for 5 h at 75 km/h.
 - A man travelling for 3 h at 20 km/h and 2 h at 40 km/h.
- B**
- The temperature is 17°C . If there is a temperature change of 4°C , state two possibilities for the new temperature.
 - In the launching of a rocket, the countdown stood at $-2:31$ at 07:30:00. At what time was the actual liftoff?
 - A student wishes to average 75% on three tests. If he has scored 72% and 68% on the first two tests, what must he score on the third test to achieve this average?
 - The area of a triangle is given by

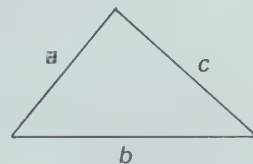
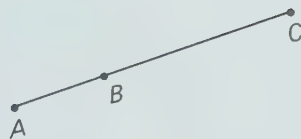
$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\text{where } s = \frac{a+b+c}{2}$$

and a , b , and c are the lengths of the sides. Find the areas of the triangles with the following sides.

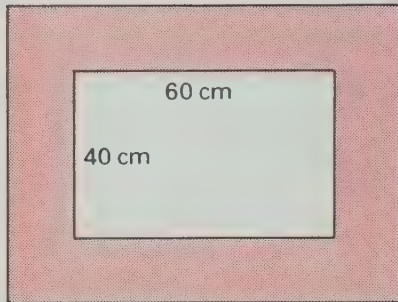
- 3 cm, 8 cm, 6 cm.
- 42 m, 25 m, 30 m.
- 22 cm, 15 cm, 25 cm.

- The sum of the squares of three consecutive numbers is 29. Find the numbers.

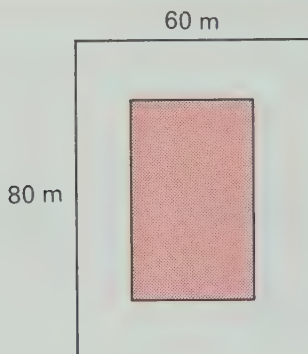


9. The sum of the squares of three consecutive even numbers is 116. Find the numbers.

10. The sum of the squares of three consecutive positive odd numbers is 83. Find the numbers.

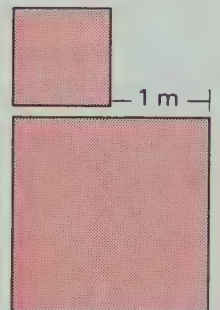


11. A picture 40 cm by 60 cm is to be surrounded by a border so that the area of the border is equal to the area of the picture. Find the width of the border.



12. Two girls agree to mow a lawn 60 m by 80 m so that the first girl cuts one half by cutting a uniform strip around the lawn. How wide a strip should the first girl cut?

13. The side of one square is 1 m longer than the side of a smaller square. The sum of their areas is 25 m^2 . Find the sides of the two squares.



D km	R km/h	T h
	x	
300		

14. Two bike riders travel 300 km from town A to town B. They return along the same route travelling 10 km/h faster. Find their speed going if the total trip took 11 h.

D km	R km/h	T h
	x	

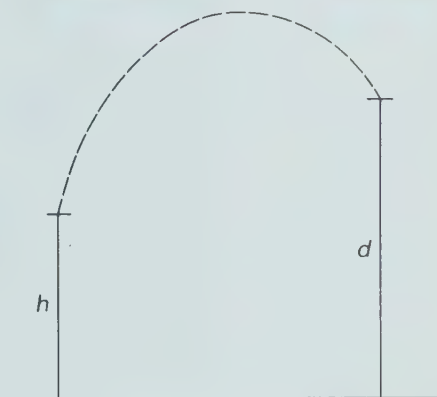
15. A small speedboat travelled 300 km along the Lake Erie shoreline and returned in 9 h. Find the average speed going out if the return trip was 15 km/h slower than going out.

16. A student swam 56 m with the current, and 48 m against the current in the same length of time. What is the swimmer's speed in still water if the current is 4 km/h? What is the swimmer's speed in m/s?

D km	R km/h	T h
	$x + 4$	
	$x - 4$	

For questions 17 to 20, the altitude d of an object t seconds after it is thrown upward with an initial velocity v metres per second from an altitude of h m is given by the formula.

$$d = h + vt - 4.9t^2$$



17. A toy rocket is fired from ground level ($h = 0$) with an upward velocity of 19.6 m/s. After how many seconds will it return to the ground?



18. A missile is fired upward from 100 m above ground level with an initial velocity of 980 m/s. After how many seconds will the missile be 9410 m above the ground?

C 19. A ball is thrown upward from a point 1.5 m above the ground with a velocity of 19.6 m/s. During what length of time is the ball higher than 16.2 m?

$$h + vt - 4.9t^2 > d$$

20. A missile is fired upward from the top of a 100 m tower with initial velocity 53.9 m/s. For what length of time is the missile above the top of the tower?

REVIEW EXERCISE

B 1. Express (a) $\frac{1}{12}$ (b) $\frac{3}{20}$ (c) $\frac{6}{11}$ (d) $\frac{1}{13}$ in decimal form.

2. Express (a) 0.325̄ (b) 3.72̄ (c) 0.245̄ as common fractions.

3. Graph the following for $x \in R$.

(a) $\{x \mid x \leq 5\}$

(b) $\{x \mid -2 < x \leq 4\}$

(c) $\{x \mid -2 \leq x < 5\}$

(d) $\{x \mid x > -3\} \cap \{x \mid x \leq 5\}$

(e) $\{x \mid x < 0\} \cap \{x \mid x \geq 3\}$

(f) $\{x \mid -1 < x \leq 2\} \cap \{x \mid 2 < x \leq 4\}$

4. Solve each of the following for x :

- (a) $5x - 7 = 18$ (b) $\frac{1}{2}x + \frac{2}{3} = \frac{3}{4}$ (c) $5 - 2(x - 2) = 1$
 (d) $2(x + 3) + 3(x - 4) = 4$ (e) $\frac{2}{3}(x - 1) - \frac{1}{2}(x + 1) = -\frac{1}{2}$
 (f) $ax + b = c$ (g) $3a + \frac{5}{x} = 2a - 7$

5. Solve the following inequalities.

- (a) $\frac{1}{4}x \leq 2$ (b) $-3x > 12$ (c) $3(x - 5) + 1 < 5$
 (d) $\frac{3 - x}{2} + \frac{x}{3} > \frac{x + 1}{4}$ (e) $\frac{1}{2}(x - 3) + \frac{1}{3}(x + 1) \geq 6$
 (f) $|x + 3| < 2$ (g) $|x - 5| \geq 1$ (h) $|2x - 1| \leq 5$

6. Solve by factoring.

- (a) $x^2 - x - 20 = 0$ (b) $x^2 - 16 = 0$ (c) $x^2 + x - 2 = 0$
 (d) $10x^2 - 11x + 3 = 0$ (e) $8x^2 + 14x = 15$ (f) $3x^2 + 11x + 6 = 0$
 (g) $(x - 2)(x^2 - 9x + 20) = 0$ (h) $(x^2 + 5x - 6)(2x - 1) = 0$

7. Solve the following correct to three significant figures.

- (a) $x^2 - 3.5x + 2.7 = 0$ (b) $0.25x^2 - 3.6x + 4.8 = 0$
 (c) $20x^2 - 4.2x - 8.5 = 0$ (d) $2.5x^2 + 4.6x - 3.2 = 0$
 (e) $4.6x^2 + 2.5x - 1 = 0$ (f) $4.7x^2 - 0.25x - 2.7 = 0$

8. Graph the solution set.

- (a) $(x - 2)(x + 3) < 0$ (b) $(x + 5)(x - 3) \leq 0$
 (c) $x^2 - x - 12 \geq 0$ (d) $2x^2 - 3x - 9 \leq 0$
 (e) $|x| \leq 3$ (f) $|x - 2| < 5$
 (g) $|x + 4| > 3$

9. Simplify.

- (a) $3\sqrt{8} - 2\sqrt{32} + \sqrt{50}$
 (b) $2\sqrt{3} - 4\sqrt{12} + 5\sqrt{27}$
 (c) $3\sqrt{5} + 2\sqrt{50} - 5\sqrt{8} + \sqrt{45}$
 (d) $3\sqrt{5}(2 + \sqrt{10})$ (e) $2\sqrt{3}(\sqrt{6} - \sqrt{2})$
 (f) $(2\sqrt{3} + \sqrt{5})(\sqrt{3} - \sqrt{5})$ (g) $(2\sqrt{3} + \sqrt{2})^2$

10. Calculate each of the following to two decimal places.

- (a) $\frac{5}{\sqrt{2}}$ (b) $\frac{7}{\sqrt{3}}$ (c) $\frac{\sqrt{2}}{\sqrt{3}}$ $\sqrt{2} \doteq 1.414$
 $\sqrt{3} \doteq 1.732$
 (d) $\frac{1}{\sqrt{5} - \sqrt{3}}$ (e) $\frac{3}{\sqrt{6} - 2}$ $\sqrt{5} \doteq 2.236$
 $\sqrt{6} \doteq 2.449$

11. Solve for x .

- (a) $\sqrt{x - 5} = 2$ (b) $\sqrt{2x - 5} = 1$ (c) $\sqrt{x + 5} + \sqrt{x} = 5$
 (d) $\sqrt{x + 5} - \sqrt{x - 3} = \sqrt{x}$

12. A tourist travels 300 km by train and 400 km by bus. He travelled 20 km/h faster by train than by bus. What was the speed of the train if the total trip took 8 h?

13. The sum of the squares of two consecutive positive numbers is 113. Find the numbers.

14. A picture 12 cm by 18 cm is to be surrounded by a border with area equal to the area of the picture. Find the width of the border.

15. The sum of the squares of two consecutive odd numbers is 74. Find the numbers.

Put the numbers from 1 to 9 in the spaces to make the statements true.

$$\square \div \square + \square = 9$$

$$\square \times \square + \square = 9$$

$$\square \times \square - \square = 9$$

REVIEW AND PREVIEW TO CHAPTER 3

EXERCISE 1

ALGEBRAIC MANIPULATION

1. Simplify:

(a) $\frac{2}{b+3} - \frac{1}{b-3}$

(b) $\frac{3x-3}{7-x} - \frac{11-2x}{49-x^2}$

2. Solve for each of the 3 variables: $\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{R}$

3. Solve for each of the 6 variables: $\frac{p_1 V_1}{T_1} = \frac{p_2 V_2}{T_2}$

4. Solve for x : $y = \frac{1+x}{1-x}$

5. Solve for y and z : $x = \frac{y+z}{yz}$

6. Solve for m and v : $E = \frac{1}{2}mv^2$, $E > 0$

7. Solve for s , b , and c : $m = \frac{(s-c)b}{(s-b)c}$

EXERCISE 2

CONTINUED FRACTIONS

Simplify the following continued fractions by expressing each of them as a simple fraction in lowest terms.

1. $\frac{1}{1 - \frac{1}{1 - \frac{1}{2}}}$

2. $3 - \frac{4}{2 + \frac{5}{3 + \frac{1}{4}}}$

3. $1 + \frac{1}{1 + \frac{1}{1+x}}$

4. $\frac{1}{1 - \frac{1}{1 - \frac{1}{\frac{1}{x}}}}$

5. $1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{x-1}}}$

6. $\frac{x}{1 - \frac{1}{x + \frac{1}{x+1}}}$

7. $\frac{\frac{2x}{1+x}}{\frac{3}{1 - \frac{1}{x}}}$

8. $\frac{\frac{x+2}{x}}{x+1 - \frac{\frac{1}{x+1}}{1 - \frac{x}{x+1}}}$

9. $\frac{\frac{x}{x+1}}{x+2 - \frac{\frac{2}{x+1}}{1 - \frac{x}{x+1}}}$

Functions

That flower of modern mathematical thought... the notion of a function.

Thomas J. McCormack

PIERRE LAPLACE (1749–1827)



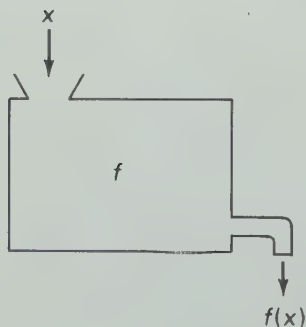
Laplace was attracted to mathematics because of its power in solving scientific problems. His own greatest contribution was in the application of mathematics to astronomy. Using the theory of functions together with Newton's Laws, he was able to show that our solar system is stable.

It has been said that probability theory owes more to Laplace than to any other person. But even his interest in probability was inspired by the need for it in mathematical astronomy.

Laplace lived at the time of the French Revolution but managed to survive quite well because he quickly changed his allegiance to whoever came to power. He even changed his book on mathematical astronomy to include glowing tributes to whichever side happened to be in power at the time. Laplace taught mathematics at the Military School and one of his students there was Napoleon Bonaparte. Napoleon was a great admirer of men of science. (He said, "The advancement and perfection of mathematics are intimately connected with the prosperity of the state.") When Napoleon came to power he awarded Laplace the Grand Cross of the Legion of Honour and gave him the powerful position of Minister of the Interior. So what did Laplace do when Napoleon fell from Power? He signed the decree which banished Napoleon, and thrived under King Louis XVIII.

3.1 FUNCTION NOTATION

distance
= speed $\times$ time



If a car travels at a speed of 80 km/h, then after half an hour the car will have travelled 40 km. If d represents distance in kilometres and t represents time in hours, then the rule that connects t and d is $d = 80t$, as long as the car's speed is a constant 80 km/h. We say that distance is a function of time because the value of d depends on the value of t . With each value of t there is associated one value of d .

Given two sets A and B , a *function* f is a rule, or correspondence, that associates with each element x in A an element, called $f(x)$, in B . A is called the *domain* of f . The *range* of f is the set of all possible values of $f(x)$ for x in A .

The symbol $f(x)$ is read " f of x ". It is the value of the function f at x . For example, the value of f when $x = 7$ is $f(7)$. If f is given by a formula, then we find $f(7)$ by substituting 7 for x .

EXAMPLE 1. If $f(x) = 7x - 1$, find (a) $f(4)$, (b) $f(-3)$, (c) $f(0)$.

Solution

(a) $f(4) = 7 \times 4 - 1 = 27$

(b) $f(-3) = 7 \times (-3) - 1 = -22$

(c) $f(0) = 7 \times 0 - 1 = -1$

EXAMPLE 2. If $g(x) = x + \frac{4}{x}$, find (a) $g(2)$, (b) $g(8)$, (c) $g(0)$.

Solution

(a) $g(2) = 2 + \frac{4}{2} = 2 + 2 = 4$

(b) $g(8) = 8 + \frac{4}{8} = 8.5$

(c) $g(0)$ is not defined because $\frac{4}{0}$ is not defined. This means that 0 does not belong to the domain of g .

EXAMPLE 3. In the example of the car, we could write $d = f(t) = 80t$. Here $f(0.5)$ means the value of the function when $t = 0.5$, i.e., the distance travelled after 0.5 h, and $f(3)$ means the value of d after 3 h.

$$f(0.5) = 80 \times 0.5 = 40$$

$$f(3) = 80 \times 3 = 240$$

Since d depends on t , we call t an *independent variable* and d a *dependent variable*.

It sometimes helps to picture a function by an arrow diagram as in Figure 3-1. Each arrow connects an element of A to an element of B . The arrow indicates that $f(x)$ is associated with x , $f(a)$ is associated with a , and so on.

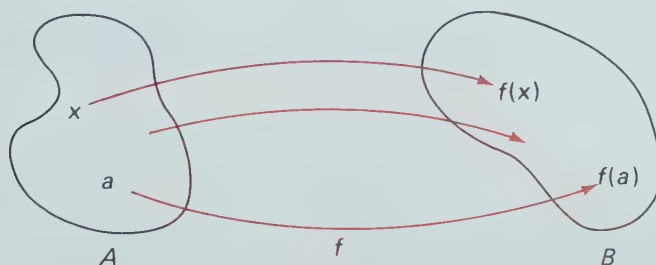


Figure 3-1

Functions are also called *mappings*, and we say that f *maps* the element x into its image $f(x)$, and f maps the set A into the set B .

Sometimes we use arrows to name a function. In Example 1 we could have written

$$f: x \rightarrow 7x - 1 \quad \text{instead of} \quad f(x) = 7x - 1.$$

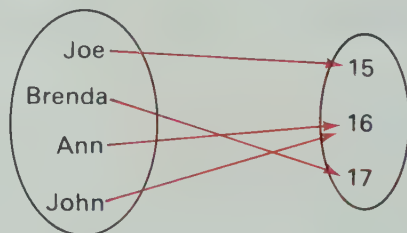
This means that each real number x is mapped into its corresponding number $7x - 1$.

EXAMPLE 4. Suppose that Joe is 15 years old, Brenda is 17, and Ann and John are both 16. Let $f(x)$ be the age of x . For instance, $f(\text{Joe}) = 15$ and $f(\text{Ann}) = 16$. State the domain and range and represent f by an arrow diagram.

Solution

Domain of f : $A = \{\text{Joe, Brenda, Ann, John}\}$

Range of f : $B = \{15, 16, 17\}$

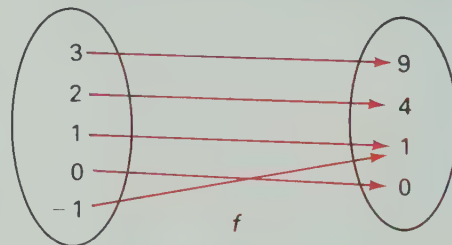


EXAMPLE 5. If f is the "squaring" function, $f: x \rightarrow x^2$, and the domain is $A = \{-1, 0, 1, 2, 3\}$, find the range of f .

Solution

$f(-1) = (-1)^2 = 1$, $f(0) = 0^2 = 0$, $f(1) = 1^2 = 1$, $f(2) = 2^2 = 4$, $f(3) = 3^2 = 9$.

The range is $B = \{0, 1, 4, 9\}$.

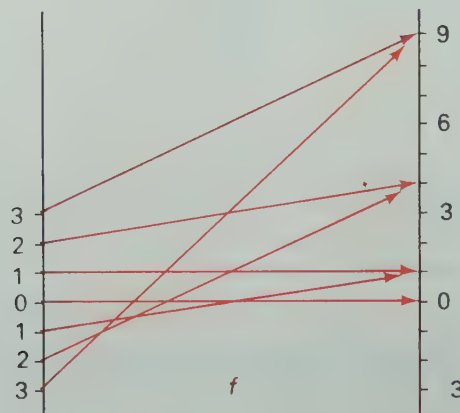


EXAMPLE 6. If $f(x) = x^2$ and the domain is R , what is the range of f ?

Solution

The range consists of all squares of real numbers and the square of a real number is never negative. Therefore, the range of f consists of all nonnegative real numbers: $\{x \in R \mid x \geq 0\}$.

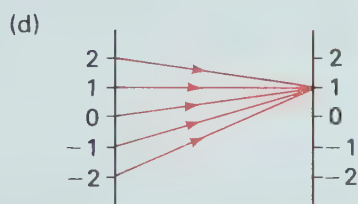
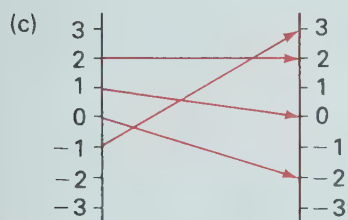
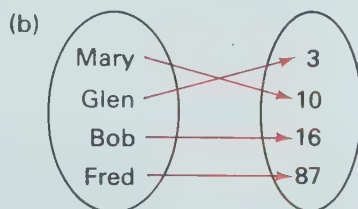
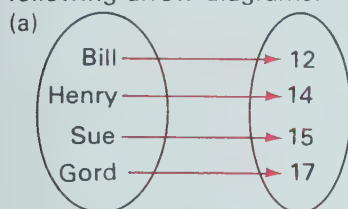
A man spends $\frac{1}{3}$ of his money and loses $\frac{2}{3}$ of the remainder. He then has \$12. How much money had he at first?



EXERCISE 3-1

- A** 1. If $f(x) = 2 + 3x$, state
 (a) $f(0)$ (b) $f(-1)$ (c) $f(2)$ (d) $f(0.2)$ (e) $f(-5)$
 (f) $f(\frac{1}{3})$ (g) $f(100)$ (h) $f(-10)$ (i) $f(a)$ (j) $f(b)$
2. If $g(x) = x^2 - 4$, state
 (a) $g(1)$ (b) $g(2)$ (c) $g(-2)$ (d) $g(0)$ (e) $g(10)$
 (f) $g(8)$ (g) $g(-8)$ (h) $g(\sqrt{5})$ (i) $g(a)$ (j) $g(-a)$
3. State $f(3)$ for each of the following functions.
 (a) $f(x) = 2x - 5$ (b) $f(x) = 1 + 9x$ (c) $f(t) = 2 - 5t$
 (d) $f(x) = x^3$ (e) $f(t) = -t$ (f) $f: x \rightarrow x^2 + 1$
 (g) $f: x \rightarrow \frac{1}{3}x - 1$ (h) $f(x) = 17$ (i) $f(x) = 8(x - 1)$

4. State the domain and range of the functions represented by the following arrow diagrams.



- B** 5. If $f(x) = x^2 + 2x + 5$, find
 (a) $f(2)$ (b) $f(-2)$ (c) $f(15)$ (d) $f(0.3)$
 (e) $f(-7)$ (f) $f(21)$ (g) $f(1.4)$ (h) $f(0.01)$

6. If $g(x) = \frac{x+3}{2x-5}$, find

- (a) $g(1)$ (b) $g(11)$ (c) $g(-9)$ (d) $g(\frac{1}{2})$
 (e) $g(0.2)$ (f) $g(100)$ (g) $g(-6.5)$ (h) $g(a)$

7. If $f(t) = t^4 + t^2 + 1$, find

- (a) $f(-1)$ (b) $f(\sqrt{3})$ (c) $f(-4)$ (d) $f(1.5)$
 (e) $f(7)$ (f) $f(-6)$ (g) $f(\sqrt[4]{2})$ (h) $f(-2.01)$

8. Find $f(4)$ for each of the following functions.

- (a) $f(x) = \sqrt{x}$ (b) $f(t) = \sqrt[3]{23+t}$ (c) $f: x \rightarrow (x+2)^3$
 (d) $f: x \rightarrow x^5$ (e) $f(x) = x - \frac{8}{x}$ (f) $f(u) = 8 + u - u^2$

9. If $f(x) = 3x^2 + x - 1$, find expressions for the following.

- (a) $f(a)$ (b) $f(a+1)$ (c) $f(2a)$
 (d) $f(a+b)$ (e) $f(-a)$ (f) $f(-x)$

10. If $f(x) = 4x - 20$, find x so that

- (a) $f(x) = 0$ (b) $f(x) = 16$ (c) $f(x) = -10$.

11. Joe and Mary Fry and their children Herman and Agnes live at 791 Main Street. Their next-door neighbour Elmer Flood lives at number 793, and Joe's mother Ethel and father Fred live across the street at number 796. Let the domain of f be the set of these 7 people and let $f(x)$ be the address of x on Main Street. What is the range of f ? Draw an arrow diagram to represent f .

12. Given the domain $A = \{-2, -1, 0, 1, 2\}$, find the range and draw arrow diagrams for the following functions.

(a) $f(x) = x^2 + 1$ (b) $g(x) = 2x$ (c) $h(x) = |x|$

13. If the domain is R , find the ranges of the following functions.

(a) $f(x) = x^2 + 1$ (b) $g(x) = 2x$ (c) $h(x) = |x|$

14. If we take the domains of the following functions to be the largest sets for which the given expressions are meaningful, find the domains in each case.

(a) $f(x) = \frac{2+x}{3-x}$ (b) $f(x) = \frac{3-x}{2+x}$

(c) $g(x) = \sqrt{x-2}$ (d) $h(x) = \sqrt{9-x^2}$

15. The emergency brake cable in a car parked on a hill suddenly breaks and the car rolls down the hill. The distance in metres, d , that it rolls in t seconds is given by the function $d = f(t) = 0.5t^2$.

(a) How far has the car rolled after 10 s?

(b) When will it splash into the lake which is at the bottom of the hill 200 m away?



16. If $f(x) = x^9 - x^4 + \frac{2}{x^2}$, find

(a) $f(3)$ (b) $f(1.01)$ (c) $f(2.537)$ (d) $f(-0.79)$

17. If $f(x) = 3x + 4$ and $g(x) = x^2 - 5$, find

(a) $f(g(2))$ (b) $g(f(2))$ (c) $f(g(-1))$
(d) $g(f(-1))$ (e) $f(g(x))$ (f) $g(f(x))$

18. If $f(x) = \frac{1}{x+1}$ and $g(x) = x + 1$, find

(a) $f(g(6))$ (b) $g(f(6))$ (c) $f(g(\frac{1}{3}))$
(d) $g(f(\frac{1}{3}))$ (e) $f(g(x))$ (f) $g(f(x))$

19. If $f(x) = x^2 + x - 12$, find all values of x such that

(a) $f(x) = 0$ (b) $f(x) = 8$ (c) $f(x) = 30$

20. Given $f(x) = x^2 - x$, find all values of t such that $f(2t - 3) = f(t + 1)$

Three men, Mike, Allen, and Ed and their wives Jane, Carol, and Sue, were playing tennis. In a doubles match, Mike partnered Allen's wife, and Carol's husband partnered Jane. Ed sat out with his sister. Who is married to whom?

3.2 GRAPHS OF FUNCTIONS

One way to picture a function is to draw an arrow diagram as we did in the preceding section. Another way is to draw its *graph*.

If the domain of a function f is the set A , then the graph of f is drawn by plotting all ordered pairs (x, y) such that $x \in A$ and $y = f(x)$. In other words, we plot all points in the plane such that $f(x \text{ component}) = y \text{ component}$.

The graph of f is the set of ordered pairs $\{(x, y) \mid y = f(x), x \in A\}$

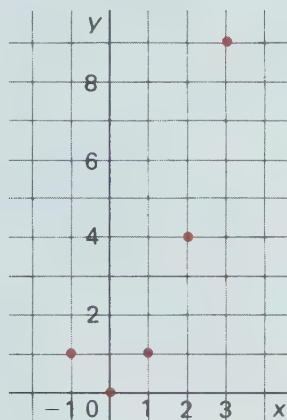
EXAMPLE 1. Draw the graph of f if $f(x) = x^2$ and the domain is $A = \{-1, 0, 1, 2, 3\}$.

Solution

$$f(-1) = 1, f(0) = 0, f(1) = 1, f(2) = 4, f(3) = 9$$

So the graph of f consists of the ordered pairs $(-1, 1) (0, 0) (1, 1) (2, 4) (3, 9)$.

Compare this picture with the picture in Example 5 of the preceding section. From either picture we can see at a glance that $f(-1) = 1$, $f(3) = 9$, and so on.

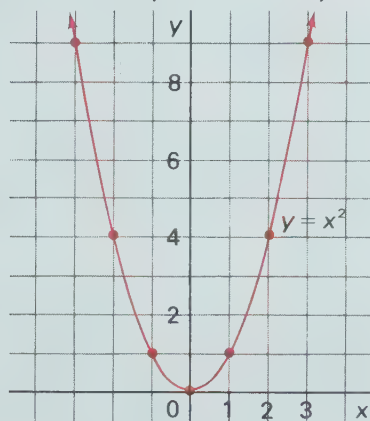


EXAMPLE 2. Draw the graph of f if $f(x) = x^2$ and the domain is R .

Solution

Here the domain consists of all real numbers and the graph $\{(x, y) \mid y = x^2, x \in R\}$ consists of infinitely many points. Obviously we cannot plot an infinite number of points individually, but we can select a few convenient values of x , calculate the corresponding values for y , and plot a few points on the graph. Then from these points we may be able to sketch the rest of the graph.

x	y
0	0
1	1
2	4
3	9
-1	1
-2	4
-3	9



EXAMPLE 3. From the graph given in Figure 3-2, state $f(2)$, $f(-1)$, $f(-3)$, $f(0)$, $f(1)$.

Solution

$$\begin{aligned} f(2) &= 1 \\ f(-1) &= -2 \\ f(-3) &= -1 \\ f(0) &= -1 \\ f(1) &= 0 \end{aligned}$$

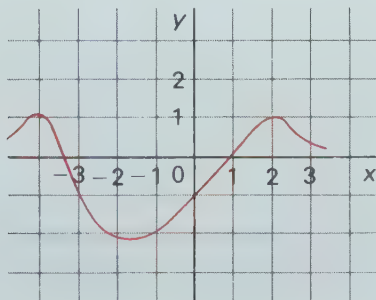


Figure 3-2

In Example 2, we could define the function f by the equation $y = x^2$, with the understanding that $y = f(x)$. Here are three ways of defining the same function f :

$$f(x) = x^2$$

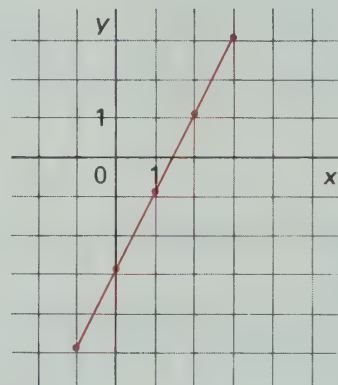
$$f: x \rightarrow x^2$$

$$y = x^2.$$

EXAMPLE 4. Sketch the graph of the function $y = 2x - 3$ if the domain is $\{x \in R \mid -1 \leq x \leq 3\}$.

Solution

x	y
-1	-5
0	-3
1	-1
2	1
3	3

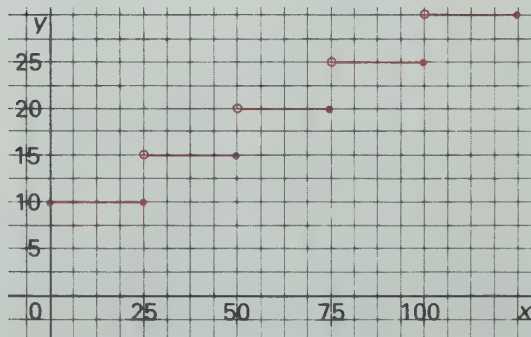


The graph is a segment of a straight line. In fact functions such as $y = -5x + 7$, $y = \frac{3}{2}x + 1$, or in general $y = mx + b$, are called *linear* functions because their graphs are straight lines. These are studied in more detail in the next chapter.

EXAMPLE 5. Postage rates for first-class letters in the country of Farland are stated as follows: "Ten cents for the first 25 g or less, five cents for each additional 25 g or fraction thereof." If x is the mass of a letter in grams, and $y = f(x)$ is the cost of postage, draw a graph of f .

Solution

○ means that the point is excluded.



$$f(x) = 10 \quad \text{if} \quad 0 \leq x \leq 25$$

$$f(x) = 15 \quad \text{if} \quad 25 < x \leq 50$$

$$f(x) = 20 \quad \text{if} \quad 50 < x \leq 75$$

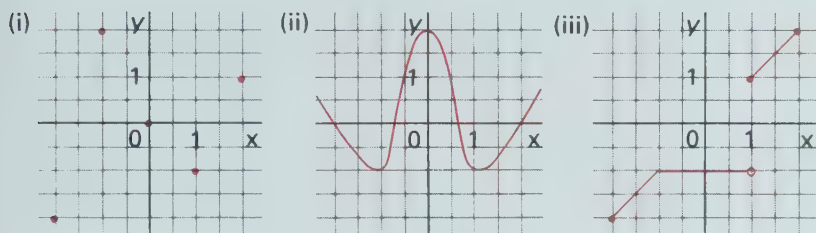
and so on.

The functions in Examples 2 and 3 have no breaks or jumps. Their graphs can be drawn without lifting the pencil from the paper, and so we call them *continuous* functions. However, the function in Example 5 is not continuous because it has breaks and jumps when $x = 25$, 50 , 75 , and so on.

EXERCISE 3-2

A 1. From each of the given graphs state

- (a) $f(0)$ (b) $f(-1)$ (c) $f(-2)$ (d) $f(1)$ (e) $f(2)$



B 2. Draw the graphs of the following linear functions with the given domains A .

- (a) $f(x) = 2x + 4$, $A = \{-2, 1, 0, 2\}$
 (b) $g(x) = 2x + 4$, $A = \mathbb{R}$
 (c) $f(x) = 7 - 2x$, $A = \{x \in \mathbb{R} \mid 1 \leq x \leq 5\}$
 (d) $h(x) = \frac{x+2}{3}$, $A = \mathbb{R}$
 (e) $f(t) = -t - 1$, $A = \mathbb{R}$
 (f) $y = 4x - \frac{1}{2}$, $A = \mathbb{R}$

3. Graph the following quadratic functions with domain $\mathbb{R}$.

- (a) $f(x) = x^2 + x + 1$
 (b) $g(x) = 1 - x^2$
 (c) $h(x) = (x - 2)^2$
 (d) $y = 3x^2 + 1$
 (e) $y = -2x^2 + x - 5$

4. Graph the following functions.

- (a) $f(x) = x^3 - 3x^2 + 1$, $x \in \{0, 1, 2, 3, 4, 5\}$
 (b) $g(x) = x + \sqrt{x - 2}$, $x \in \{2, 3, 6, 11\}$
 (c) $h(x) = \frac{1}{x}$, $x \in \mathbb{R}, x \neq 0$
 (d) $f: x \rightarrow \sqrt{x}$, $x \geq 0$
 (e) $y = 2 - \frac{1}{x}$, $1 \leq x \leq 7$
 (f) $y = |x|$, $x \in \mathbb{R}$
 (g) $y = |x - 2|$, $x \in \mathbb{R}$

5. Sketch a graph of the population of Canada as a function of time by using the following census figures.

Year	1901	1911	1921	1931	1941	1951	1961	1971	1981
Population (in millions)	5.4	7.2	8.8	10.4	11.5	14.0	18.2	21.6	

What is the only common word in the English language that contains three consecutive pairs of identical letters.

6. The following table gives the number n of inmates admitted to penitentiaries in Canada in 1968–69 according to the grade reached in school.

Grade	less than 8	8	9	10	11	12	13 and above
n	1273	886	550	473	199	160	85

Use this table to graph n as a function of amount of education.



7. Graph the following functions.

(a) $f(x) = 1.09x^3 + 0.87x^2 - 4.54x - 5.23$, $x \in R$

(b) $g(x) = \frac{x}{\sqrt{x^2 + 7}}$, $x \in R$, $x \geq 0$

(c) $h(x) = \sqrt[4]{x^4 + 2x^2 + 10}$, $x \in R$.

8. Let $[x]$ be the largest integer which is less than or equal to x , $x \in R$. For example, $[2.6] = 2$, $[2] = 2$, $[\pi] = 3$, $[-2.6] = -3$.

(a) Draw the graph of the function f given by $f(x) = [x]$, $-5 \leq x \leq 5$.

(b) Is f continuous?

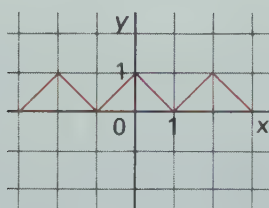
3.3 NEW FUNCTIONS FROM OLD FUNCTIONS

Suppose we know what the graph of a given function $y = f(x)$ looks like. Can we then immediately sketch the graphs of such new functions as $y = 2f(x)$, $y = f(x) + 3$, and $y = 2f(x) + 3$? In the following investigation you will discover some rules for graphing these functions.

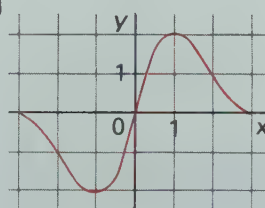
INVESTIGATION 3-3

We shall work with the following three functions.

(i)



(ii)



(iii)

$$y = x^2$$

$$(-3 \leq x \leq 3)$$

1. (a) For each of the three functions $y = f(x)$ above complete the following table in your notebook.

x	$f(x)$	$f(x)+1$	$f(x)-2$	$f(x)+3$
-3				
-2				
-1				
0				
1				
2				
3				

(b) For each of the 3 given functions, use the table in (a) to graph the 4 functions $y = f(x)$, $y = f(x)+1$, $y = f(x)-2$, $y = f(x)+3$. (Use the same axes for all 4 functions.)

(c) State a rule for graphing the function $y = f(x)+d$, where d is a constant, given the graph of $y = f(x)$.

2. (a) For each of the 3 functions (i), (ii), (iii) above, complete the following table in your notebook.

x	$f(x)$	$2f(x)$	$4f(x)$	$-3f(x)$
-3				
-2				
-1				
0				
1				
2				
3				

(b) For each of the 3 given functions, use the table in (a) to graph the 4 functions $y = f(x)$, $y = 2f(x)$, $y = 4f(x)$, $y = -3f(x)$ using the same axes.

(c) State a rule for graphing the function $y = cf(x)$, where c is a constant, given the graph of $y = f(x)$.

3. (a) For each of the 3 functions (i), (ii), (iii) above, complete the following table in your notebook.

x	$f(x)$	$2f(x)+1$	$4f(x)-2$	$-3f(x)+3$
-3				
-2				
-1				
0				
1				
2				
3				

(b) For each of the 3 given functions, use the table in (a) to graph the functions $y = f(x)$, $y = 2f(x) + 1$, $y = 4f(x) - 2$, $y = -3f(x) + 3$.

(c) State a rule for graphing the function $y = cf(x) + d$, where c and d are constants, given the graph of $y = f(x)$.

EXERCISE 3-3

1. The graphs of 3 functions $y = f(x)$ are given below. Copy these graphs and use the rules discovered in Investigation 3-3 to graph the following functions.

(a) $y = f(x) + 2$

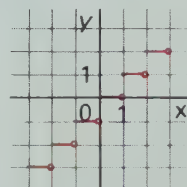
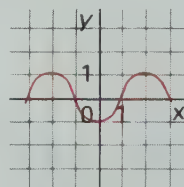
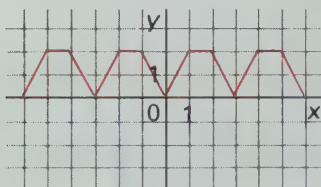
(b) $y = f(x) - 4$

(c) $y = 3f(x)$

(d) $y = 3f(x) + 1$

(e) $y = -f(x)$

(f) $y = 2 - f(x)$



2. In each part of this question, graph the first function and then use the rules discovered in Investigation 3-3 to graph the remaining functions.

(a) $y = x^2$, $y = -x^2$, $y = 3x^2$, $y = 3x^2 + 1$ ($x \in \mathbb{R}$)

(b) $y = x$, $y = 2x$, $y = 2x + 4$ ($x \in \mathbb{R}$)

(c) $y = \sqrt{x}$, $y = -2\sqrt{x}$, $y = 2\sqrt{x} - 1$ ($x \in \mathbb{R}$, $x \geq 0$)

(d) $y = \frac{1}{x}$, $y = \frac{3}{x}$, $y = 1 + \frac{3}{x}$ ($x \in \mathbb{R}$, $x \neq 0$)

(e) $y = x^3$, $y = x^3 + 5$, $y = -2x^3 + 5$ ($-2 \leq x \leq 2$)

(f) $y = |x|$, $y = -|x|$, $y = 3|x| + 2$ ($x \in \mathbb{R}$)

3.4 RELATIONS

The statements

"Allan is married to Sharon,"

"George is the brother of Felix,"

"Juliet loves Romeo,"

"2 is less than 5,"

"7 is a factor of 21,"

all involve the idea of a relationship. In each case the relation is given by the words in italics. A relation connects the members of two sets.

Consider the relation "drinks" which connects the sets $A = \{\text{Henry, Sue}\}$ and $B = \{\text{cola, root beer, orange}\}$. Suppose that Henry drinks only cola and orange, and Sue drinks only root beer and orange. Then the relation "drinks" can be described by the set of ordered pairs

$\{(\text{Henry, cola}), (\text{Henry, orange}), (\text{Sue, root beer}), (\text{Sue, orange})\}$.

Notice that this is a subset of the Cartesian product $A \times B$.

A relation is a set of ordered pairs. The *domain* of the relation is the set of all first components. The *range* of the relation is the set of all second components.

In the above example the domain is A and the range is B .

EXAMPLE 1. $A = B = \{\text{Mary, Joan, Bill, Brian, Peter}\}$. Mary loves Bill, Bill loves Joan, Joan loves Brian, Brian loves Mary, and Peter loves both Mary and Joan. Express the relation "loves" as a set of ordered pairs. What are the domain and the range?

Solution

$\{(\text{Mary, Bill}), (\text{Bill, Joan}), (\text{Joan, Brian}), (\text{Brian, Mary}), (\text{Peter, Mary}), (\text{Peter, Joan})\}$

Domain = A

Range = $\{\text{Mary, Joan, Bill, Brian}\}$

EXAMPLE 2. List the relation "is less than" on the set $A = \{10, 9, 3, 7, 4\}$ as a set of ordered pairs. State the domain and range.

Solution

$\{(3, 4), (3, 7), (3, 9), (3, 10), (4, 7), (4, 9), (4, 10), (7, 9), (7, 10), (9, 10)\}$

Domain = $\{3, 4, 7, 9\}$

Range = $\{4, 7, 9, 10\}$

We can picture a relation by drawing its graph, i.e., by plotting the ordered pairs which make up the relation. For instance Figure 3-3 shows the graph of the relation in Example 2.

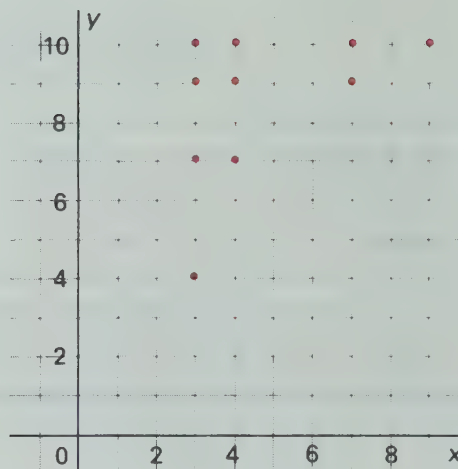
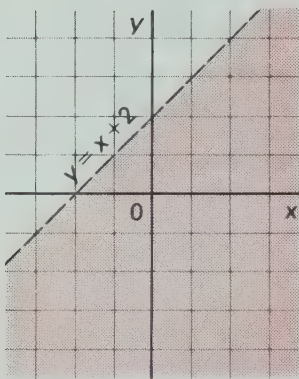


Figure 3-3

EXAMPLE 3. Graph the relation $\{(x, y) \in R \times R \mid y < x + 2\}$. (Unless otherwise stated, all the relations that we study will be subsets of $R \times R$. Thus we could refer to this relation simply as “ $y < x + 2$.”)



Solution

We already know how to graph the line $y = x + 2$. We want to plot all ordered pairs (x, y) such that $y < x + 2$. If $y = x + 2$ for points *on* the line, then $y < x + 2$ for points *below* the line because the value of y decreases as we go below the line. Let us check this: $(0, 0)$ is obviously a point below the line and $0 < 0 + 2$. Thus the graph of $y < x + 2$ is the shaded region below the line $y = x + 2$.

Functions and relations both have graphs. What is the connection between functions and relations? We can consider any function f to be a special kind of relation because when we graph f we are plotting the set of ordered pairs $(x, f(x))$. It is a special set of ordered pairs since for each first component x there is only one second component, namely $f(x)$.

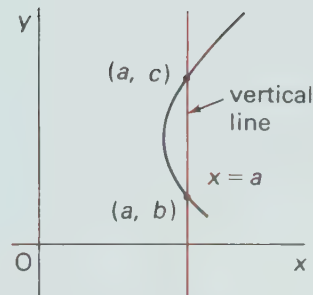
Not every relation is a function because in some relations a given first component can have more than one second component. The relation in Example 1 is not a function because Peter loves both Mary and Joan. The relation in Example 2 is not a function because, for instance, the ordered pairs $(3, 4)$ and $(3, 7)$ both occur. Likewise the relation in Example 3 is not a function.

In order to tell if a given graph represents a function or a relation we use the following test.

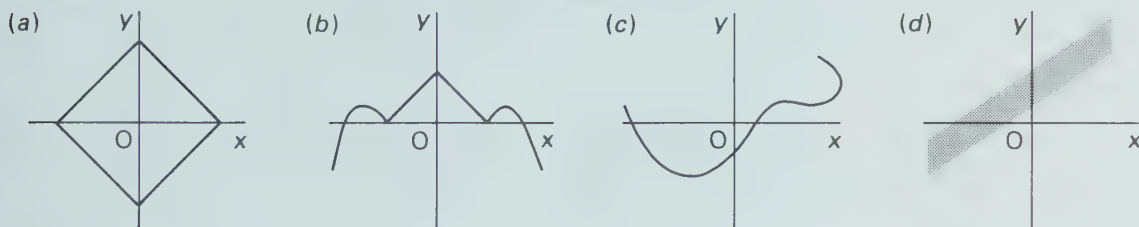
Vertical Line Test

If it is possible to draw a vertical line which intersects a graph in more than one point, then the graph does *not* represent a function.

This test works because if $x = a$ intersects the graph in two points (a, b) and (a, c) , then we have two ordered pairs with the same first component.

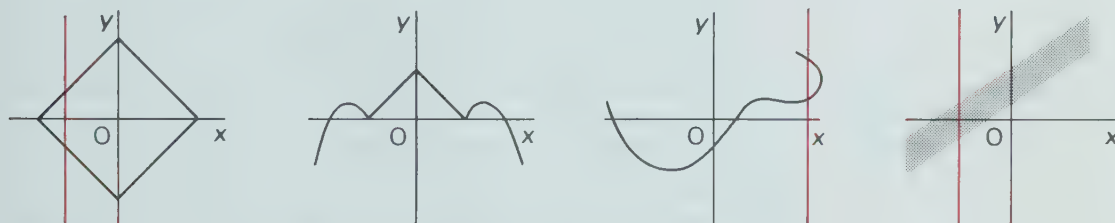


EXAMPLE 4. Which of the following graphs represent functions?



Solution

In (b) every vertical line intersects the graph only once. But in (a), (c), (d) the indicated vertical lines intersect the graphs more than once. Thus only (b) represents a function.



EXAMPLE 5. Graph the relation $x^2 + y^2 = 25$.

Solution

Remember that $x^2 + y^2 = 25$ is short for

$$\{(x, y) \mid x^2 + y^2 = 25, (x, y) \in \mathbb{R} \times \mathbb{R}\}.$$

The graph of this relation is the circle with radius 5 and centre $(0, 0)$. It is not a function. Note that if we try to solve the equation $x^2 + y^2 = 25$ for y we get

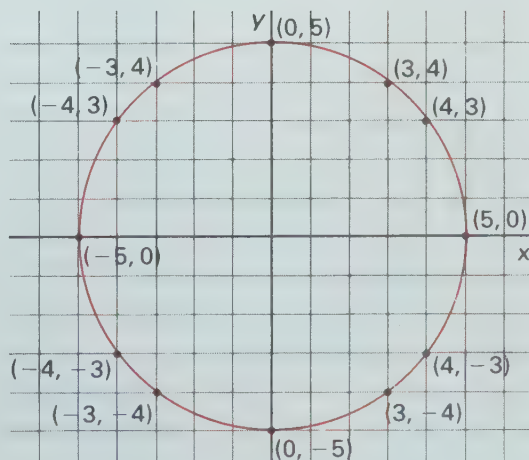
$$y^2 = 25 - x^2,$$

$$y = \pm\sqrt{25 - x^2}.$$

This represents *two* functions:

The function $y = \sqrt{25 - x^2}$ represents the top half of the circle.

The function $y = -\sqrt{25 - x^2}$ represents the bottom half of the circle.

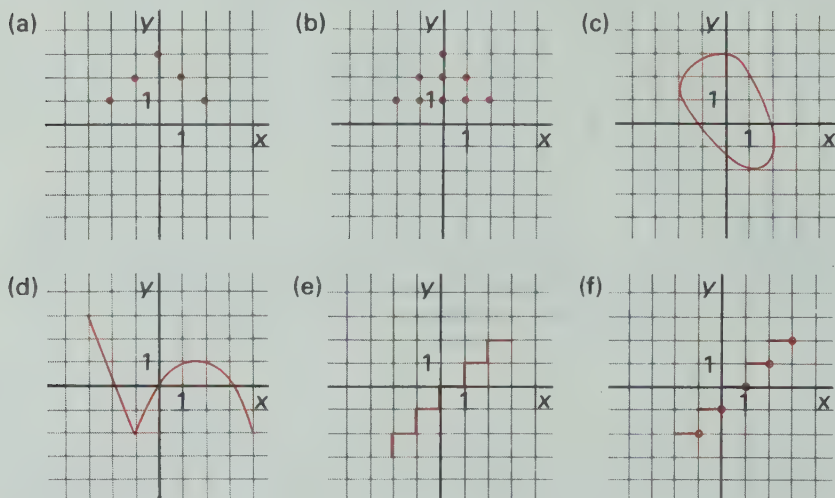


EXERCISE 3-4

- A** 1. State the domain and range of the following relations. Which of them are functions?

- (a) $\{(\text{elephant, peanut}), (\text{monkey, banana}), (\text{elephant, banana})\}$
 (b) $\{(3, 17), (4, 0), (5, 1), (6, 0)\}$
 (c) $\{(\pi, 1), (\pi, 2), (\pi, 3)\}$
 (d) $\{(-2, 0), (0, 0), (2, 0), (4, -3)\}$
 (e) $\{(\text{Sandra, 82}), (\text{Pat, 84}), (\text{Jane, 86}), (\text{Pat, 82})\}$

2. State the domain and range of the relations whose graphs are given. Which of them represent functions?



- B** 3. The five children in the Smith family are Joe, Jane, Judy, Jim, and John, who were born in that order.

- (a) Express the relation "is younger than" as a set of ordered pairs. State the domain and range of this relation.
 (b) Express the relation "is a sister of" as a set of ordered pairs. State the domain and range of this relation.

4. If $A = \{2, 4, 6, 8, 10\}$ and $B = \{3, 5, 7, 9\}$, express the relation "is greater than" as a set of ordered pairs in $A \times B$.

- (b) State the domain and range and graph this relation.

5. (a) If $A = \{2, 3, 4, 5, 6, 7, 8, 9\}$, express the relation "is a factor of" as a set of ordered pairs in $A \times A$.

- (b) Graph this relation.

6. Graph the following relations.

- (a) $\{(x, y) \in R \times R \mid y \geq x\}$ (b) $\{(x, y) \in R \times R \mid y > x + 1\}$
 (c) $\{(x, y) \in R \times R \mid y \geq 2\}$ (d) $\{(x, y) \in R \times R \mid x < 5\}$
 (e) $\{(x, y) \in R \times R \mid y < 2x - 3\}$
 (f) $\{(x, y) \in R \times R \mid y \leq 3 - 2x\}$
 (g) $\{(x, y) \in R \times R \mid x + y < 1\}$
 (h) $\{(x, y) \in R \times R \mid 1 < x + y < 2\}$

7. Graph the following relations in $R \times R$. Which are functions?

- (a) $x = y^2$ (b) $y = x^2$ (c) $y > x^2$

A, B and C all play each other once in hockey. A partial table of results is

	G	W	L	T	Goals For	Goals Against
A	2	1			4	
B	2				3	0
C	2		1	2		

Find the score in each game.

- (d) $y = |x|$ (e) $y \leq |x|$ (f) $|y| = x$
 (g) $y = 1 - |x|$ (h) $x = 1 - |y|$ (i) $x^2 + y^2 = 1$
 (j) $x + y = 1$ (k) $|x| + |y| = 1$ (l) $|x| = 1$

8. (a) Graph the following relations in $R \times R$ using the same axes and a large scale.

- (i) $x^2 + y^2 = 1$ (ii) $x^4 + y^4 = 1$
 (iii) $x^6 + y^6 = 1$ (iv) $x^8 + y^8 = 1$

(b) What shape does the graph of $x^{2n} + y^{2n} = 1$ approach as n becomes large?

3.5 INVERSES OF RELATIONS AND FUNCTIONS

Consider the relation "is a factor of" on $A \times B$, where $A = \{2, 3, 4, 5\}$, and $B = \{4, 5, 6\}$. It consists of the set of ordered pairs

$$F = \{(2, 4), (2, 6), (3, 6), (4, 4), (5, 5)\}$$

The domain is A and the range is B .

Now consider the relation "is a multiple of" on $B \times A$. It consists of the set of ordered pairs

$$M = \{(4, 2), (6, 2), (6, 3), (4, 4), (5, 5)\}$$

Here the domain is B and the range is A .

Notice that each ordered pair in M can be obtained by changing the order of the components in F . We say that F is the *inverse* of M and we write $M = F^{-1}$. The domain of M is equal to the range of F , and the range of M is equal to the domain of F .

$$2^{-1} = \frac{1}{2} \text{ but } F^{-1} \text{ does not mean } \frac{1}{F}.$$

We find the inverse F^{-1} of any relation F by interchanging the components (i.e., by reversing the order of the components).

If $(a, b) \in F$, then $(b, a) \in F^{-1}$.

EXAMPLE 1. If $F = \{(2, 1), (3, 1), (4, 1), (3, 2), (4, 2)\}$ graph both F and F^{-1} and state their domains and ranges.

Solution

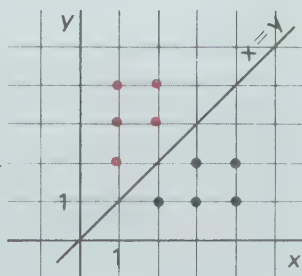
$$F^{-1} = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4)\}$$

Domain of F : $\{2, 3, 4\}$ Range of F : $\{1, 2\}$

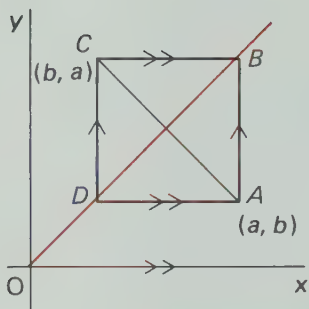
Domain of F^{-1} : $\{1, 2\}$ Range of F^{-1} : $\{2, 3, 4\}$

The graph of F is shown in black.

The graph of F^{-1} is shown in red.



You can see that in Example 1 the graph of F^{-1} could be obtained by reflecting the graph of F in the line $y = x$. If you placed a mirror along the line $y = x$, then the graph of F^{-1} would be the reflection, or mirror image, of the graph of F . Is this always true?

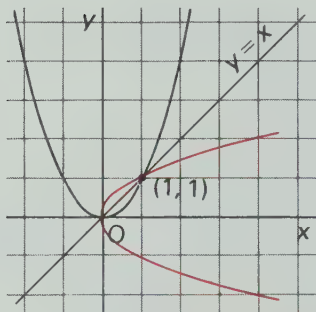


If F is any relation and $A(a, b) \in F$, then $C(b, a) \in F^{-1}$. It follows that $ABCD$ is a square. (Why?) Therefore CA is the perpendicular bisector of BD . (Why?)

The graph of any inverse relation F^{-1} is the reflection of the graph of F in the line $y = x$.

EXAMPLE 2. Any function f can be considered as a relation and so it has an inverse relation f^{-1} . If $f(x) = x^2$, graph f^{-1} .

Solution



First we graph f as in section 3.2 (shown in black). Then we reflect in the line $y = x$ to get the graph of f^{-1} (shown in red).

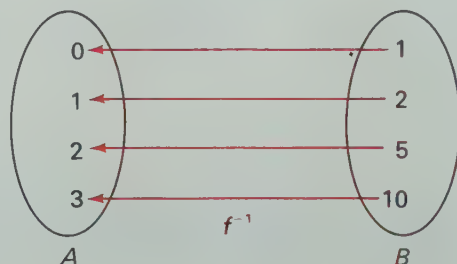
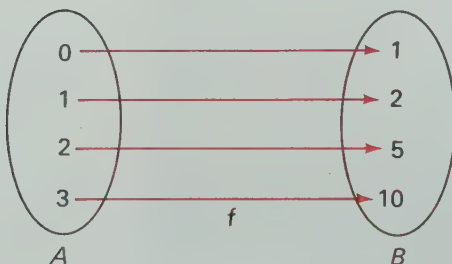
Note that the inverse relation f^{-1} is *not* a function because it fails the vertical line test.

The inverse f^{-1} of a function f is not necessarily a function.

EXAMPLE 3. If $f(x) = x^2 + 1$ and the domain of f is $A = \{0, 1, 2, 3\}$, find f^{-1} .

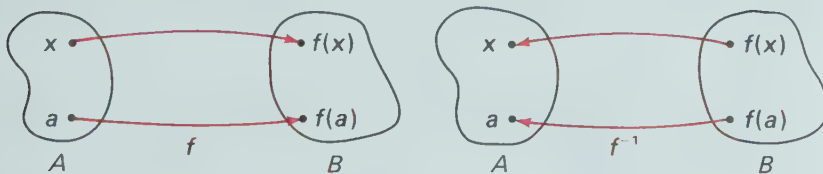
Solution

$f(0) = 1$, $f(1) = 2$, $f(2) = 5$, $f(3) = 10$. The graph of f is $\{(0, 1), (1, 2), (2, 5), (3, 10)\}$. So the graph of f^{-1} is $\{(1, 0), (2, 1), (5, 2), (10, 3)\}$. This time f^{-1} is actually a function and we can write $f^{-1}(1) = 0$, $f^{-1}(2) = 1$, $f^{-1}(5) = 2$, $f^{-1}(10) = 3$.



In terms of arrow diagrams, we simply reverse the direction of the arrows to get f^{-1} from f .

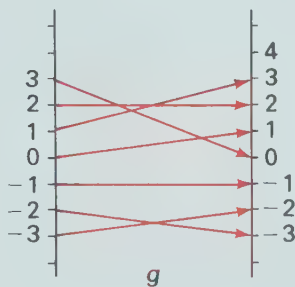
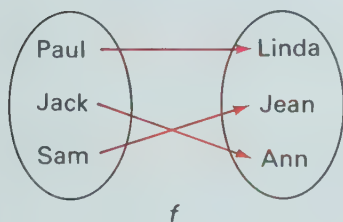
This is true in general, if f and f^{-1} are both functions.



Since f maps x into $f(x)$, the inverse function f^{-1} maps $f(x)$ into x ; f^{-1} reverses the effect of f .

EXERCISE 3-5

- A** 1. Give names to the inverses of the following relations.
- | | |
|------------------------|-----------------------|
| (a) "loves" | (b) "owns" |
| (c) "is a multiple of" | (d) "is a brother of" |
| (e) "eats" | (f) "is younger than" |
2. (a) State the ordered pairs of the inverses of the following relations.
 (b) State the domain and range of each inverse.
- $\{(0, 2), (0, 3), (4, 6), (8, 2)\}$
 - $\{(1, -2), (2, -1), (3, 0), (1, 1)\}$
 - $\{(-1, 1), (-2, 2), (-3, 3)\}$
 - $\{(3, \frac{1}{3}), (4, \frac{1}{4}), (3, 3), (4, 4)\}$
3. Arrow diagrams are given for two functions f and g .



Are f^{-1} and g^{-1} both functions?

State

- | | | | |
|----------------------|----------------------------|--------------------------|---------------------|
| (a) $f(\text{Jack})$ | (b) $f^{-1}(\text{Linda})$ | (c) $f^{-1}(\text{Ann})$ | (d) $f(\text{Sam})$ |
| (e) $g(-3)$ | (f) $g(-1)$ | (g) $g^{-1}(0)$ | (h) $g^{-1}(-3)$ |
| (i) $g^{-1}(3)$ | (j) $g(1)$ | (k) $g(g^{-1}(3))$ | (l) $g^{-1}(g(0))$ |

- B** 4. In each part of this question graph the relation and its inverse using the same axes but different colours.
- $\{(-4, 1), (-3, 1), (-2, 1), (-2, 2), (-1, 2), (-1, 3), (-1, 4)\}$.
 - $\{(-2, 0), (-1, 1), (0, -1), (0, 2), (1, 0)\}$
 - $\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x = y^2\}$
 - $\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = 2x\}$
 - $\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x + y = 0\}$
 - $\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y > 1 - x\}$
 - $\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 = 1\}$

5. Draw arrow diagrams for each of the following functions f with domain A . State whether or not f^{-1} is a function. If it is, draw an arrow diagram for f^{-1} .

(a) $f(x) = x^4 - 4$, $A = \{0, 1, 2, 3, 4\}$

(b) $f(x) = x^4 - 4$, $A = \{-2, -1, 0, 1, 2\}$

(c) $f(x) = 17 - x^3$, $A = \{-2, -1, 0, 1, 2\}$

(d) $f(x)$ = the largest prime factor of x , $A = \{2, 3, 4, 5, 6, 7, 8, 9\}$

6. Graph the following functions f and their inverses f^{-1} . In which cases is f^{-1} a function?

(a) $f(x) = x + 1$, $x \in \mathbb{R}$

(b) $f(x) = |x|$, $x \in \mathbb{R}$

(c) $f(x) = 2$, $x \in \mathbb{R}$

(d) $f(x) = \frac{1}{x}$, $x \neq 0$

(e) $f(x) = x^2 + 1$, $x \in \mathbb{R}$

(f) $f(x) = \sqrt{x}$, $x \geq 0$

C 7. Find a formula for the inverse of each of the following functions.

(a) $f(x) = x + 1$, $x \in \mathbb{R}$

(b) $g(x) = x^2$, $x > 0$

3.6 APPLICATIONS

Nature herself exhibits to us measurable and observable quantities in definite mathematical dependence; the conception of a function is suggested by all the processes of nature where we observe natural phenomena varying according to distance or time.

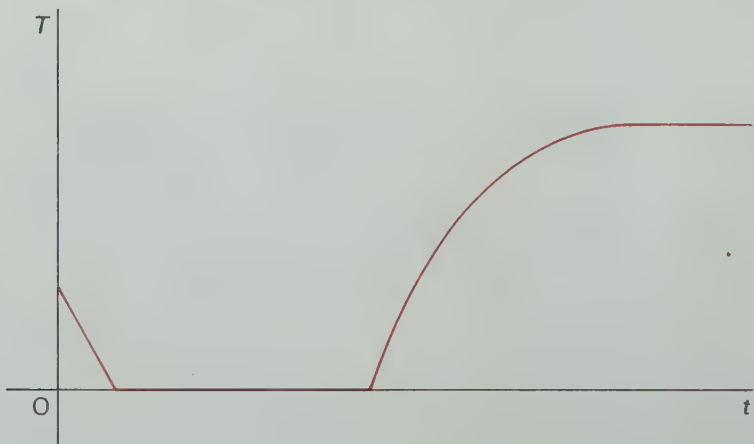
T. J. Merz

In this section we shall graph some of the functions which occur in science or in everyday situations.

EXAMPLE 1. You put some ice cubes in a glass, fill the glass with cold water and then let the glass sit on a table. The temperature T of the water (in degrees Celsius) is a function of the time t (in seconds) that the glass has been sitting on the table. Sketch a graph of T as a function of t .

Solution

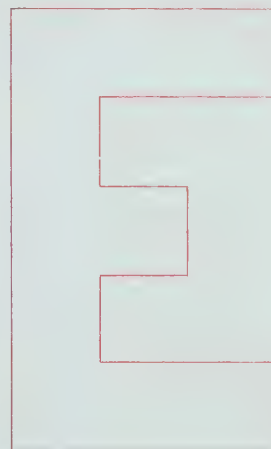
The cold water gradually becomes colder as the ice melts, and so T decreases until $T = 0$ is reached. After a while all the ice melts and T increases until it levels off at room temperature.



EXERCISE 3-6

- B** 1. The price P (in dollars) of a new house on a given lot depends on the area A (in square metres) of floor space in the house. Sketch a graph of P as a function of A .
2. You fill an electric kettle with cold water and plug in the kettle. After the kettle boils, you then pull the plug out. The temperature T of the water (in degrees Celsius) is a function of the time t (in seconds) that has passed since the kettle was plugged in. Sketch a graph of T as a function of t and state the domain and range of this function.
3. The number n of swim suits sold in a store depends on the time of year. Let t be the time measured in weeks (from the beginning of the year). Sketch a graph of n as a function of t .
4. Your height h (in centimetres) depends on your age A (in years). Sketch a graph of h as a function of A .
5. The price P (in cents) of a bottle of ketchup depends on the volume V (in millilitres) of ketchup in the bottle. Sketch a graph of P as a function of V .
6. A train goes from Toronto to Windsor and stops for 2 minutes in Dundas, 5 minutes in Brantford, 2 minutes in Woodstock, 10 minutes in London, and 5 minutes in Chatham. The distance d (in kilometres) travelled by the train depends on the time t (in minutes) since the train left Toronto. Sketch a graph of d as a function of t .
7. A biologist observes that the number of bacteria in a laboratory dish doubles every day. If N is the number of bacteria (in millions) and $N = 1$ after the first day, sketch a graph of N as a function of time t (in days). Can you give a formula for N as a function of t ?
8. A farmer finds that the value V of his crop (in dollars) depends on the amount x of fertilizer (in cubic metres) that he uses. Sketch a graph of V as a function of x .
9. On a sunny day the length L (in metres) of the shadow of a telephone pole depends on the time of day t (in hours). Sketch a graph of L as a function of t . State the domain and range of this function.
10. Nancy bought a 1 kg bag of sugar and used it only for her morning cup of coffee in which she put 10 g of sugar. If S is the amount of sugar remaining (in grams) after t days, sketch a graph of S as a function of t . State the domain and range of this function.
11. When you blow up a round balloon, the diameter D (in centimetres) of the balloon depends on the number N of breaths that you have blown into it. Sketch a graph of D as a function of N . Can you give a formula for D as a function of N ?
- C** 12. The mass m (in kilograms) of a given kind of fish depends on its length l (in centimetres). Sketch a graph of m as a function of l . Can you give a formula for m as a function of l ?

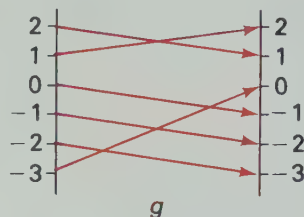
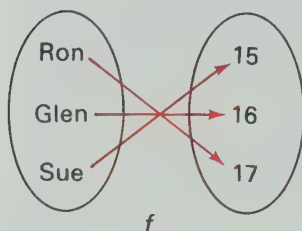
Make four cuts in the letter E and arrange the five pieces to form a square.



REVIEW EXERCISE

- A** 1. If $f(x) = 1 - 2x$ and $g(x) = x^2 + 3$, state
- (a) $f(2)$ (b) $g(2)$ (c) $f(-5)$ (d) $g(-9)$ (e) $f(0.3)$
 (f) $f(\frac{1}{2})$ (g) $g(\frac{1}{2})$ (h) $g(7)$ (i) $f(a)$ (j) $g(b)$

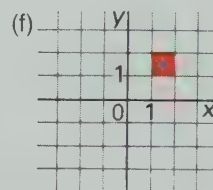
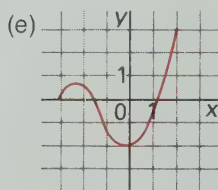
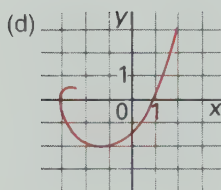
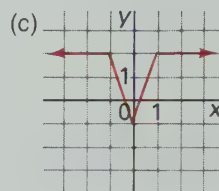
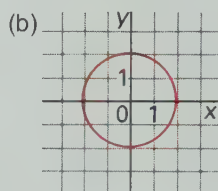
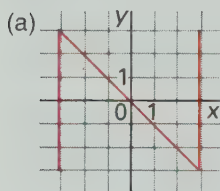
2. Arrow diagrams are given for two functions f and g .



State

- (a) $f(\text{Sue})$ (b) $f^{-1}(16)$ (c) $f^{-1}(15)$ (d) $f(\text{Ron})$
 (e) $g(-2)$ (f) $g(2)$ (g) $g^{-1}(0)$ (h) $g^{-1}(1)$
 (i) $f(\text{Glen})$ (j) $g(0)$ (k) $g^{-1}(-2)$ (l) $f(f^{-1}(17))$

3. State the domain and range of the relations whose graphs are given. Which of them represent functions?



- B** 4. If $f(x) = x^3 - 2x^2 + 3x - 4$ and $g(x) = \frac{1-x^2}{1+x^2}$, find

- (a) $f(2)$ (b) $g(\frac{1}{2})$ (c) $g(17)$ (d) $f(-5)$
 (e) $f(12)$ (f) $g(0.1)$ (g) $f(\pi)$ (h) $g(f(1))$

5. If $f(x) = 2x^2 - 3x + 1$, find expressions for the following.

- (a) $f(a)$ (b) $f(a-1)$ (c) $f(a-b)$
 (d) $f(3a)$ (e) $f(-a)$ (f) $f(-x)$

6. Draw the graphs of the following functions with the given domains A .

- (a) $f(x) = 1 - 6x$ $A = R$
 (b) $g(x) = 1 - 6x + x^2$ $A = \{0, 1, 2, 3, 4, 5, 6\}$
 (c) $h(x) = (x+3)^2$ $A = \{x \in R \mid -6 \leq x \leq 0\}$

- (d) $f(t) = |2t|$ $A = R$
 (e) $f: x \rightarrow \sqrt{x-1}$ $A = \{x \in R \mid x \geq 1\}$
 (f) $g: x \rightarrow 0.5x + 1.5$ $A = R$
 (g) $y = \frac{1}{x-1}$ $A = \{x \in R \mid x > 1\}$

7. In each part of this question, graph the first function and then use the rules discovered in Investigation 3-3 to graph the remaining functions.

- (a) $y = x^2, y = 2x^2, y = 2x^2 - 2$ ($x \in R$)
 (b) $y = \frac{1}{x}, y = -\frac{1}{x}, y = 1 - \frac{1}{x}$ ($x \in R, x \neq 0$)
 (c) $y = \frac{1}{x^2}, y = -\frac{1}{x^2}, y = 1 - \frac{1}{x^2}$ ($x \in R, x \neq 0$)

8. (a) If $A = \{2, 4, 6, 8, 10, 12\}$, express the relation "is a multiple of" as a set of ordered pairs in $A \times A$.

(b) Graph the relation.

9. Graph the following relations in $R \times R$.

- (a) $y \geq 1 - x$ (b) $x \geq -2$ (c) $x = -y^2$
 (d) $x < -y^2$ (e) $0 < y < 1$ (f) $x^2 + y^2 = 4$
 (g) $x^2 + y^2 < 4$ (h) $x^2 + y^2 > 4$ (i) $x = |y| - 4$

10. Graph the following functions f and their inverses f^{-1} . In which cases is f^{-1} a function?

- (a) $f(x) = 2x - 3$ ($x \in R$) (b) $f(x) = \sqrt{1 - x^2}$ ($-1 \leq x \leq 1$)
 (c) $f(x) = \sqrt{x - 1}$ ($x \geq 1$) (d) $f(x) = 0$ ($x \in R$)

11. When you turn on a hot water faucet the temperature T of the water (in degrees Celsius) depends on the time t (in seconds) since the tap was turned on. Sketch a graph of T as a function of t .

- C** 12. If $f(x) = x(x-1)$, find all values of x such that
 (a) $f(x) = 0$ (b) $f(x) = 6$ (c) $f(x) = 20$



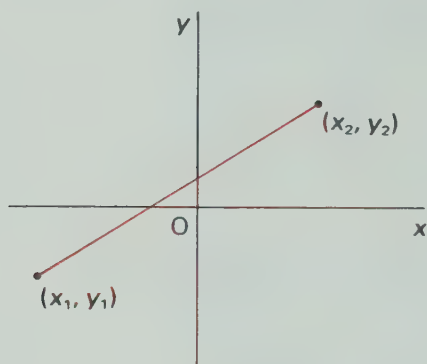
13. If $f(x) = \frac{x^3 + x^2 + 17x - 1}{x^4 + 1}$, find

- (a) $f(2)$ (b) $f(-0.1)$ (c) $f(123)$ (d) $f(1.736)$.

REVIEW AND PREVIEW TO CHAPTER 4

EXERCISE 1

SLOPE



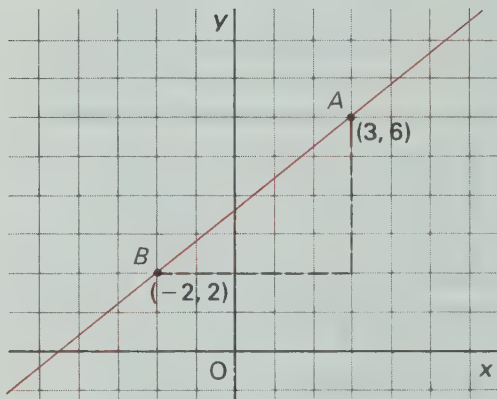
$$\begin{aligned}\text{Slope} = m &= \frac{\Delta y}{\Delta x} \\ &= \frac{y_2 - y_1}{x_2 - x_1}\end{aligned}$$

Horizontal lines have a slope of 0

Vertical lines have no slope.

Lines that rise from left to right have a positive slope.

Lines that fall from left to right have a negative slope.



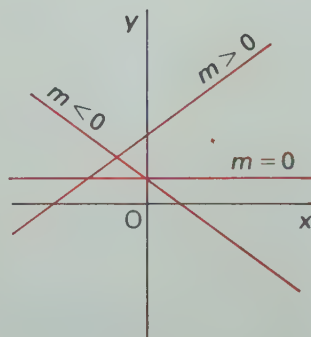
$$\begin{aligned}m_{AB} &= \frac{6 - 2}{3 - (-2)} \\ &= \frac{4}{5}\end{aligned}$$

1. Determine the slope of the line containing the given points.

- | | |
|------------------------|------------------------|
| (a) (3, 2), (10, 14) | (b) (1, 3), (4, 7) |
| (c) (3, 3), (-5, 2) | (d) (-1, 0), (-3, 4) |
| (e) (-6, -7), (-1, -4) | (f) (-4, 6), (8, -3) |
| (g) (-6, 10), (-11, 7) | (h) (5, -9), (-4, -11) |
| (i) (5, -12), (0, -6) | (j) (7, -3), (7, 6) |
| (k) (-3, -2), (9, -2) | (l) (5, -7), (16, 3) |
| (m) (-4, -3), (-4, 7) | (n) (2, -1), (-5, 6) |
| (o) (6, 0), (-8, 12) | (p) (-9, -13), (1, 1) |
| (q) (-2, -4), (-5, 6) | (r) (a, b), (c, d) |

2. Graph the line containing the given point and having the given slope.

- | | |
|---------------------------------|--------------------------------|
| (a) (3, 2), $m = \frac{3}{4}$ | (b) (1, 2), $m = \frac{2}{3}$ |
| (c) (-1, 3), $m = \frac{2}{5}$ | (d) (3, 0), $m = -\frac{1}{2}$ |
| (e) (0, -4), $m = -\frac{2}{3}$ | (f) (-1, -2), $m = 3$ |
| (g) (-5, 4), $m = -2$ | (h) (-2, 5), $m = 0$ |
| (i) (2, -4), $m = 1$ | (j) (3, 6), no slope |



3. Find 2 other points on the line containing the given point and having the given slope.

- (a) $(3, 2)$, $m = \frac{2}{3}$ (b) $(-2, 2)$, $m = \frac{1}{2}$
 (c) $(-1, -1)$, $m = -\frac{3}{4}$ (d) $(2, -3)$, $m = 0$
 (e) $(-2, 3)$, no slope (f) $(4, 0)$, $m = 2$

4. Determine the value of k so that the line containing the given points has the given slope.

- (a) $(3, 2)$, $(5, k)$, $m = 2$ (b) $(-1, k)$, $(3, 15)$, $m = 3$
 (c) $(-2, 4)$, $(k, -2)$, $m = \frac{1}{2}$ (d) $(k, 5)$, $(6, -1)$, $m = -3$
 (e) $(-3, 4)$, $(2, k)$, $m = 0$ (f) $(k, 7)$, $(-3, -2)$, no slope

5. Show that the points $A(4, 5)$, $B(2, 1)$ and $C(-2, -7)$ are collinear by showing that $m_{AB} = m_{BC}$.

6. Use slopes to determine which of the following sets of points are collinear.

- (a) $(1, 0)$, $(3, 5)$, $(-1, -5)$ (b) $(-3, 4)$, $(-1, 2)$, $(4, -4)$
 (c) $(1, -5)$, $(-3, -1)$, $(6, -8)$ (d) $(3, 3)$, $(1, -1)$, $(-4, -11)$
 (e) $(-1, 9)$, $(2, -6)$, $(3, -11)$ (f) $(-2, 2)$, $(4, 1)$, $(-6, 4)$

EXERCISE 2

1. Determine the missing coordinate so that the given point lies on the given line.

- (a) $x + y = 7$; $(3, \square)$, $(\square, 6)$, $(-3, \square)$, $(\square, -2)$
 (b) $x + y = -2$; $(5, \square)$, $(\square, 3)$, $(-3, \square)$, $(\square, -6)$
 (c) $x - y = 3$; $(4, \square)$, $(\square, 6)$, $(-4, \square)$, $(\square, -1)$
 (d) $x - y = -4$; $(6, \square)$, $(\square, 5)$, $(-3, \square)$, $(\square, -5)$
 (e) $y - x = 4$; $(3, \square)$, $(\square, 3)$, $(-2, \square)$, $(\square, -3)$
 (f) $y = 2x + 3$; $(4, \square)$, $(\square, 5)$, $(-2, \square)$, $(\square, -7)$
 (g) $y = -x - 2$; $(5, \square)$, $(\square, 10)$, $(-8, \square)$, $(\square, -12)$
 (h) $2x + 3y = 12$; $(3, \square)$, $(\square, 4)$, $(-3, \square)$, $(\square, -6)$
 (i) $4x - 5y = 20$; $(0, \square)$, $(\square, 4)$, $(-5, \square)$, $(\square, -12)$
 (j) $3x - 2y - 6 = 0$; $(2, \square)$, $(\square, 3)$, $(-4, \square)$, $(\square, -6)$
 (k) $5x + 2y + 6 = 0$; $(2, \square)$, $(\square, 2)$, $(-4, \square)$, $(\square, -3)$

2. Determine coordinates of 3 points contained in each of the following lines.

- (a) $x + y = 6$ (b) $x + y = -3$
 (c) $x - y = 4$ (d) $x - y = -1$
 (e) $y = 3x + 1$ (f) $x = 2y - 1$
 (g) $y = -2x + 3$ (h) $2x + y - 8 = 0$
 (i) $\frac{1}{2}x + y = 7$ (j) $2x + 3y = 12$

3. Determine whether the given point lies on the given line.

- (a) $2x + 5y = 29$; $(2, 5)$ (b) $3x - 4y = 12$; $(7, 2)$
 (c) $y = -3x + 5$; $(2, -2)$ (d) $4x - 3y - 5 = 0$; $(2, 1)$
 (e) $3y - 4x = 1$; $(3, 2)$ (f) $10x + 3y - 6 = 0$; $(2, -4)$
 (g) $3x - 4 = 5y$; $(-2, -2)$ (h) $y = \frac{1}{2}x - 7$; $(2, -6)$

The Straight Line

If a man's wit be wandering, let him study the mathematics.

Francis Bacon

4.1 DEFINING AND GRAPHING THE STRAIGHT LINE

A straight line is a set of all points such that for any two points on the line $\frac{\Delta y}{\Delta x}$ is the same. We have called the constant value $\frac{\Delta y}{\Delta x}$ the slope.

In the next section we will prove that

The graph of every equation in the form $Ax + By + C = 0$, where A , B and C are constants and not both A and B are zero, is always a straight line.

To graph a straight line we must find ordered pairs that satisfy the equation of the line. There are two common methods of doing this.

Method 1. Solving for y

The equation is solved for y and ordered pairs are determined by substituting values for x .

EXAMPLE 1. Draw the graph of $3x + 2y - 4 = 0$

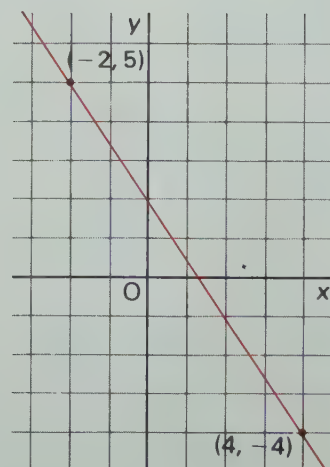
Solution

Solve the equation for y .

$$\begin{aligned} 3x + 2y - 4 &= 0 \\ 2y &= 4 - 3x \\ y &= \frac{4 - 3x}{2} \end{aligned}$$

Two points are sufficient to determine a straight line

x	y
-2	5
4	-4



EQUATIONS OF STRAIGHT LINES

$$y = -3x + 2$$

$$2x + y = 5$$

$$4x - 3y + 2 = 0$$

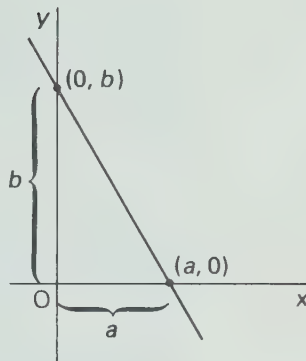
$$5x = -2y - 3$$

$$x = \frac{1}{2}y + 7$$

Unless otherwise stated,
 $x, y \in \mathbb{R}$

Method 2. Determining the Intercepts

A particular graph may cross either one or both of the axes. The x -intercept, a , is the x -coordinate of the point where the line intersects the x -axis. At this point, $y=0$. The y -intercept, b , is the y -coordinate where the line intersects the y -axis. At this point, $x=0$. When the equation of a straight line is expressed in the form $Ax+By+C=0$, then the x -intercept is $-\frac{C}{A}$ and the y -intercept is $-\frac{C}{B}$, where $A, B \neq 0$.



EXAMPLE 2. Draw the graph of $3x-2y-6=0$ using the intercept method.

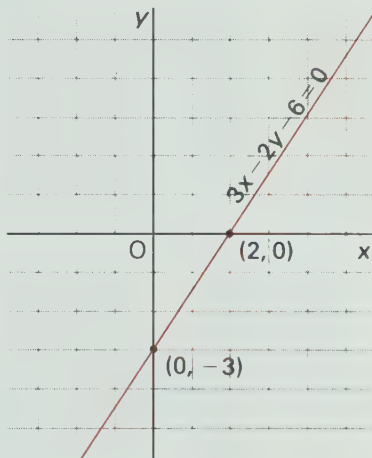
Solution

To find the x -intercept, let $y=0$.

$$\begin{aligned} 3x-2y-6 &= 0 \\ 3x-2(0)-6 &= 0 \\ 3x &= 6 \\ x &= 2 \end{aligned}$$

To find the y -intercept, let $x=0$.

$$\begin{aligned} 3x-2y-6 &= 0 \\ 3(0)-2y-6 &= 0 \\ -2y &= 6 \\ y &= -3 \end{aligned}$$



EXERCISE 4-1

A 1. State 3 ordered pairs that satisfy each of the following.

- | | |
|------------------------|------------------------|
| (a) $y=3x+2$ | (b) $y=2x-5$ |
| (c) $y=-4x+5$ | (d) $y=-7x$ |
| (e) $y=\frac{x+3}{2}$ | (f) $y=\frac{1-2x}{3}$ |
| (g) $y=5$ | (h) $x=-4$ |
| (i) $y=-\frac{x-1}{4}$ | (j) $y=-\frac{x}{3}+5$ |

2. State the x - and y -intercepts, if they exist, for each of the following.

- | | |
|------------------|-----------------|
| (a) $5x+3y=15$ | (b) $x+3y=6$ |
| (c) $2x-y=8$ | (d) $4x-7y=28$ |
| (e) $4x-3y-12=0$ | (f) $y+6=0$ |
| (g) $x=-2$ | (h) $2x=5y+10$ |
| (i) $x-2y=2$ | (j) $2x-3y-1=0$ |

3. Describe the graph of the line that passes through $(5, 7)$ and $(5, -2)$.

4. Describe the graph of the line that passes through $(6, -2)$ and $(-4, -2)$.

WORD LADDER

Start with the word "comb" and change one letter at a time to form a new word until you reach "hair". The best solution has the fewest steps.

c o m b

h a i r

5. State the equation of the horizontal line that passes through $(-5, 6)$.

6. State the equation of the vertical line that passes through $(4, 7)$.

B 7. Write each of the following in the form $Ax + By + C = 0$, where A , B and C are integers.

(a) $y = 3x - 4$

(b) $3x = 2y + 7$

(c) $8 = x - 3y$

(d) $y - 5 = 2(x - 3)$

(e) $y + 3 = -4(x + 1)$

(f) $y + 3 = \frac{1}{2}(x - 4)$

(g) $y - \frac{1}{2} = -\frac{1}{3}(x + 5)$

(h) $\frac{x}{3} = \frac{y}{4} - 6$

(i) $\frac{3}{4}y = \frac{2}{5}x - 8$

8. Solve the following equations for y .

(a) $2x + y = 7$

(b) $x + 2y = 3$

(c) $3x + y - 4 = 0$

(d) $2x - y = 6$

(e) $5x - y - 1 = 0$

(f) $3x - 2y = 0$

(g) $5x + 2y - 9 = 0$

(h) $x - 5y = 6$

(i) $2x = 3y + 4$

(j) $3y + 7 = 0$

9. Graph each of the following.

(a) $y = 3x + 2$

(b) $2x - 3y = 6$

(c) $4x + 5y - 20 = 0$

(d) $x + 3y = 7$

(e) $x = \frac{1}{2}y - 4$

(f) $y - 2 = 3(x - 1)$

(g) $2x = 3$

(h) $5y + 10 = 0$

(i) $\frac{x}{2} + \frac{y}{3} = 1$

(j) $\frac{x}{4} - \frac{y}{2} = 1$

(k) $3 - 2(x + y) = 2$

(l) $\frac{x+1}{2} - \frac{y-1}{3} = 1$

(m) $2x - 3(x - 4) + y = 5$

(n) $3(x - y) - 2(x + 2y) = 5$

C 10. State the value of k so that the given point lies on the graph of the given equation.

(a) $2x + ky = 21$; $(3, 5)$

(b) $kx - 3y = 11$; $(-1, -5)$

(c) $2kx + 5y - 6 = 0$; $(-1, 2)$

(d) $x + 4y = 2k - 3$; $(1, 1)$

(e) $(1 + k)x + ky = k + 2$; $(-2, 4)$

(f) $3x + 7y = k^2 + 6$; $(0, 1)$

4.2 DETERMINING EQUATIONS OF LINES

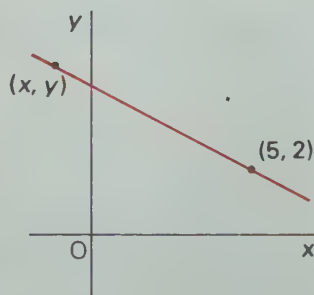
The line graphed at the right passes through the point $(5, 2)$ and has a slope $m = -\frac{1}{2}$. To determine the equation of the line let (x, y) be any point on the line other than $(5, 2)$. By the definition of slope

$$\frac{y - 2}{x - 5} = -\frac{1}{2}$$

$$y - 2 = -\frac{1}{2}(x - 5)$$

$$2y - 4 = -x + 5$$

$$x + 2y - 9 = 0$$



In general, if we let (x_1, y_1) be the given point, m the given slope and (x, y) any other point on the line, then

$$\frac{y - y_1}{x - x_1} = m$$

$$y - y_1 = m(x - x_1)$$

This is called the **point-slope form** for the equation of a straight line,

EXAMPLE 1. Determine the equation of the line passing through the points $(4, 3)$ and $(-2, 5)$. Express the equation in the form $Ax + By + C = 0$.

Solution

$$\begin{aligned} \text{Slope} = m &= \frac{\Delta y}{\Delta x} = \frac{5 - 3}{-2 - 4} \\ &= \frac{2}{-6} \\ &= -\frac{1}{3} \end{aligned}$$

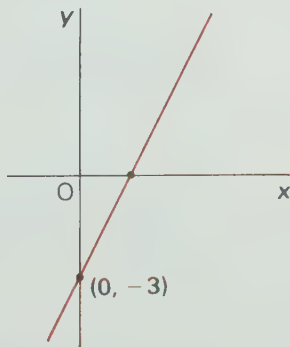
Select either point, say $(4, 3)$, to be (x_1, y_1) and use the point-slope form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 3 &= -\frac{1}{3}(x - 4) \\ y - 3 &= -\frac{1}{3}x + \frac{4}{3} \\ 3y - 9 &= -x + 4 \\ x + 3y - 13 &= 0 \end{aligned}$$

When the given point lies on the y -axis the result is an important case of the point-slope form.

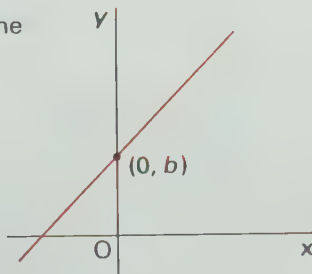
The line graphed at the right passes through the point $(0, -3)$ and has a slope $m = 2$. To determine the equation of the line we use the point-slope form for the equation of a line.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y + 3 &= 2(x - 0) \\ y + 3 &= 2x \\ y &= 2x - 3 \end{aligned}$$



In general, if we let $(0, b)$ be the point on the y -axis and m the given slope then

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - b &= m(x - 0) \\ y - b &= mx \\ y &= mx + b \end{aligned}$$



In each case the constant term (-3 or b) is the y -intercept of the line and the coefficient of x , (2 or m), is the slope of the line.

The general equation

$$y = mx + b$$

is called the **slope y -intercept form** of the equation of the line.

What we have shown above is that if (x, y) is a point on the line with slope m and y -intercept b then its equation is $y = mx + b$.

Conversely we can show that any equation of the form $y = mx + b$ represents a straight line. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be points which lie on $y = mx + b$.

Then

$$y_1 = mx_1 + b$$

and

$$y_2 = mx_2 + b$$

$$\begin{aligned} \therefore \text{the slope of } P_1P_2 &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1} \\ &= \frac{m(x_2 - x_1)}{x_2 - x_1} \\ &= m \end{aligned}$$

It takes 24 h for Ed and Alex to do a job together. How long will it take each to do the job alone if Ed works $\frac{2}{3}$ as fast as Alex?

Therefore for any two points P_1 and P_2 on the line $\frac{\Delta y}{\Delta x}$ is the same so $y = mx + b$ represents a straight line.

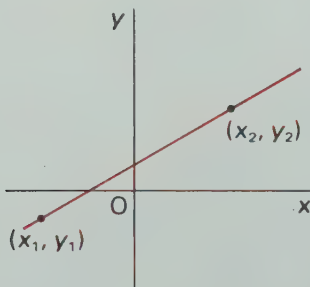
$Ax + By + C = 0$ is a straight line since we can write it in the form $y = mx + b$.

There are two cases, namely when $B = 0$ and $B \neq 0$.

If $B = 0$, $Ax + By + C = 0$ reduces to $x = -\frac{C}{A}$ whose graph is a straight line parallel to the y -axis.

If $B \neq 0$, then $Ax + By + C = 0$ can be written in the form $y = mx + b$, namely $y = -\frac{A}{B}x - \frac{C}{B}$.

There are three other forms for the equation of a straight line.



If a line passes through the points (x_1, y_1) and (x_2, y_2) then its equation can be written as

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$$

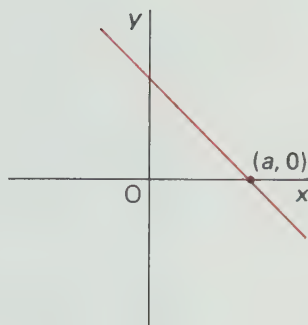
This is called the **two-point form** for the equation of a line.

If a line passes through a point on the x -axis, say $(a, 0)$, with slope m , then

$$y - y_1 = m(x - x_1)$$

$$y - 0 = m(x - a)$$

$$y = m(x - a)$$



This is called the **slope x -intercept form** for the equation of a line.

If a line has x -intercept a and y -intercept b , then

the slope of the line $m = -\frac{b}{a}$ and

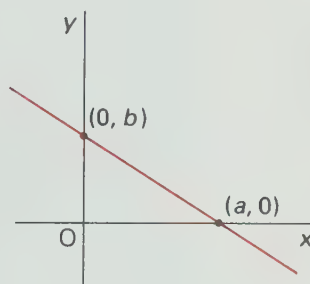
$$y - y_1 = m(x - x_1)$$

$$y - 0 = -\frac{b}{a}(x - a)$$

$$y = -\frac{b}{a}x + b$$

$$\frac{b}{a}x + y = b$$

$$\frac{x}{a} + \frac{y}{b} = 1$$



This is called the **intercept form** for the equation of a line.

EXAMPLE 2. Determine the slope and y -intercept of the line $2x + 3y = 10$.

Solution

Solve the equation for y .

$$2x + 3y = 10$$

$$3y = -2x + 10$$

$$y = -\frac{2}{3}x + \frac{10}{3}$$

The equation is now in the form $y = mx + b$ where $m = -\frac{2}{3}$ and $b = \frac{10}{3}$.
Therefore the slope is $-\frac{2}{3}$ and the y -intercept is $\frac{10}{3}$.

EXERCISE 4-2

A 1. State the point-slope form of the equation of the line through the given point and having the given slope.

(a) $(3, 2)$; $m = 4$

(b) $(1, 4)$; $m = 6$

(c) $(-2, 5)$; $m = 1$

(d) $(-7, -2)$; $m = 5$

(e) $(-3, -4)$; $m = -2$

(f) $(6, 0)$; $m = -4$

(g) $(0, 5)$; $m = 2$

(h) $(-1, -2)$; $m = 0$

2. State the slope y -intercept form of the equation of the line with the given slope m and y -intercept b .

- (a) $m = 2; b = 3$ (c) $m = 5; b = -3$ (e) $m = -\frac{2}{3}; b = 0$
 (b) $m = -4; b = 5$ (d) $m = \frac{1}{2}; b = -7$ (f) $m = 0; b = -6$

3. State the slope and y -intercept of the following linear functions.

- (a) $y = 3x - 7$ (b) $y = -\frac{1}{2}x + 5$
 (c) $y = -5x - 4$ (d) $y = 7x$
 (e) $y = \frac{2x-1}{3}$ (f) $y + 7 = 2x$
 (g) $y + 3x = 4$ (h) $2x - 3 = y$
 (i) $5x + y - 2 = 0$ (j) $7x = 3 + y$
 (k) $4x - y = 5$ (l) $3x + 2y = 5$
 (m) $x + 3y = -2$ (n) $6x - y - 4 = 0$

B 4. Determine an equation of the line through the given point and having the given slope. Express the equation in the form

$$Ax + By + C = 0.$$

- (a) $(2, 1); m = 3$ (b) $(-3, 4); m = 5$
 (c) $(4, -5); m = -1$ (d) $(-1, -2); m = 7$
 (e) $(5, 0); m = -6$ (f) $(0, -5); m = \frac{1}{2}$
 (g) $(-1, 6); m = 0$ (h) $(4, -3);$ no slope
 (i) $(-6, 4); m = -\frac{1}{2}$ (j) $(-7, -1); m = \frac{1}{3}$

5. Write the equation of the linear function through $(-4, 5)$ with slope

- (a) 2 (b) -4 (c) 1.5 (d) -0.5 (e) -2.5

6. Write the equation of the line having slope $\frac{4}{3}$ and containing the point

- (a) $(0, 0)$ (b) $(-3, 2)$ (c) $(-3, 5)$ (d) $(5, -3)$ (e) $(0, -6)$

7. Determine an equation of the line through the given point and having the given slope. Express the equation in the form $y = mx + b$ and determine two other points on the line.

- (a) $(1, 2); m = 3$ (b) $(-1, -1); m = -2$
 (c) $(3, 0); m = 2$ (d) $(4, -2); m = -0.5$
 (e) $(0, -3); m = \frac{1}{2}$ (f) $(-5, -6); m = 0$
 (g) $(-3, 4); m = \frac{3}{4}$ (h) $(10, 20); m = 0.2$

8. Determine an equation of the line through the given points. Express the equation in the form $Ax + By + C = 0$.

- (a) $(1, 2), (2, 4)$ (b) $(5, 6), (7, 8)$
 (c) $(2, 3), (-2, -1)$ (d) $(-1, 4), (5, 1)$
 (e) $(-2, -1), (4, -2)$ (f) $(-6, 4), (-1, -2)$
 (g) $(10, 8), (-3, 1)$ (h) $(2, 5), (7, -4)$
 (i) $(3, -1), (-6, -5)$ (j) $(-6, 7), (1, -3)$
 (k) $(2\frac{1}{2}, \frac{1}{2}), (-3, -2\frac{1}{2})$ (l) $(-\frac{3}{4}, -1\frac{1}{2}), (2, 2\frac{1}{4})$
 (m) $(-3.8, 1.2), (1.2, 2.7)$ (n) $(-4.5, 6), (-1.5, -4)$
 (o) $(-4, 0), (0, -8)$ (p) $(4, 6), (4, -2)$
 (q) $(-2, -3), (4, -3)$ (r) $(1.2, 1.2), (-3.6, -3.6)$

9. Express each of the following equations in the form $y = mx + b$. State the slope and y -intercept and determine two other points on the line.

- (a) $4x + y = 7$ (b) $3x + 2y = 5$
 (c) $2x - 3y - 1 = 0$ (d) $x - 2y = 4$
 (e) $4y - x = 2$ (f) $2x + 3y = 0$

Place the digits from 1 to 9 in the circles to make a correct addition sum. No two consecutive digits must be in adjacent circles.



(g) $3 = x - 2y$

(h) $y - 7 = 0$

10. Write an equation for the linear function f if

(a) $f(3) = 2$ and $f(5) = 8$

(c) $f(-1) = -3$ and $f(0) = 1$

(b) $f(1) = 2$ and $f(-1) = -2$

(d) $f(-2) = 3$ and $f(-4) = 4$

11. Determine an equation of the line with the given slope and x -intercept.

(a) $m = 3, a = 4$

(b) $m = -2, a = 5$

(c) $m = -1, a = -3$

(d) $m = 5, a = -2$

(e) $m = \frac{1}{4}, a = 0$

(f) $m = 0.2, a = -1$

(g) $m = -0.5, a = 1.4$

(h) $m = -2.6, a = -5.2$

12. Determine an equation of the line whose x - and y -intercepts are respectively

(a) 3 and 6

(c) 2 and -5

(e) -8 and 4

(b) -2 and 4

(d) -1 and -6

(f) 3 and -7

C 13. Determine an equation of the line through $(-3, 4)$ and having the same y -intercept as $2x - 3y = 6$.**14.** Determine an equation of the line through $(-3, -2)$ and having the same slope as the graph of $2x + 3y = 7$.**15.** Determine an equation of the line having the same x -intercept as $4x - 3y - 12 = 0$ and the same slope as the graph of $2x + 5y = 7$.**16.** Determine an equation of the line through $(-1, 3)$ and having its x -intercept equal to its y -intercept.

4.3 PARALLEL AND PERPENDICULAR LINES

When two different lines in a plane have the same slope, or no slope (vertical lines), the lines are parallel. This can be proved as follows.

Given: slope $l_1 = \text{slope } l_2 = m$

Required: To prove $l_1 \parallel l_2$

Proof: If l_1 and l_2 have no slope, both are parallel to the y -axis. Hence $l_1 \parallel l_2$.

Likewise if $m = 0$, l_1 and l_2 are parallel to the x -axis. Hence $l_1 \parallel l_2$.

If $m \neq 0$, then l_1 and l_2 meet the x -axis at A and D respectively.

Locate B and E to the right of A and D respectively so that $AB = DE = 1$

Draw perpendiculars at B and E to intersect l_1 and l_2 at C and F respectively, so that

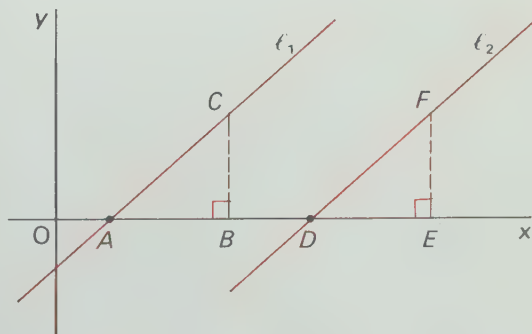
$$\angle ABC = \angle DEF = 90^\circ$$

$$\therefore CB = m = FE \quad \text{and}$$

$$\triangle ABC \cong \triangle DEF \text{ (SAS)}$$

$$\therefore \angle CAB = \angle FDE$$

$$\therefore l_1 \parallel l_2 \text{ (TPT)}$$



Conversely, by reversing the above argument, it can be proved that two parallel lines in a plane either have the same slope or have no slope (vertical lines).

If two nonvertical lines are parallel, they have the same slope; conversely, if two distinct nonvertical lines have the same slope, they are parallel.

EXAMPLE 1. Determine an equation of the line containing $(-4, 6)$ and parallel to the graph of $3x + 2y = 7$.

Solution

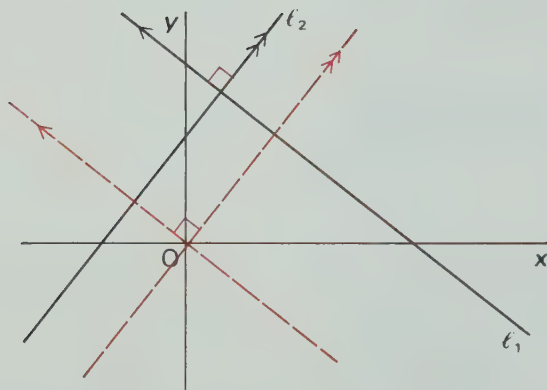
Determine the slope of $3x + 2y = 7$ by expressing the equation in the form $y = mx + b$.

$$\begin{aligned} 3x + 2y &= 7 \\ 2y &= -3x + 7 \\ y &= -\frac{3}{2}x + \frac{7}{2} \end{aligned}$$

The slope of the required line containing $(-4, 6)$ is $-\frac{3}{2}$.
Finding the equation:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 6 &= -\frac{3}{2}(x + 4) \\ 2y - 12 &= -3x - 12 \\ 3x + 2y &= 0 \end{aligned}$$

Two lines which intersect at right angles are called perpendicular lines. The slopes of perpendicular lines are negative reciprocals of each other. This can be proved as follows.



Given: $l_1 \perp l_2$

Required: To prove $m_1 = -\frac{1}{m_2}$

Proof

If l_1 and l_2 are the original lines, consider the parallel lines that pass through the origin. Since the lines are parallel the angles don't change. Locate A and C as shown so that

$$OA = OC = 1$$

$\angle BOA = \angle COD$ (each is the complement of $\angle COB$)

$$\angle BAO = \angle DCO = 90^\circ$$

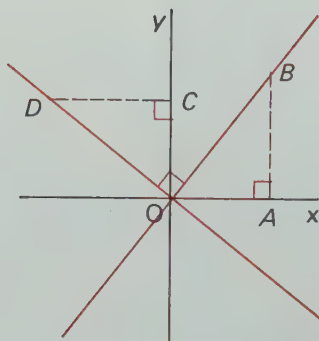
$$\therefore \triangle ABO \cong \triangle CDO \text{ (ASA)}$$

$$\therefore AB = DC$$

The coordinates of B are $(1, m_2)$ since the slope of l_2 is m_2 .

Since $AB = CD$, the coordinates of D are $(-m_2, 1)$.
 $\therefore$ The slope of the line joining $D(-m_2, 1)$ to $O(0, 0)$ is

$$\begin{aligned} m_1 &= \frac{\Delta y}{\Delta x} = \frac{1 - 0}{-m_2 - 0} \\ &= -\frac{1}{m_2} \end{aligned}$$



Conversely, it can be proved that if the slopes of two lines are negative reciprocals of each other the lines are perpendicular. This theorem may also be stated as follows.

If two lines l_1 and l_2 with slopes $m_1 \neq 0$ and $m_2 \neq 0$ are perpendicular then $m_1 m_2 = -1$; conversely, if the slopes of two lines are such that $m_1 m_2 = -1$ then the lines are perpendicular.

EXAMPLE 2. Determine the equation of the line containing $(4, -2)$ and perpendicular to the graph of $3x - 2y + 4 = 0$.

Solution

Determine the slope of $3x - 2y + 4 = 0$

$$\begin{aligned} 3x - 2y + 4 &= 0 \\ -2y &= -3x - 4 \\ y &= \frac{3}{2}x + 2 \end{aligned}$$

Slope of given line is $\frac{3}{2}$

Slope of required line is $-\frac{1}{\frac{3}{2}} = -\frac{2}{3}$

$$\left(\frac{3}{2}\right)\left(-\frac{2}{3}\right) = -1$$

The slope of the line containing $(4, -2)$ is $-\frac{2}{3}$.

Determining the equation:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y + 2 &= -\frac{2}{3}(x - 4) \\ 3y + 6 &= -2x + 8 \\ 2x + 3y - 2 &= 0 \end{aligned}$$

EXERCISE 4-3

- A** 1. Given the slope m of a line l , state the slope of a line (i) parallel to l (ii) perpendicular to l

- (a) $m = 2$ (b) $m = \frac{1}{4}$ (c) $m = -\frac{3}{4}$ (d) $m = -4$
 (e) $m = -\frac{3}{2}$ (f) $m = \frac{5}{6}$ (g) $m = -1$ (h) $m = 0$

2. Given the slopes of two lines, classify the lines as parallel, perpendicular, or neither parallel nor perpendicular.

- (a) $m_1 = 3, m_2 = \frac{1}{3}$ (b) $m_1 = \frac{3}{4}, m_2 = \frac{6}{8}$
 (c) $m_1 = -2, m_2 = \frac{1}{2}$ (d) $m_1 = -\frac{1}{2}, m_2 = \frac{1}{2}$
 (e) $m_1 = \frac{4}{3}, m_2 = -\frac{3}{4}$ (f) $m_1 = 0, m_2 = 1$
 (g) $m_1 = 1, m_2 = -1$ (h) $m_1 = \frac{1}{4}, m_2 = \frac{6}{24}$

3. Classify the lines whose equations are given as parallel, perpendicular, or neither parallel nor perpendicular.

- (a) $y = 2x + 5, y = 2x - 1$ (b) $y = \frac{1}{2}x + 3, y = -2x - 3$
 (c) $y = 3x - 1, y = -3x + 1$ (d) $y = x + 5, y = -x - 4$
 (e) $2y = 6x + 7, y = 3x - 5$ (f) $x = 7, x = -1$
 (g) $x = -6, y = 4$ (h) $3x + y = 5, y = \frac{1}{3}x - 8$

- B** 4. Classify the lines determined by the two pairs of points as parallel, perpendicular, or neither parallel nor perpendicular.

- (a) $(-2, 8), (3, 7)$ and $(4, 3), (9, 2)$
 (b) $(0, 1), (-5, 4)$ and $(5, 3), (0, 5)$

(c) $(2, 5)$, $(8, 7)$ and $(-3, 1)$, $(-2, -2)$

(d) $(4, 6)$, $(-3, -1)$ and $(6, -3)$, $(4, 5)$

5. Determine an equation of the line satisfying the given conditions:

(a) through $(4, 6)$ and parallel to the graph of $y = 3x + 4$

(b) containing $(-2, -3)$ and parallel to the graph of $y + 2x = 6$

(c) through $(-1, 5)$ and perpendicular to the graph of $y = -3x + 7$

(d) containing $(-3, -2)$ and perpendicular to $y - 2x + 6 = 0$

(e) through $(5, 2)$ and parallel to $3x - 5y = 6$

(f) containing $(-5, 6)$ and perpendicular to $2x + 3y - 7 = 0$

(g) through $(-\frac{1}{2}, \frac{1}{3})$ and parallel to $3x - 4y + 7 = 0$

(h) having the same x -intercept as $3x + 5y - 15 = 0$ and parallel to $5x + 2y = 7$

(i) having the same y -intercept as $2x - 3y = -6$ and perpendicular to $4x - y = 6$

(j) containing $(-1, -3)$ and parallel to $x = 3$

(k) through $(5, -6)$ and parallel to $y = 7$

(l) with x -intercept 4 and y -intercept -5

6. Find an equation for the line

(a) through $(-5, 6)$ and parallel to $y = 7x + 5$

(b) with y -intercept 3 and perpendicular to $2x + 3y = 5$

(c) with slope $-\frac{3}{2}$ and x -intercept 4

(d) with slope $\frac{1}{5}$ and y -intercept -2

(e) through $(-1, 4)$ and perpendicular to the x -axis.

(f) through $(5, -2)$ and parallel to the x -axis.

7. Show that the triangle whose vertices have the coordinates $(3, 3)$, $(8, 17)$ and $(11, 5)$ is a right triangle. (Prove that two of the sides lie on perpendicular lines.)

8. Prove using slopes that $(-8, 4)$, $(3, 2)$, $(9, 10)$ and $(-2, 12)$ are the vertices of a parallelogram.

9. Prove using slopes that $(-1, 1)$, $(1, -1)$, $(6, 4)$ and $(4, 6)$ are the vertices of a rectangle.

10. For each pair of equations, find a value of k so that the graph of the first equation is parallel to that of the second.

(a) $y = kx + 3$; $3y = 2x + 4$

(b) $3x - ky = 7$; $y = -2x + 5$

(c) $4x + ky - 2 = 0$; $3x - 2y = 5$

11. For each pair of equations, find a value of k so that the graph of the first equation is perpendicular to that of the second.

(a) $y = kx - 4$; $3y = 2x + 7$

(b) $5x - ky = 3$; $y = -3x + 2$

(c) $4x + ky = 6$; $5x - 2y + 5 = 0$

C 12. Determine an equation of the line containing $(5, 4)$ and having its x -intercept equal to its y -intercept.

13. If $A(4, -1)$, $B(-4, 5)$ and $C(3, 6)$ are the vertices of $\triangle ABC$, determine

(a) an equation of the line through A and parallel to BC .

(b) an equation of the line containing the altitude from A to BC .

(c) an equation of the line containing the altitude from B to AC .

A small boat is floating in a bathtub. Which will raise the level of the water in the tub more, dropping a penny into the tub or into the boat?

14. Find an equation of the line containing (s, t) and parallel to the graph of $Ax + By + C = 0$, where $A \neq 0$ and $B \neq 0$.

4.4 APPLICATIONS

In this section we will study some applications of linear functions. Some liberties will be taken in using equations and graphs to solve problems since mathematical models may not fit real world situations exactly. However, the models are close enough for most practical purposes.

In the problems that follow, assume that a linear function is the best mathematical model that fits the situation. When the domain is N it will be convenient to draw the graph as though $x \in R$ instead of showing the function as a series of points in the plane.

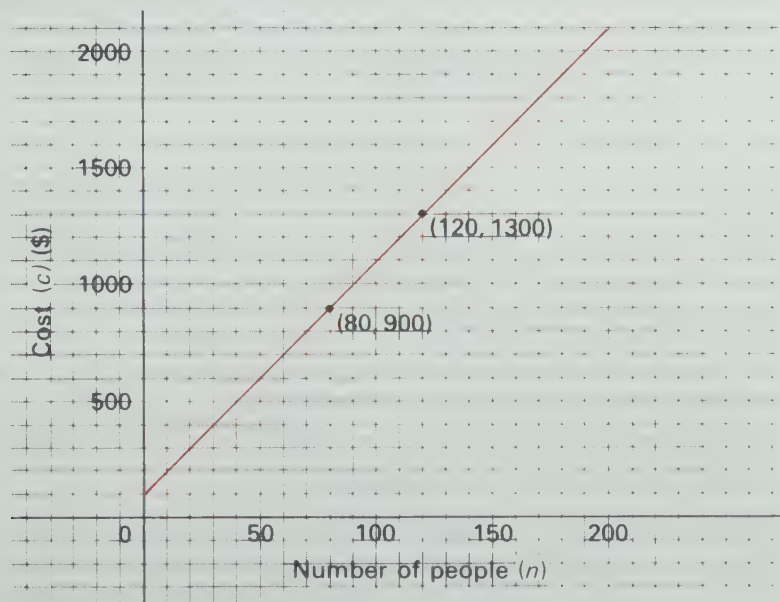
I APPLICATIONS OF THE FORM $y = mx + b$

EXAMPLE 1. The Mainway Restaurant has banquet facilities for up to 200 people. When the owner quotes a price for a banquet he is including the room rent plus the cost of the meal. A banquet for 80 people will cost \$900. For 120 people the price is \$1300.

- Plot a graph of cost versus number of people.
- From the graph, determine the cost of a banquet for 150 people.
- Determine an equation for this function. Express it in the form $c = mn + b$ where c represents the cost and n the number of people.
- Use the equation to determine the cost of a banquet for 90 people.
- What quantity does the slope of the line represent?
- What meaning does the “ y -intercept” have?

Solution

(a)



(b) \$1600

$$\begin{aligned} \text{(c)} \quad m &= \frac{\Delta y}{\Delta x} = \frac{1300 - 900}{120 - 80} \\ &= \frac{400}{40} \\ &= 10 \\ c - c_1 &= m(n - n_1) \\ c - 900 &= 10(n - 80) \\ c - 900 &= 10n - 800 \\ c &= 10n + 100 \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad c &= 10n + 100 \\ &= 10(90) + 100 \\ &= 900 + 100 \\ &= 1000 \end{aligned}$$

$\therefore$ cost for 90 people is \$1000

(e) The slope represents the price per meal.

(f) The y-intercept represents the room rent.

II DIRECT VARIATION: APPLICATIONS OF THE FORM $y = mx$

The type of linear function associated with an equation of the form $y = mx$ is called a *direct variation*. We say that y varies directly as x or y is directly proportional to x . The constant m is called the constant of proportionality or constant of variation. In this case, since $y = mx$ then $\frac{y}{x} = m$ or the ratio of $\frac{y}{x}$ is constant.

EXAMPLE 2. Since light travels much faster than sound, during a thunderstorm you see the lightning before you hear the thunderclap. The time interval between the flash and the thunderclap is directly proportional to the distance between you and the storm.

- Write an equation for this direct variation function.
- If the thunderclap from lightning 1980 m away takes 6 s to reach you, determine the constant of proportionality.
- If the time interval is 10 s, how far away is the storm centre?
- Plot a graph of time versus distance.
- What quantity does the slope (constant) represent?
- Use the graph to determine the time interval if the storm is 2.5 km away.

Solution

(a) Distance is directly proportional to time.

$$d = mt$$

(b)

$$d = mt$$

$$1980 = 6m$$

$$m = 330$$

(c)

$$d = 330t$$

$$\text{or } \frac{d}{t} = 330$$

$$d = 330(10)$$

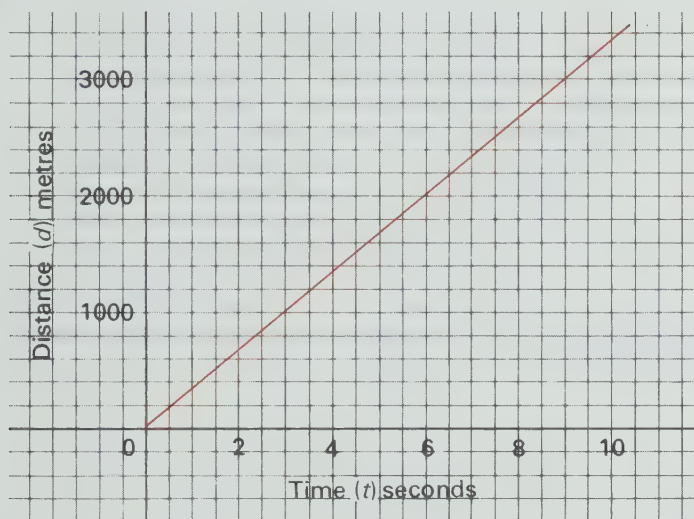
$$\frac{d}{10} = 330$$

$$= 3300$$

$$d = 3300$$

The storm is 3300 m away.

(d)



(e) The approximate speed of sound, 330 m/s.

(f) Approximately 7.5 s.

III. FUNCTIONS WITH SEVERAL DEFINING SENTENCES

The following example will illustrate a mathematical model that cannot be defined using just one defining equation.

EXAMPLE 3. A cab company charges \$1.00 for the first 0.6 km and \$0.40 for each additional 0.2 km.

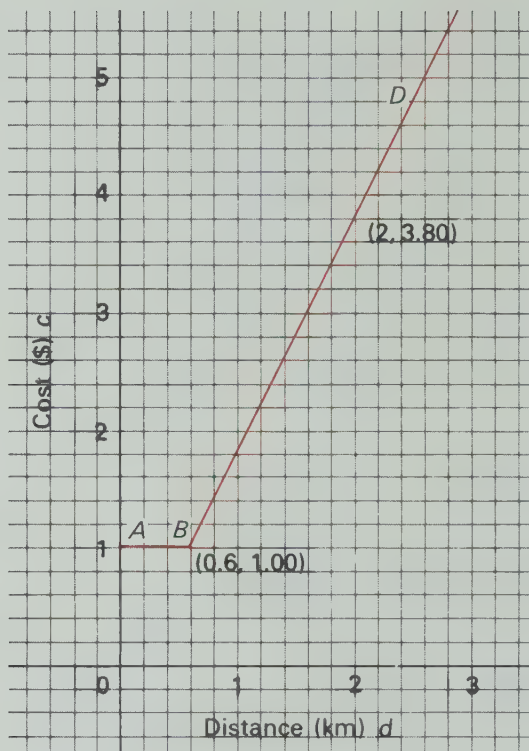
(a) Draw the graph of cost versus distance.

(b) Determine the equations which define this function.

(c) How much will it cost to go 15 km by cab?

Solution

(a)



(b) From A to B the function is a constant one whose equation is $c = 1.00$ or $y = 1.00$. From B to D and beyond the function is not a constant one. Here

$$\begin{aligned} m &= \frac{3.80 - 1.00}{2 - 0.6} \\ &= \frac{2.80}{1.4} \\ &= 2 \end{aligned}$$

and

$$\begin{aligned} c - c_1 &= m(d - d_1) \\ c - 3.80 &= 2(d - 2) \\ c - 3.80 &= 2d - 4 \\ c &= 2d - 0.20 \end{aligned}$$

Therefore the defining equation is

$$c = \begin{cases} 1.00, & 0 < d \leq 0.6 \\ 2d - 0.2, & d > 0.6 \end{cases}$$

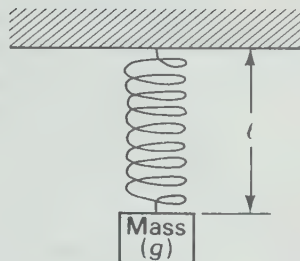
(c)

$$\begin{aligned} c &= 2d - 0.20 \\ &= 2(15) - 0.20 \\ &= 30 - 0.20 \\ &= 29.80 \end{aligned}$$

It will cost \$29.80 to go 15 km.

EXERCISE 4-4

- B** 1. In order to determine the "law of stretch" for a particular coil spring the following test results were obtained. When a mass of 40 g was added the length was 60 cm. For an 80 g mass the length was 90 cm.



- Plot a graph of length versus mass.
- From the graph determine the length for a mass of 70 g.
- Determine an equation for this function. Express it in the form $y = mx + b$.
- Use the equation to determine the length for a mass of 60 g.
- What quantity does the slope of the line represent?
- What meaning does the y -intercept have?

2. In the 18th century a French scientist, Jacques Charles, discovered that when he plotted the volume of a fixed amount of gas versus the temperature a straight line resulted. At 27°C , the volume was 400 cm^3 and at 81°C the volume was 472 cm^3 .

- Plot a graph of volume versus temperature.
- From the graph determine the volume at 50°C .
- From the graph determine the temperature if the volume is 450 cm^3 .
- Determine an equation for this function. Express it in the form $v = mt + b$ where v represents the volume and t the temperature.
- Use the equation to determine the volume when the temperature is 63°C .
- What quantity does the slope of the line represent?
- What meaning does the " y -intercept" have?

3. Normally, property tax is directly proportional to the assessed value of the property. In Satellite City, the tax is \$1200 on a property with an assessed value of \$48 000.

- Write an equation for this direct variation function.
- Determine the constant of proportionality.
- If the assessed value of a property is \$57 000, what is the property tax?
- If the property tax is \$900, what is the assessed value of the property?
- If 1 mill = \$0.001, what is the mill rate for Satellite City?

4. In Silver City, a driver must pay a fine of \$60 for driving 55 km/h and \$135 for driving at 80 km/h.

- Plot a graph of fine versus speed for this linear function.
- From the graph determine the fine for driving 70 km/h.
- Determine an equation for this function.
- Use the equation to determine the fine for driving 60 km/h.
- What quantity does the slope of the line represent?
- What is the legal speed limit?

You are given 8 identical coins and told that one of them is lighter than the rest. Using a simple balance twice, find the light coin.



5. If there are 1.8 g of haemoglobin in 15 cm^3 of normal human blood, how many grams of haemoglobin would 28 cm^3 of the same blood contain?

6. The dividend on 500 shares of a stock is \$875 per year. What is the dividend on 160 shares of the same stock?

7. The Pollution Control Department determined that a factory increased the pollution in a lake by 8 ppm (parts per million) in fourteen months. At the same rate, how long would it take to increase the pollution level by 26 ppm?

8. A pharmacist read the following instructions on a bottle of medicine: "Mix with water in the ratio of 5 mL of water per 3 mL of medicine." How much water should be mixed with 37 mL of medicine?

9. The quality control inspector at a gravel processing factory determined that 632 kg of washed gravel remain after 800 kg of unwashed gravel are washed. How many kg of washed gravel will remain if 1300 kg of unwashed gravel are washed?

10. An automobile company used the following procedure to determine the gasoline consumption of a new car. The tank was filled with gas and a test driver drove around the test track at a constant highway speed. After 90 km, 85 L of gas remained in the tank. After 360 km, 40 L of gas remained.

- Plot a graph of litres of gas remaining versus distance driven.
- From the graph, determine the number of litres remaining after 180 km.
- Determine an equation for this linear function.
- Use the equation to determine the number of litres remaining after 240 km.
- What quantity does the slope of the line represent?
- What is the capacity of the gas tank?
- At the test speed, how many kilometres could you expect to drive on a full tank of gas?
- Express the gasoline consumption as L/100 km.

11. The resistance in an electrical wire is directly proportional to the length of the wire. If 1800 m of wire has a resistance of 25Ω , what is the resistance of 1350 m of wire?

12. The Birko Electrical Supply Company claims that its heater for home hot water tanks will take 4 h to heat cold water to the required hot water temperature. An inspector filled a tank with cold water and found that after 2 h the water temperature was 34°C , and after 3 h, 45.5°C .

- Plot a graph of temperature versus time.
- From the graph determine the water temperature after 2.5 h.
- Determine an equation for this linear function.
- Use the equation to determine the water temperature after one hour.
- What quantity does the slope of the line represent?
- What meaning does the "y-intercept" have?

13. The number of dollars per month it costs to operate a car depends on the number of kilometres you drive the car per month. According to a recent survey for medium sized cars, it costs about \$94/mon if you drive 200 km/mon and about \$134/mon if you drive 1000 km/mon.

- Plot a graph of cost/mon versus distance/mon.
- Determine an equation for this linear function.
- Use the equation to determine the cost/mon if you drive 800 km/mon.
- What quantity does the slope of the line represent?
- What meaning does the “y-intercept” have? What “non driving” costs account for the y-intercept being greater than zero?

14. A car rental agency charges \$8.00 for the first 20 km and \$0.20 for each additional kilometre.

- Draw a graph of cost versus distance.
- Determine the equations which define this function.
- How much will it cost to go 40 km?
- If your bill was \$29.00, how many kilometres did you drive?

15. A long distance call costs \$2.50 for the first 3 min and \$0.30 for each additional minute.

- Draw a graph of cost versus time.
- Determine the equations which define this function.
- How much will it cost to talk for 7 min?
- How long did you talk if the cost was \$4.60?

16. A newspaper advertised a job that paid \$9.60/h for the first 40 h/week and time and a half for overtime.

- Draw a graph of wages versus time.
- Determine the equations which define this function.
- If you work 45 h in one week, what will your salary be—before deductions?

17. A soft drink wholesale outlet charges \$5.00 a case for orders between 1 and 100 cases, \$4.50 a case for orders between 101 and 200 cases and \$4.00 a case for orders over 200 cases.

- Draw a graph of cost versus number of cases.
- Determine the equations which define this function.
- Determine the cost of
 - 87 cases
 - 153 cases
 - 213 cases

C 18. In Satellite City, speeding fines are determined as follows:

Speed Excess	Fine
1 to 10 km/h	\$2 for each km/h
11 to 20 km/h	\$4 for each km/h
21 to 30 km/h	\$6 for each km/h
over 30 km/h	\$10 for each km/h

- (a) Plot a graph of fine versus "speed excess" for this function.
- (b) From the graph determine the fine for driving
 - (i) 6 km/h over the limit.
 - (ii) 18 km/h over the limit.
 - (iii) 22 km/h over the limit.
- (c) Determine an equation for this function.

REVIEW EXERCISE

B 1. Graph each of the following.

- | | |
|-------------------------------------|------------------------|
| (a) $y = 2x + 3$ | (b) $y = -x + 4$ |
| (c) $y = 5x - 2$ | (d) $3x + 2y = 6$ |
| (e) $x + 3y - 9 = 0$ | (f) $2(x - 1) + y = 5$ |
| (g) $\frac{x}{3} - \frac{y}{2} = 1$ | (h) $3x - 6 = 0$ |

2. Determine an equation of the line through the given point and having the given slope.

- | | |
|-------------------------|-------------------------|
| (a) (2, 4); $m = 1$ | (b) (-2, 3); $m = 4$ |
| (c) (3, -3); $m = -2$ | (d) (4, -2); $m = 0$ |
| (e) (-5, 7); $m = 4$ | (f) (-3, -7); $m = -6$ |
| (g) (-4, -4); $m = 0.5$ | (h) (3, -2); $m = -0.2$ |

3. Determine an equation of the line through the given points.

- | | |
|----------------------|------------------------|
| (a) (2, 3), (3, 5) | (b) (3, -2), (4, 6) |
| (c) (-1.0), (-2, -3) | (d) (-6, -2), (3, -2) |
| (e) (2, -1), (-3, 4) | (f) (1.2, 5), (1.3, 6) |
| (g) (2, -3), (2, 7) | (h) (6, 5), (-1, -1) |

4. Determine an equation of the line satisfying the given conditions:

- (a) through (-5, 1) and having slope $m = 0.5$
- (b) through (-6, 2) and (5, -3)
- (c) through (1, 6) and parallel to $y = 3x + 4$
- (d) containing (-5, 0) and perpendicular to $y = -2x + 3$
- (e) having an x-intercept of 4 and a y-intercept of -3
- (f) having slope $m = 2$ and an x-intercept of 5
- (g) having a y-intercept of 2 and slope $m = \frac{1}{2}$

5. Determine an equation of the line satisfying the given conditions:

- (a) containing (-1, -3) and parallel to $3x + 2y = 7$
- (b) containing (0, -7) and perpendicular to $4x - 3y + 8 = 0$
- (c) parallel to $3x + y = 8$ and having the same x-intercept as $4x + 5y - 20 = 0$
- (d) containing (-1, 6) and parallel to $y = -2$
- (e) containing (3, -4) and perpendicular to $y = 5$
- (f) through (-2, -5) and perpendicular to $2x - 3y - 5 = 0$
- (g) through (-7, -1) and perpendicular to the line through (-4, 2) and (1, 1)

6. Find the value of k so that the graph of $3x + ky + 5 = 0$ is parallel to $4x - y = 7$

7. Find the value of k so that the graph of $7x - 2ky = 6$ is perpendicular to $4x + 2y - 1 = 0$

WORD LADDER

Start with the word "pity" and change one letter at a time to form a new word until you reach "good". The best solution has the fewest steps.

p i t y

good

8. Show that the triangle whose vertices have coordinates $(3, -4)$, $(4, 6)$ and $(-7, -3)$ is a right triangle.
9. Show that $(-1, -3)$, $(5, 6)$, $(1, 5)$ and $(3, -2)$ are the vertices of a parallelogram.
10. The dividend on 750 shares of stock is \$2625 per year. What is the dividend on 284 shares of the same stock?
11. At the Quarterdeck Restaurant a banquet for 70 people costs \$1900 and a banquet for 130 people costs \$3100.
 - (a) Plot a graph of cost versus number of people.
 - (b) Determine an equation for this function and express it in the form $c = mn + b$ where c represents the cost and n the number of people.
 - (c) Determine the cost of a banquet for 91 people.
 - (d) What quantity does the slope of the line represent?
 - (e) What meaning does the y -intercept have?
12. Sylvia's salary as a commercial artist is \$18.60/h plus time and a half for overtime. Overtime is any time over 35 h/wk.
 - (a) Draw a graph of wages versus time.
 - (b) Determine the equations which define this function.
 - (c) If she works 42 h in one week, what is her salary before deductions?
13. In Culver City the property tax is \$2220 on a property with an assessed value of \$60 000.
 - (a) Write an equation for this direct variation function.
 - (b) What is the constant of proportionality?
 - (c) If the assessed value of a property is \$53 000, what is the property tax?
 - (d) If the property tax is \$1554, what is the assessed value of the property?

REVIEW AND PREVIEW TO CHAPTER 5

EXERCISE 1

1. Add

- | | | |
|---|---|--|
| (a) $\begin{array}{r} 3x+5y+7 \\ 4x+2y-6 \\ \hline \end{array}$ | (b) $\begin{array}{r} 3a-4b-6 \\ 2a-5b-7 \\ \hline \end{array}$ | (c) $\begin{array}{r} 4m+3n-3 \\ -4m+5n-7 \\ \hline \end{array}$ |
| (d) $\begin{array}{r} 4t-3x-8 \\ -8t+3x-11 \\ \hline \end{array}$ | (e) $\begin{array}{r} 7x+6-4y \\ 8x-3y+2 \\ \hline \end{array}$ | (f) $\begin{array}{r} 6a-5-4b \\ 7b-6a \\ \hline \end{array}$ |
| (g) $7x+3y-4$ and $2x-5y+6$ | | |
| (h) $3a-5-4b$ and $7b-3a-2$ | | |
| (i) $7m-3r-6$ and $8m-7+3r$ | | |
| (j) $\begin{array}{r} 4a-3b-7 \\ 6x+5b-8 \\ \hline \end{array}$ | (k) $\begin{array}{r} 3m-6t-4 \\ 3a+5b \\ \hline \end{array}$ | (l) $\begin{array}{r} 8x^2-7x-6 \\ 4x^2+10x-5 \\ \hline \end{array}$ |

2. Subtract.

- | | | |
|---|---|---|
| (a) $\begin{array}{r} 3x+5y+6 \\ 2x+3y+4 \\ \hline \end{array}$ | (b) $\begin{array}{r} 3a-2b-4 \\ 2a-3b+7 \\ \hline \end{array}$ | (c) $\begin{array}{r} 6m-3n-5 \\ 7m+2n+6 \\ \hline \end{array}$ |
| (d) $\begin{array}{r} 7x-6y+8 \\ 4x-6y+7 \\ \hline \end{array}$ | (e) $\begin{array}{r} 3a+4b+2 \\ 7a+4b-8 \\ \hline \end{array}$ | (f) $\begin{array}{r} -3m-4n-8 \\ -7m-2n-6 \\ \hline \end{array}$ |
| (g) $\begin{array}{r} 5x-4y+6 \\ 2x+6y \\ \hline \end{array}$ | (h) $\begin{array}{r} 8a-4+3b \\ 6a-b \\ \hline \end{array}$ | (i) $\begin{array}{r} 2x+7m \\ 3m-5-4x \\ \hline \end{array}$ |
| (j) $\begin{array}{r} 4a-5 \\ 3b-7a+6 \\ \hline \end{array}$ | (k) $\begin{array}{r} 7b \\ 4a-2b-6 \\ \hline \end{array}$ | (l) $\begin{array}{r} 7x^2-8x \\ 8x^2-5+3x \\ \hline \end{array}$ |

3. (a) Subtract $3a+7b-4$ from the sum of $4a-7b+6$ and $3a-11b-14$.
(b) Subtract $4x-6y-7$ from 0.
(c) From $3x+8y-2$ subtract the sum of $8x-5y+6$ and $3x-2y$.
(d) What expression added to $5x+3y+6$ equals $7x-5y+4$?
(e) What expression decreased by $4x^2-3x+1$ equals $2x^2-5x$?
(f) Subtract the sum of $4x-6x^2+7$ and $5x^2-2+3x$ from 0.
(g) Subtract the sum of $4a-3b$ and $-6a-2b$ from 1.

EXERCISE 2

1. Solve the following for x .

- | | |
|------------------------------|-------------------------------|
| (a) $x+5=14$ | (b) $x-7=-8$ |
| (c) $4x=12$ | (d) $3x-15=0$ |
| (e) $2x+7=19$ | (f) $5x-4=26+3x$ |
| (g) $2(x+1)-4=18$ | (h) $5(x-7)-3(x+5)=10$ |
| (i) $2(x-1)-(x+5)=37$ | (j) $15=3(x-5)+6$ |
| (k) $3(2x-1)=4(x-6)-5$ | (l) $3(x+6)-2x=1-(2x+3)$ |
| (m) $4(1-x)=5-(x+5)+11$ | (n) $2(1-2x)-(3x+5)=6-(3x-4)$ |
| (o) $5(x+3)-4+3(x-1)=8$ | (p) $5x-3(2x+1)-7=0$ |
| (q) $3(x+1)-4(x-3)-(x+5)=10$ | (r) $5=2(3x-1)-4(x+2)-11$ |

2. Solve the following.

(a) $\frac{x}{2} - \frac{x}{3} = 5$

(b) $\frac{x+1}{3} = 6$

(c) $\frac{x+2}{4} - \frac{x-1}{3} = 1$

(d) $\frac{x-1}{2} + x = 5$

(e) $\frac{x-6}{5} = \frac{x+3}{4} + 2$

(f) $\frac{x}{4} - \frac{x+1}{2} = 8$

(g) $5(x+1) - \frac{x+2}{2} = 3$

(h) $0.4(x-8) + 3 = 4$

(i) $0.5x - 0.1(x-3) = 4$

(j) $1.5(x-3) - 2(x-0.5) = 10$

(k) $1\frac{1}{2}x - \frac{x-4}{3} = 7$

(l) $1.2(10x-5) - 2(4x+7) = 8$

(m) $(x+2)(x-3) - x^2 = 7$

(n) $(x+5)(x-4) = (x+3)(x+2)$

(o) $2(x-1)(x+1) - (2x-1)(x+3) = 0$

(p) $(x-3)(x-4) - (x+2)(x+5) = 6$

(q) $3(x-1)(x+1) - 2x^2 = x(x+1) - 6$

(r) $2(1-x^2) + (x+3)(2x-5) = 8$

3. Solve the following for x.

(a) $x + y = 7$

(b) $y = 3x - 2$

(c) $3y - 2x = 6$

(d) $5 - 3x = 4y$

(e) $y = \frac{x+1}{2}$

(f) $y = \frac{2}{3}x - 4$

(g) $\frac{x}{2} + \frac{y}{3} = 1$

(h) $\frac{3}{2}x - 7 = 2y$

(i) $\frac{x+y}{2} = 5$

4. Solve the following for y.

(a) $3y - 2x = 4$

(b) $4x + 3y - 6 = 0$

(c) $3x - y + 7 = 0$

(d) $2x - 3y = 6$

(e) $\frac{1}{2}y - x = 6$

(f) $\frac{x+y}{3} = 5$

(g) $\frac{2x+3y}{7} = 4$

(h) $3y = \frac{x+y}{2}$

(i) $\frac{x-y}{3} = \frac{x+y}{7}$

5. Solve the following for the variable indicated.

(a) $A = \frac{1}{2}bh$; h

(b) $P = 2(l + w)$; w

(c) $E = I^2RT$; I

(d) $v = u + at$; t

(e) $s = ut + \frac{1}{2}at^2$; a

(f) $y = m(x - a)$; x

(g) $\frac{x}{a} + \frac{y}{b} = 1$; b

(h) $y = mx + b$; m

(i) $s = \frac{n}{2}(a + l)$; a

(j) $\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$; T_2

Linear Systems

Each problem that I solved became a rule which served afterwards to solve other problems.

Descartes

JOHN VON NEUMANN (1903–1957)



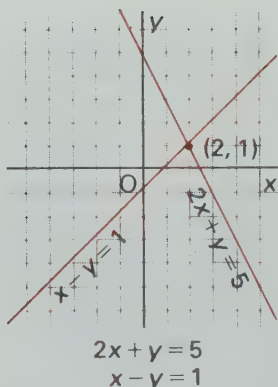
John von Neumann will be remembered as one of the most brilliant mathematicians of the twentieth century. He was born in Hungary and received his doctorate in mathematics from the University of Budapest at the age of 20. In 1930 he came to the United States to teach at Princeton University and joined Einstein as one of the permanent members of the Institute for Advanced Study when it was founded in 1933. Most mathematicians like to work out problems with pencil and paper, but von Neumann had a reputation for being able to solve complicated problems and perform lengthy calculations very quickly in his head. He could converse easily in seven languages. Von Neumann was a very versatile mathematician. Aside from his contributions to pure mathematics, he was one of the pioneers in such areas as linear programming, game theory, and the development of the computer and the atomic bomb.

Linear programming grew out of the problems of military supply in the Second World War, for example the problems of air-lifting forces and supplies and deploying aircraft strategically. In 1947 von Neumann was one of a group of mathematicians assembled by the United States Air Force to use mathematical methods to solve these problems. The results were the techniques of linear programming. It was then realized that these techniques, although developed for military problems, would be very useful in traffic analysis, nutrition, communications, business and industry.

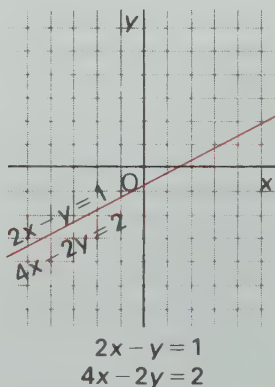
5.1 SYSTEMS OF LINEAR EQUATIONS IN TWO VARIABLES

When you graph two linear equations in two variables, the resulting lines may (a) intersect at one point, (b) coincide or (c) be parallel.

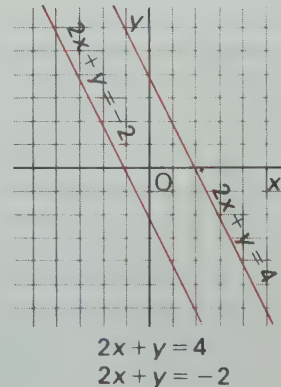
(a)



(b)



(c)



When two equations place two conditions on the same two variables at the same time, we say that they form a system of simultaneous equations. A solution of a system of equations is an ordered pair which satisfies both of the equations in the system.

Systems that have solutions, such as (a) and (b), are called *consistent*. A system, such as (a), that has one solution is called *independent*. A system, such as (b), that has infinitely many solutions is called *dependent*. A system, such as (c), that has no solution is called *inconsistent*. A system of simultaneous equations may be solved graphically.

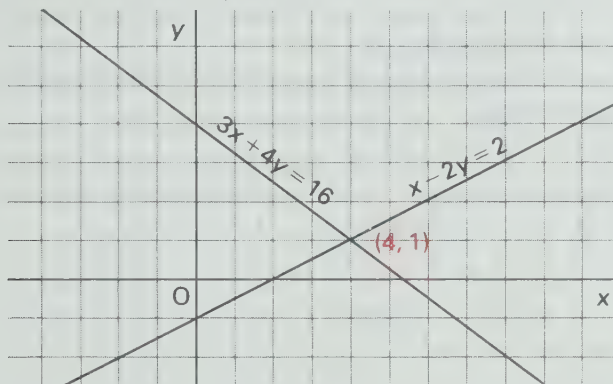
EXAMPLE 1. Solve the following system graphically.

$$x - 2y = 2 \quad \textcircled{1}$$

$$3x + 4y = 16 \quad \textcircled{2}$$

Solution

Draw the graph of each equation.



From the graph, the solution is the ordered pair (4, 1).

check in ①

$$\text{L.S.} = x - 2y$$

$$= 4 - 2(1)$$

$$= 4 - 2$$

$$= 2$$

$$\text{R.S.} = 2$$

check in ②

$$\text{L.S.} = 3x + 4y$$

$$= 3(4) + 4(1)$$

$$= 12 + 4$$

$$= 16$$

$$\text{R.S.} = 16$$

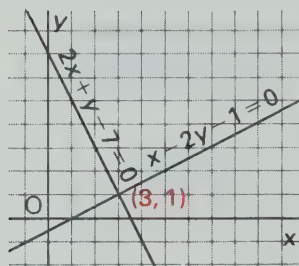
Graphing is not always the best way to find a solution, especially when the solutions are not integers. We will now investigate two algebraic methods of solving simultaneous equations. However there are two important facts that should be reviewed before this is done.

I MULTIPLES OF AN EQUATION

There are many ordered pairs that satisfy the equation $2x - y - 4 = 0$. The same ordered pairs satisfy $4x - 2y - 8 = 0$ or $2(2x - y - 4) = 0$. The equation $4x - 2y - 8 = 0$ is called a *multiple* of $2x - y - 4 = 0$.

In general, if (x_1, y_1) satisfies the equation $Ax + By + C = 0$, then it will also satisfy $k(Ax + By + C) = 0$, where k is any constant.

II EQUIVALENT SYSTEMS



Two systems are said to be *equivalent* if they have the same solution set. Consider the following system of equations.

$$2x + y - 7 = 0 \quad \text{①}$$

$$x - 2y - 1 = 0 \quad \text{②}$$

The ordered pair (3, 1) is the solution of this system.

If we add equations ① and ② we get

$$(2x + y - 7) + (x - 2y - 1) = 0.$$

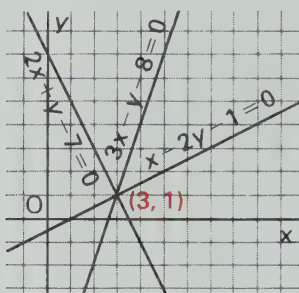
This equation is also satisfied by (3, 1) and may be simplified to

$$3x - y - 8 = 0.$$

In other words

$$\begin{cases} 2x + y - 7 = 0 \\ x - 2y - 1 = 0 \end{cases} \quad \begin{cases} 2x + y - 7 = 0 \\ 3x - y - 8 = 0 \end{cases} \quad \begin{cases} x - 2y - 1 = 0 \\ 3x - y - 8 = 0 \end{cases}$$

are *equivalent systems* since they have the same solution.



If we extend this idea to include the multiples of an equation then we obtain the following.

If $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ are the equations of two intersecting lines l_1 and l_2 , and h and k are arbitrary constants, then the equation

$$h(A_1x + B_1y + C_1) \pm k(A_2x + B_2y + C_2) = 0$$

represents a system of lines passing through the point of intersection of l_1 and l_2 .

Stated another way,

If you add or subtract any multiples of two equations whose graphs intersect, you will get another equation whose graph contains their point(s) of intersection.

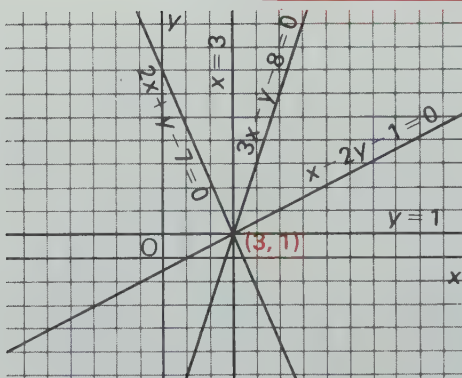


Figure 5-1

The set of all such sums or differences of multiples of two linear equations has as graph a *family of lines* with a common point. The lines shown in Figure 5-1 all belong to the family that contains (3, 1).

To solve a pair of linear equations by addition or subtraction, we choose multiples that eliminate one of the variables when the equations are added or subtracted. By doing this we can reduce the original system to an equivalent one that can be solved by inspection.

ALGEBRAIC METHOD I—SOLVING BY ADDITION OR SUBTRACTION

EXAMPLE 2. Solve the following system.

$$2x - 3y = 13 \quad \textcircled{1}$$

$$3x + 4y = -6 \quad \textcircled{2}$$

Solution

$$2x - 3y = 13 \quad \textcircled{1}$$

$$3x + 4y = -6 \quad \textcircled{2}$$

$$\textcircled{1} \times 4 \quad 8x - 12y = 52$$

$$\textcircled{2} \times 3 \quad 9x + 12y = -18$$

$$\begin{array}{r} \text{Adding} \quad 17x = 34 \\ \quad \quad \quad x = 2 \end{array}$$

Substituting $x = 2$ in $\textcircled{1}$

$$2(\mathbf{2}) - 3y = 13$$

$$-3y = 9$$

$$y = -3$$

We have now reduced the original system to an equivalent one, namely

$$x = 2$$

$$y = -3$$

From this system the solution set $\{(2, -3)\}$ can be easily read.

ALGEBRAIC METHOD II—SOLVING BY SUBSTITUTION

EXAMPLE 3. Solve the following system of equations

$$3x + y = 1 \quad \textcircled{1}$$

$$2x + 5y = 18 \quad \textcircled{2}$$

Solution

To eliminate a variable by substitution you (a) solve one of the equations for one of the variables and (b) substitute the result in the other equation.

$$3x + y = 1 \quad \textcircled{1}$$

$$2x + 5y = 18 \quad \textcircled{2}$$

Solving for y in $\textcircled{1}$

$$3x + y = 1$$

$$y = 1 - 3x$$

In how many ways can you spell the word mathematics starting from the top and working down through the array?



Substituting for y in ②

$$2x + 5y = 18$$

$$2x + 5(1 - 3x) = 18$$

$$2x + 5 - 15x = 18$$

$$-13x = 13$$

$$x = -1$$

Since

$$y = 1 - 3x$$

$$y = 1 - 3(-1)$$

$$= 1 + 3$$

$$= 4$$

check in ①

check in ②

$$\text{L.S.} = 3x + y$$

$$\text{L.S.} = 2x + 5y$$

$$= 3(-1) + (4)$$

$$= 2(-1) + 5(4)$$

$$= -3 + 4$$

$$= -2 + 20$$

$$= 1$$

$$= 18$$

$$\text{R.S.} = 1$$

$$\text{R.S.} = 18$$

$$\therefore x = -1, y = 4$$

EXERCISE 5-1

A 1. Solve the following for x and y

(a) $x + y = 12$

(b) $x + y = 9$

(c) $x + y = 10$

(d) $x - y = 2$

$x - y = 2$

$x - y = 1$

$x - y = 6$

$x + y = 6$

(e) $3x = 6$

(f) $x - y = 7$

(g) $x + 2y = 5$

(h) $x - y = 0$

$x + y = 7$

$2y = 4$

$x - 2y = 1$

$x + y = 10$

(i) $2x + y = 7$

(j) $x + 2y = 9$

(k) $x + 2y = 4$

(l) $x - 2y = 0$

$x + y = 4$

$x + y = 5$

$x - y = 1$

$3x + 2y = 16$

B 2. In each of the following determine whether the system is consistent and independent, consistent and dependent, or inconsistent.

(a) $2x + 3y = 12$

(b) $3x - 2y = 8$

(c) $x + 3y = 4$

$4x + 6y = 24$

$6x - 4y = 21$

$5x - y = 4$

(d) $2x + 5y = -1$

(e) $5x = 3y - 8$

(f) $2x = y - 14$

$2x - 3y = 7$

$15x - 9y + 24 = 0$

$6y - 11 = 12x$

3. Make up two inconsistent systems, two consistent and dependent systems, and two consistent and independent systems.

4. Solve, and check, the following systems of equations.

(a) $3x + 2y = 14$

(b) $2x + 3y = 41$

$5x - y = 6$

$4x + 5y = 71$

(c) $7x - 2y = -21$

(d) $3x - 2y = 3$

$8x - 3y = -19$

$4x + 3y = -30$

(e) $5x + 2y = 24$

(f) $2x - 3y = 3$

$2x + 3y = 25$

$3x - 5y = 3$

(g) $4x + 7y = 61$

(h) $3x - 5y = 33$

$3x - 2y = -5$

$2x + 7y = -40$

(i) $5x - 2y = 3$

(j) $3x + 5y = -37$

$2x - 7y = 26$

$4x - 7y = 19$

5. Solve, and check, the following systems of equations.

- | | |
|--|--|
| (a) $5x = 3y - 29$
$2x + 7y = 13$ | (b) $3x - 2y = 6$
$6x = 11 + 4y$ |
| (c) $3x = 5y - 21$
$2y = -4x - 28$ | (d) $5x - 7y + 71 = 0$
$8x - 9y + 96 = 0$ |
| (e) $13 = 4x - 3y$
$5x = 59 + 6y$ | (f) $y = 4x + 7$
$6x + 2y = 7$ |
| (g) $5x + 3y = -14$
$3x - 6y = -11$ | (h) $2x = 9 - 6y$
$6x + 18y - 27 = 0$ |
| (i) $4x - 5 = 15y$
$6x = 4 + 5y$ | (j) $3x - 7y = 41$
$5x - 3y = 51$ |

6. Solve each system by the substitution method.

- | | | |
|------------------------------------|--------------------------------------|--|
| (a) $2x + y = 5$
$3x - 4y = 2$ | (c) $5x - 2y = -4$
$4x + y = -11$ | (e) $3y - x + 10 = 0$
$3x + 4y = -22$ |
| (b) $x + 6y = 17$
$3x - 7y = 1$ | (d) $5x - y = 11$
$3x + 4y = 2$ | (f) $2x + 5y = 31$
$x + 2 = 0$ |

7. Solve the following systems of equations.

- | | |
|--|--|
| (a) $\frac{x}{2} + \frac{y}{4} = 4$
$\frac{x}{4} + \frac{y}{8} = 2$ | (b) $\frac{x}{2} - \frac{4y}{3} = 5$
$\frac{5x}{2} + y = 2$ |
| (c) $0.3x + 0.5y = 6$
$0.4x - y = -2$ | (d) $8x + 7y = 3.7$
$5x - 3y = 0.1$ |
| (e) $5x - 3y + 3.6 = 0$
$2x + 5y = -0.2$ | (f) $8x = 3y - 1.9$
$3x - 2 = 5y$ |
| (g) $0.6x - 0.5y = 3.8$

$1.4x + 0.1y = 3.8$ | (h) $\frac{x}{2} - \frac{y}{3} = -1$
$\frac{x}{4} + \frac{y}{2} = 5\frac{1}{2}$ |
| (i) $3(x - 1) - 4(y + 2) = -5$
$4(x + 5) - (y - 1) = 16$ | (j) $2(1 - 2x) - 3(1 - y) = -16$
$4(2y + 1) - 3(x + 3) = -22$ |
| (k) $\frac{x+3}{3} - \frac{y-1}{2} = 0$
$\frac{x-6}{3} + \frac{y+1}{6} = 0$ | (l) $\frac{x-1}{4} - \frac{y-3}{5} = 2$
$\frac{x+1}{3} + \frac{y+6}{2} = 4$ |

8. Determine the vertices of the triangle whose sides lie on the lines $x + 3y = 13$, $x - y = 1$ and $5x - y = -15$.

C 9. Solve the following systems of equations.

- | | |
|---|--|
| (a) $\frac{1}{x} + \frac{1}{y} = 5$
$\frac{2}{x} - \frac{3}{y} = -5$ | (b) $\frac{2}{x} - y = 5$
$\frac{1}{x} + 2y = 10$ |
| (c) $\frac{3}{y} - \frac{2}{x} = 8$

$\frac{4}{x} - \frac{2}{y} = -8$ | (d) $\frac{3}{2} - \frac{4}{1} = 17$
$\frac{1}{x} - \frac{2}{y} = -5$ |

5.2 APPLICATIONS OF LINEAR SYSTEMS IN TWO VARIABLES

Many practical problems can be expressed as a system of linear equations in two variables.

The following terms are used in problems involving aircraft.

air speed: the speed of the plane in still air.

ground speed: the speed of the plane in relation to the ground.

wind speed: the speed of the wind in relation to the ground.

tail wind: a wind blowing in the same direction as the plane is flying.

head wind: a wind blowing in the opposite direction to that in which the plane is flying.

EXAMPLE 1. *With a tail wind a plane took 4 h to fly 1920 km. Flying back against the same wind and with the same air speed the plane took 1 h longer. Find the wind speed and the plane's air speed.*

Solution

Let x be the plane's air speed (in km/h)

Let y be the wind speed (in km/h)

We use the relationship Distance = Speed $\times$ Time, $D = ST$, and arrange the given facts in a chart.

	Distance (km)	Ground Speed (km/h)	Time (h)
with tail wind	1920	$x + y$	4
with head wind	1920	$x - y$	5

The distance with a tail wind is 1920 km

$$4(x + y) = 1920 \quad \text{①}$$

The distance with a head wind is 1920 km

$$5(x - y) = 1920 \quad \text{②}$$

$$\text{①} \div 4$$

$$x + y = 480 \quad \text{①}$$

$$\text{②} \div 5$$

$$x - y = 384 \quad \text{②}$$

Adding

$$2x = 864$$

$$x = 432$$

Substitute in ①

$$432 + y = 480$$

$$y = 48$$

The air speed is 432 km/h and the wind speed is 48 km/h.

check: When flying with the wind the plane's ground speed is $432 + 48 = 480$ km/h and the time to travel 1920 km is $\frac{1920}{480} = 4$ h.
 When flying against the wind the plane's ground speed is $432 - 48 = 384$ km/h and the time to travel 1920 km is $\frac{1920}{384} = 5$ h.

EXAMPLE 2. A certain alloy contains 9% silver. Another alloy is 14% silver. How much of each type should be combined to make 10 kg of an alloy that contains 12% silver?

Solution

Let x represent the amount of 9% alloy used.

Let y represent the amount of 14% alloy used.

$$x + y = 10 \quad \text{① alloy equation}$$

$$0.09x + 0.14y = (0.12)10 \quad \text{② silver equation}$$

$$\text{①} \times 9 \quad \quad \quad 9x + 9y = 90 \quad \text{①}$$

$$\text{②} \times 100 \quad \quad \quad 9x + 14y = 120 \quad \text{②}$$

$$\begin{array}{r} \text{subtracting} \\ \hline -5y = -30 \\ y = 6 \end{array}$$

substituting in ①

$$x + 6 = 10$$

$$x = 4$$

4 kg of 9% alloy and 6 kg of 14% alloy should be combined to give 10 kg of 12% alloy.

check: There must be 10 kg of alloy

$$4 + 6 = 10$$

There must be 1.2 kg of silver in the alloy

$$0.09(4) + 0.14(6)$$

$$= 0.36 + 0.84$$

$$= 1.2$$

Put the numbers from 1 to 9 in the spaces to make the statements true.

$$\square - \square \div \square = 1$$

$$\square - \square \times \square = 1$$

$$\square + \square - \square = 1$$

EXERCISE 5-2

- B**
- Find two numbers whose sum is 763 and whose difference is 179.
 - Find two numbers whose sum is 947 and whose difference is 177.
 - The sum of two numbers is 181. Three times the larger plus twice the smaller is 459. Find the numbers.
 - Five times the larger of two numbers plus 4 times the smaller is 271. Three times the larger less twice the smaller is 57. Find the numbers.
 - Six times the larger of two numbers less 3 times the smaller is 270. Five times the larger less twice the smaller is 261. Find the numbers.

6. If the difference in the measure of two complementary angles is 24° , what are the measures of the angles?
7. If the difference in the measure of two supplementary angles is 62° , what are the measures of the angles?
8. Jill had \$12 100 to invest. She invested part of it in bonds paying 8%/a and the remainder in a second mortgage paying 9%/a. After one year the total income from the investments was \$1043. How much was invested at each rate?
9. Terry invested her inheritance of \$335 000, part at 7%/a and the remainder at 10%/a. After one year the total interest from these investments was \$24 500. How much did she invest at each rate?
10. A parking lot contained 102 vehicles (cars and buses). Each car is charged \$3 and each bus \$10. The total revenue was \$418. How many buses were on the lot?
11. During March, stock X gained or lost x dollars in value per share and stock Y , y dollars in value per share. On stocks X and Y , two shareholders realized the following loss and gain during April.

Shareholder	Shares of X	Shares of Y	Monthly Gain or Loss
Paterson	50	100	-\$50
Santini	100	120	\$60

Find x and y .

12. During September, stock M gained or lost m dollars in value per share and stock N , n dollars in value per share. On stocks M and N , two shareholders realized the following gain and loss during September.

Shareholder	Shares of M	Shares of N	Monthly Gain or Loss
MacDonald	210	130	\$450
Brown	60	90	-\$30

Find m and n .

13. During November, stock R gained or lost r dollars in value per share and stock S , s dollars in value per share. On stocks R and S , two shareholders realized the following loss and gain during November.

Shareholder	Shares of R	Shares of S	Monthly Gain or Loss
Evers	130	60	-\$69
Schultz	70	90	\$84

Find r and s .

14. Two bottles contain hydrochloric acid, one 40% strength and the other 30% strength. How much must be taken from each bottle to make 20 L of 34% strength?

15. A chemist wants to get 100 L of 47% alcohol solution by volume by mixing 40% alcohol solution with 50% alcohol solution. How many litres of each should he use?

16. A chemical supply company received an order for 1500 L of 28% salt solution. To fill the order it was necessary to mix 40% salt solution with 20% salt solution. How many litres of each should be mixed?

17. With a certain tail wind a jet aircraft takes 3 h to travel 1890 km. Flying against the same wind, the plane makes the return trip in 3.5 h. Find the wind speed and the plane's speed.

18. A private plane took 2 h to make a 600 km trip when flying with the wind. The return trip, flying against the same wind, took 2.5 h. Find the speed of the wind and the plane's air speed.

19. It took 3 h for a Coast Guard patrol boat to travel 48 km up a river and 2 h for the return trip. Find the speed of the current in the river and the speed of the boat in stillwater.

20. Find A and B so that the graph of $Ax + By = 20$ contains the points $(2, 4)$ and $(-1, 8)$.

21. Find A and B so that the graph of $Ax + By = -10$ contains the points $(-2, 1)$ and $(2, 9)$.

22. Find m and b so that the graph of $y = mx + b$ contains the points $(2, 2)$ and $(-4, -16)$.

23. Find m and b so that the graph of $y = mx + b$ contains the points $(2, 5)$ and $(-6, 1)$.

24. Find a and c so that the graph of $y = ax^2 + c$ contains the points $(2, 8)$ and $(-1, -1)$.

C 25. Find two numbers whose sum is m and whose difference is n .

26. Two kilometres upstream from his starting point, a rower passed a raft floating with the current. He rowed upstream for one more hour, and then rowed back and reached his starting point just as the raft arrived. Find the speed of the current.

5.3 SYSTEMS WITH NON-NUMERICAL SOLUTIONS

In some systems of equations the coefficients of the variables for which you are to solve may not be numerical. These systems are solved in the same way as you solve "numerical systems", but the solutions are found in terms of the other variables, called parameters, in the equations.

EXAMPLE 1. Solve the following system for x and y in terms of c and d .

$$cx + dy = 2cd \quad \text{①}$$

$$dx + cy = c^2 + d^2 \quad \text{②}$$

Solution

$$cx + dy = 2cd \quad \text{①}$$

$$dx + cy = c^2 + d^2 \quad \text{②}$$

$$\text{①} \times c$$

$$c^2x + cdy = 2c^2d$$

$$\text{②} \times d$$

$$d^2x + cdy = c^2d + d^3$$

subtracting

$$c^2x - d^2x = c^2d - d^3$$

factoring

$$x(c^2 - d^2) = d(c^2 - d^2)$$

$$x = d \text{ (when } c \neq \pm d)$$

substitute

$$x = d \text{ in ①}$$

$$cx + dy = 2cd$$

$$c(d) + dy = 2cd$$

$$cd + dy = 2cd$$

$$dy = cd$$

$$y = c \text{ (when } d \neq 0)$$

check in ①

check in ②

$$\text{L.S.} = cx + dy$$

$$\text{L.S.} = dx + cy$$

$$= c(d) + d(c)$$

$$= d(d) + c(c)$$

$$= cd + cd$$

$$= d^2 + c^2$$

$$= 2cd$$

$$= c^2 + d^2$$

$$\text{R.S.} = 2cd$$

$$\text{R.S.} = c^2 + d^2$$

$$\therefore x = d \text{ and } y = c$$

The 8 L jug is filled with wine. The 5 L and 3 L jugs are empty. Divide the wine into two equal parts by decanting.



EXERCISE 5-3

A 1. If we wish to eliminate y by addition or subtraction, state what each equation should be multiplied by in the following systems.

$$(a) \quad x - 2y = a \quad \text{①}$$

$$(b) \quad 3x + 4y = m \quad \text{①}$$

$$x + 3y = b \quad \text{②}$$

$$5x + 6y = n \quad \text{②}$$

$$(c) \quad ax + by = c \quad \text{①}$$

$$(d) \quad mx - ny = q \quad \text{①}$$

$$dx + ey = f \quad \text{②}$$

$$nx + my = r \quad \text{②}$$

B 2. Solve the following systems for x and y in terms of the other variables in the equations.

$$(a) \quad x - y = a + b$$

$$(b) \quad x + y = a$$

$$x + y = a - b$$

$$x - y = b$$

$$(c) \quad 2x + 3y = 13a$$

$$(d) \quad ax + by = 2$$

$$3x - 4y = -6a$$

$$ax - by = 0$$

$$(e) \quad ax + by = a^2 - b^2$$

$$(f) \quad dx + cy = -2cd$$

$$bx - ay = 2ab$$

$$cx + dy = -c^2 - d^2$$

$$(g) \quad mx + ny = m^2 + n^2$$

$$(h) \quad ax - by = a^2 + b^2$$

$$my - nx = m^2 + n^2$$

$$x - y = 2a$$

$$\begin{aligned} \text{(i)} \quad mx + ny &= t \\ rx + sy &= w \end{aligned}$$

$$\begin{aligned} \text{(j)} \quad a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

3. (a) Solve for v and x

$$av + bx = 1$$

$$cv + dx = 0$$

(b) Solve for w and y

$$aw + by = 0$$

$$cw + dy = 1$$

5.4 SYSTEMS OF LINEAR EQUATIONS IN THREE OR MORE VARIABLES

The *ordered triple* $(4, 2, -1)$ is a solution of the equation $2x + 3y - 4z = 18$ since

$$2(4) + 3(2) - 4(-1) = 18 \text{ is a true statement}$$

Some other solutions of this equation are $(9, 0, 0)$, $(5, 0, -2)$ and $(0, 2, -3)$. There are many others.

In solving problems it is sometimes necessary to solve systems of 3 equations in three variables. The method is similar to the method of elimination by addition or subtraction that was used in solving two equations in two variables.

EXAMPLE 1. Solve

$$3x - 4y + 5z = 2 \quad \textcircled{1}$$

$$4x + 5y - 3z = -5 \quad \textcircled{2}$$

$$5x - 3y + 2z = -11 \quad \textcircled{3}$$

Solution

We first reduce the system of 3 equations in 3 variables to a system of 2 equations in 2 variables by taking two different pairs of equations and eliminating the same variable from each. It doesn't matter which variable you decide to eliminate first.

Eliminating y from $\textcircled{1}$ and $\textcircled{2}$

$$\begin{array}{rcl} \textcircled{1} \times 5 & 15x - 20y + 25z & = 10 \\ \textcircled{2} \times 4 & 16x + 20y - 12z & = -20 \\ \hline \text{adding to get } \textcircled{4} & 31x + 13z & = -10 \quad \textcircled{4} \end{array}$$

Eliminating y from $\textcircled{2}$ and $\textcircled{3}$

$$\begin{array}{rcl} \textcircled{2} \times 3 & 12x + 15y - 9z & = -15 \\ \textcircled{3} \times 5 & 25x - 15y + 10z & = -55 \\ \hline \text{adding to get } \textcircled{5} & 37x + z & = -70 \quad \textcircled{5} \end{array}$$

We have now reduced the system to one of two equations in two variables, which we can solve by the methods of section 5.1.

$$\begin{array}{rcl} & 31x + 13z & = -10 \quad \textcircled{4} \\ & 37x + z & = -70 \quad \textcircled{5} \\ & \hline & 31x + 13z & = -10 \quad \textcircled{4} \\ \textcircled{5} \times 13 & 481x + 13z & = -910 \quad \textcircled{5} \\ \hline \text{subtracting} & -450x & = 900 \\ & x & = -2 \end{array}$$

substituting $x = -2$ in ⑤

$$\begin{aligned}37x + z &= -70 \\37(-2) + z &= -70 \\-74 + z &= -70 \\z &= 4\end{aligned}$$

substituting $x = -2$ and $z = 4$ in ①

$$\begin{aligned}3x - 4y + 5z &= 2 \\3(-2) - 4y + 5(4) &= 2 \\-6 - 4y + 20 &= 2 \\-4y &= -12 \\y &= 3\end{aligned}$$

check in ①

$$\begin{aligned}\text{L.S.} &= 3x - 4y + 5z \\&= 3(-2) - 4(3) + 5(4) \\&= -6 - 12 + 20 \\&= 2 \\ \text{R.S.} &= 2\end{aligned}$$

check in ②

$$\begin{aligned}\text{L.S.} &= 4x + 5y - 3z \\&= 4(-2) + 5(3) - 3(4) \\&= -8 + 15 - 12 \\&= -5 \\ \text{R.S.} &= -5\end{aligned}$$

check in ③

$$\begin{aligned}\text{L.S.} &= 5x - 3y + 2z \\&= 5(-2) - 3(3) + 2(4) \\&= -10 - 9 + 8 \\&= -11 \\ \text{R.S.} &= -11 \\x &= -2, y = 3 \text{ and } z = 4.\end{aligned}$$

Systems of 4 or more variables may be solved in the same manner.

EXERCISE 5-4

- A** 1. Is the given ordered triple a solution of the given equation?
- | | |
|--------------------------------------|--------------------------------------|
| (a) $3x + y + 2z = 6$; (1, 1, 1) | (b) $2x - 3y + z = 7$; (3, 0, 1) |
| (c) $2x + 3y - z = 10$; (3, 2, 1) | (d) $4x + 5y + 2z = 5$; (2, 0, -2) |
| (e) $4x - 2y + 3z = -9$; (0, 3, -1) | (f) $5x + 3y - 2z = 6$; (2, -1, 0) |
| (g) $2x - 3y - 4z = 8$; (1, -1, -1) | (h) $2x - 3y - z = -5$; (-2, 1, -2) |
- B** 2. Solve the following systems of equations.
- | | |
|--|--|
| (a) $x + y + 3z = 12$
$x - y + 4z = 11$
$2x + y + 3z = 13$ | (b) $3x + 2y + z = 14$
$4x + 3y + 2z = 20$
$5x + 4y + 3z = 26$ |
| (c) $4x + 3y - z = -7$
$3x - 2y + 3z = -10$
$x + y - z = -2$ | (d) $3x + 5y - z = 47$
$2x - y + 3z = -2$
$4x + y - 2z = 30$ |
| (e) $3x - 4y + 5z = 26$
$6x - 2y - 3z = -39$
$x + 3y - 4z = -31$ | (f) $4x - 3y - 2z = 31$
$5x - y - 4z = 28$
$7x - 3y + 5z = 26$ |

(g) $7x - 5y + 4z = 57$

$9x + 3y - 5z = 45$

$8x - 4y + 7z = 60$

(i) $3x - 2y + z + 3 = 0$

$5x + 3y = 26 - 2z$

$4x + 5z = 2y - 8$

(k) $4x - 3y + 6z = -9$

$2x + 4y - 3z = -10$

$3x + 2y - 4z = -11$

(h) $4r + 3s - 2t = -21$

$5r - s - t = -15$

$3r - 2s + 5t = -16$

(j) $3x + 2y - 3z = -4$

$5x + y + 2z = -16$

$2x - 3y + z = 0$

(l) $2x + 3y - z = 10$

$5x - 2y + 3z = 3$

$4x + 2y - 5z = 32$

3. Solve the following.

(a) $3x - 2y + 5z = 1$

$4x + 5y - 3z = 17$

$7x - 3y + 2z = 36$

(c) $4x - 3y + 8z = 5$

$6x + 9y + 16z = 4$

$3x - 12y - 2z = 5$

(e) $3x + 2y = 10$

$3y - 4z = -4$

$5x - 2z = 34$

(g) $4x - 2y + 3z = 0.4$

$2x - 5y + z = 1.5$

$3x - 4y - 2z = 1.9$

(b) $4x - y + 3z = -11$

$6x - 3y - 2z = -3$

$8x - 2y + 5z = -19$

(d) $x + y = 4$

$y + z = -1$

$x - z = 5$

(f) $0.4x + 0.6y - 0.3z = 1.2$

$0.3x - 0.4y + 0.5z = 2.1$

$1.1x - 0.2y - 0.1z = 2.5$

(h) $3w + 2x - 3y + z = 1$

$4w - x + 2y + 3z = 19$

$2w + 3x - 4y + 2z = -1$

$w - 4x + y - 4z = -7$

(i) $3w + 2x - 3y + z = 2$

$4w - 3x - y - z = -3$

$3w - 2x + 2y - 4z = -14$

$w - 3x + 2y - 3z = -8$

(j) $\frac{x}{4} + \frac{y}{3} - \frac{z}{2} = 8$

$\frac{x}{2} - \frac{y}{2} + \frac{z}{6} = 2$

$\frac{x}{6} - \frac{y}{3} + \frac{z}{3} = -2$

Find two different whole numbers x and y so that

$x^y = y^x$

4. During July, stock X gained or lost x dollars in value per share, stock Y , y dollars in value per share, and stock Z , z dollars in value per share. On stock X , Y and Z , three shareholders realized the following losses or gains during July.

Shareholder	Shares of X	Shares of Y	Shares of Z	Monthly Gain or Loss
McWilliams	100	150	80	-\$170
Baswick	60	110	70	-\$140
Siena	90	40	100	\$160

Find x , y and z .

5. During March, stock R gained or lost r dollars in value per share, stock S , s dollars in value per share and stock T , t dollars in value per share. Three shareholders realized the following losses or gains during March.

Shareholder	Shares of <i>R</i>	Shares of <i>S</i>	Shares of <i>T</i>	Monthly Gain or Loss
Mathews	80	120	50	\$330
Kwong	60	130	70	\$230
Yardley	50	80	120	-\$50

Find r , s and t .

6. Find a , b and c so that the graph of $y = ax^2 + bx + c$ contains the points $(2, 0)$, $(-1, 6)$ and $(3, 2)$.

7. Find a , b and c so that the graph of $y = ax^2 + bx + c$ contains the points $(1, 0)$, $(-1, -6)$ and $(2, 9)$.

8. A 960 m^2 parking lot was filled with a total of 103 cars, vans and buses. Cars were charged \$3, vans \$4 and buses \$20 and the total revenue was \$380. If cars require 8 m^2 of space, vans 10 m^2 and buses 40 m^2 , how many vans were on the lot?

C 9. Solve the following system.

$$u + v + w + x + y = 2$$

$$v + w + x + y + z = 1$$

$$w + x + y + z + u = 0$$

$$x + y + z + u + v = -1$$

$$y + z + u + v + w = 7$$

$$z + u + v + w + x = 6$$

5.5 BASIC MATRIX OPERATIONS

We have seen that systems of equations may involve several variables. Many problems that businesses, governments, and scientific research organizations deal with today use systems of equations in many variables. Solving problems like this can become complicated, so it has become necessary to find convenient ways of solving them.

To solve any system of equations we must work with rows and columns. In presenting information generally, we often use tables, graphs, and designs located in terms of rows and columns. In working with these rectangular displays of information, it is often necessary to create a new display that is a modification of the first one. This may involve adding, subtracting, multiplying, or dividing, modifying the first information with new information. The method of performing these mathematical operations on rectangular displays is called matrix algebra. It offers ways of solving systems of equations and of working with information that we can use ourselves or can instruct computers to use.

A *matrix* is a rectangular array of numbers enclosed by parentheses. Examples of matrices are:

$$A = \begin{pmatrix} 3 & 1 & 4 \\ 5 & -2 & 0 \end{pmatrix} \quad B = \begin{pmatrix} -5 & 2 \\ -3 & -1 \end{pmatrix} \quad C = \begin{pmatrix} 4 & -1 \\ 1 & 0 \\ -3 & 2 \end{pmatrix}$$

$$D = (-5 \quad 6 \quad 3 \quad -1)$$

The individual numbers in a matrix are called the *entries* or *elements* of the matrix.

The number of rows (horizontal) and columns (vertical) determine the *dimensions* of a matrix. For the examples of matrices given above, the matrix A has dimensions 2×3 (read "two by three"). The matrix B is a 2×2 matrix. The matrices C and D are 3×2 and 1×4 respectively. Notice that the number of rows is always given first.

A matrix having only one row is called a *row matrix*.

A matrix with only one column is called a *column matrix*.

If a matrix has the same number of rows and columns we call it a *square matrix*.

To add two matrices we just add the corresponding elements.

$$\begin{pmatrix} 3 & -2 & 6 \\ 5 & -7 & 0 \end{pmatrix} + \begin{pmatrix} -3 & 6 & 1 \\ 5 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 3+(-3) & -2+6 & 6+1 \\ 5+5 & -7+0 & 0+5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 4 & 7 \\ 10 & -7 & 5 \end{pmatrix}$$

You can add two matrices only if they are of the same dimensions. It can be shown that addition of matrices is commutative and associative.

The negative of a matrix M is the matrix $-M$, each of whose entries is the negative of the corresponding entry in M .

$$\text{If } M = \begin{pmatrix} 3 & -5 \\ -6 & 2 \end{pmatrix}, \text{ then } -M = \begin{pmatrix} -3 & 5 \\ 6 & -2 \end{pmatrix}$$

As with real numbers, we define subtraction of matrices in terms of addition.

$$\begin{pmatrix} 3 & -5 \\ 2 & -4 \end{pmatrix} - \begin{pmatrix} 6 & 7 \\ -3 & -4 \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ 2 & -4 \end{pmatrix} + \begin{pmatrix} -6 & -7 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -3 & -12 \\ 5 & 0 \end{pmatrix}$$

$$A - B = A + (-B)$$

Multiplication of a matrix by a real number is the operation that multiplies every entry of the matrix by the number.

$$3 \begin{pmatrix} 4 & -2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3(4) & 3(-2) \\ 3(1) & 3(0) \end{pmatrix} = \begin{pmatrix} 12 & -6 \\ 3 & 0 \end{pmatrix}$$

It may seem that multiplication of matrices should be done in a manner similar to that of addition of matrices. However, experience has shown that the way which is most useful is what is often called *row-by-column* multiplication.

The product of two matrices exists only if the first matrix has the same number of columns as the number of rows in the second matrix.

You obtain the entries of the product matrix by adding the products of the elements of a row of the first matrix with the corresponding elements of a column of the second matrix. The product of two 2×2 matrices is illustrated below.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

EXAMPLE 1. Find the product:

$$\begin{pmatrix} 3 & 2 \\ -1 & -3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -3 & 2 \\ 4 & -2 \end{pmatrix}$$

Solution

Reason

$$\begin{pmatrix} 3 & 2 \\ -1 & -3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -3 & 2 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -9 & 4 \\ -3 & 2 \end{pmatrix} \quad \begin{matrix} 3(-3)+2(4)=-1 \\ 3(2)+2(-2)=2 \end{matrix}$$

$$\begin{pmatrix} 3 & 2 \\ -1 & -3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -3 & 2 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -9 & 4 \\ -3 & 2 \end{pmatrix} \quad \begin{matrix} -1(-3)+(-3)(4)=-9 \\ -1(2)+(-3)(-2)=4 \end{matrix}$$

$$\begin{pmatrix} 3 & 2 \\ -1 & -3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -3 & 2 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -9 & 4 \\ -3 & 2 \end{pmatrix} \quad \begin{matrix} 1(-3)+0(4)=-3 \\ 1(2)+0(-2)=2 \end{matrix}$$

$$\begin{pmatrix} 3 & 2 \\ -1 & -3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -3 & 2 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -9 & 4 \\ -3 & 2 \end{pmatrix} \quad \begin{matrix} 1(-3)+0(4)=-3 \\ 1(2)+0(-2)=2 \end{matrix}$$

$$\begin{pmatrix} 3 & 2 \\ -1 & -3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -3 & 2 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -9 & 4 \\ -3 & 2 \end{pmatrix} \quad \begin{matrix} 1(-3)+0(4)=-3 \\ 1(2)+0(-2)=2 \end{matrix}$$

$$\begin{pmatrix} 3 & 2 \\ -1 & -3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -3 & 2 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -9 & 4 \\ -3 & 2 \end{pmatrix} \quad \begin{matrix} 1(-3)+0(4)=-3 \\ 1(2)+0(-2)=2 \end{matrix}$$

Using a piece of paper, make a "road map" with the sections numbered as shown.

1	8	7	4
2	3	6	5

Now fold it so that the numbers are in order 1 to 8.

EXAMPLE 2. Find AB and BA when $A = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}$.

Solution

$$AB = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} -11 & 16 \\ -1 & 2 \end{pmatrix}$$

$$BA = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ -2 & -9 \end{pmatrix}$$

Notice that $AB \neq BA$.

In general, multiplication of matrices is not commutative.

An **identity matrix**, I , is a square matrix which, when multiplied by another square matrix M , leaves the matrix M unchanged. It can be shown that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the 2×2 identity matrix.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

EXERCISE 5-5

A 1. State the dimensions of each matrix.

(a) $\begin{pmatrix} 3 & 5 & 2 \\ 4 & 0 & -6 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 4 \\ -6 & 5 \\ -3 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ (d) $(5 \ 6 \ 8 \ -4)$

(e) (7) (f) $\begin{pmatrix} 2 & 3 \\ -1 & 6 \end{pmatrix}$ (g) $(1 \ 4)$ (h) $\begin{pmatrix} 4 & 6 & -9 \\ 2 & 8 & 0 \\ 1 & -1 & 2 \end{pmatrix}$

2. State the indicated sum, difference, or product.

(a) $\begin{pmatrix} 5 & 3 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 5 & -2 \end{pmatrix}$ (b) $\begin{pmatrix} 6 & 8 \\ 5 & 7 \end{pmatrix} - \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$ (c) $2 \begin{pmatrix} 3 & 5 \\ -2 & 4 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & -4 \\ 2 & 0 \\ -5 & -2 \end{pmatrix} + \begin{pmatrix} 5 & -6 \\ -3 & 4 \\ 1 & 3 \end{pmatrix}$ (e) $-3 \begin{pmatrix} 4 & 2 & -5 \\ 1 & -3 & 0 \end{pmatrix}$

(f) $\begin{pmatrix} 6 & -4 \\ -3 & 0 \end{pmatrix} - \begin{pmatrix} 5 & -2 \\ -7 & 5 \end{pmatrix}$

B 3. Express as a single matrix.

(a) $3 \begin{pmatrix} 4 & 2 \\ -1 & 4 \end{pmatrix} + \begin{pmatrix} -5 & 3 \\ 8 & 2 \end{pmatrix}$ (b) $2 \begin{pmatrix} 1 & 3 \\ 6 & -4 \\ -2 & 7 \end{pmatrix} - \begin{pmatrix} -5 & 2 \\ 4 & -3 \\ -6 & -5 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & -1 & 4 \\ 8 & 2 & -2 \end{pmatrix} + 3 \begin{pmatrix} 0 & 5 & -2 \\ 4 & -1 & -1 \end{pmatrix}$ (d) $5 \begin{pmatrix} 4 & 2 \\ -1 & 3 \end{pmatrix} - 4 \begin{pmatrix} 6 & -7 \\ 2 & 5 \end{pmatrix}$

(e) $2 \begin{pmatrix} 1 \\ 4 \\ -6 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - 4 \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$ (f) $-2 \begin{pmatrix} 1 & 7 \\ -3 & 5 \end{pmatrix} - \begin{pmatrix} 6 & -3 \\ -8 & 2 \end{pmatrix}$

4. Write each product as a single matrix, where possible.

(a) $\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 5 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 5 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ (c) $\begin{pmatrix} 6 & -2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 1 \\ 3 & 4 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 4 & -5 \\ 1 & 2 \end{pmatrix}$ (e) $(3 \ 2 \ -1) \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}$ (f) $\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 4 & -1 & -3 \\ 1 & 2 & 4 \end{pmatrix}$

(g) $\begin{pmatrix} 3 & 1 & -2 \\ 4 & 2 & 1 \\ 3 & 5 & -4 \end{pmatrix} \begin{pmatrix} 5 & -1 & -2 \\ 3 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ (h) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & -2 \\ 4 & 3 \end{pmatrix}$ (i) $\begin{pmatrix} 1 \\ 0 \\ 6 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}$

(j) $\begin{pmatrix} 2 & 3 & 1 \\ -1 & 0 & 2 \\ 4 & -3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (k) $\begin{pmatrix} 6 & 1 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix}$ (l) $\begin{matrix} 5 & -3 \\ 1 & 2 \end{matrix} (4 \ 6)$

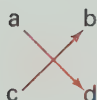
$$(m) \begin{pmatrix} -2 & -4 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ -2 & 3 \end{pmatrix} \quad (n) \begin{pmatrix} 2 & -1 \\ -3 & -4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad (o) \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} \begin{pmatrix} 3 & 1 & 0 \end{pmatrix}$$

C 5. Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ -1 & -2 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}$

- Find AB and BA .
- Find $(A+B)C$ and $AC+BC$.
- Find $(AB)C$ and $A(BC)$.
- Find $(A+B)(A-B)$ and A^2-B^2 .
- Find $(A+B)^2$ and $A^2+2AB+B^2$.
- Find $(A-B)^2$ and $A^2-2AB+B^2$.

5.6 DETERMINANTS AND INVERSES OF 2×2 MATRICES

Associated with each square matrix M is a number called the *determinant* of M .



$$\text{If } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ then } \det M = ad - bc$$

EXAMPLE 1. If $M = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ and $N = \begin{pmatrix} -4 & -3 \\ 2 & 1 \end{pmatrix}$, find $\det M$ and $\det N$.

Solution

$$\det M = 3(2) - (1)(5) = 6 - 5 = 1$$

$$\det N = (-4)(1) - (-3)(2) = -4 + 6 = 2$$

Two real numbers whose product is 1 are said to be multiplicative inverses of each other. Similarly, any two matrices M and N such that $MN = NM = I$ are called *inverses*. We usually denote N by M^{-1} , which is called the inverse of M .

EXAMPLE 2. Find the inverse of $M = \begin{pmatrix} 2 & -1 \\ 5 & -3 \end{pmatrix}$

Solution

$$\text{Let } M^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Since a matrix times its inverse equals the identity matrix

$$\begin{pmatrix} 2 & -1 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Using the definition of matrix multiplication

$$\begin{pmatrix} 2 & -1 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2a - c & 2b - d \\ 5a - 3c & 5b - 3d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

If we now place corresponding entries equal, we have

$$2a - c = 1 \quad \textcircled{1} \quad \text{and} \quad 2b - d = 0 \quad \textcircled{3}$$

$$5a - 3c = 0 \quad \textcircled{2} \quad \text{and} \quad 5b - 3d = 1 \quad \textcircled{4}$$

$$\textcircled{1} \times 3 \quad 6a - 3c = 3 \quad \textcircled{3} \times 3 \quad 6b - 3d = 0$$

$$5a - 3c = 0 \quad 5b - 3d = 1$$

$$\text{subtracting} \quad a = 3 \quad \text{subtracting} \quad b = -1$$

$$\text{substituting in } \textcircled{2} \quad \text{substituting in } \textcircled{4}$$

$$5(\mathbf{3}) - 3c = 0 \quad 5(\mathbf{-1}) - 3d = 1$$

$$15 - 3c = 0 \quad -5 - 3d = 1$$

$$-3c = -15 \quad -3d = 6$$

$$c = 5 \quad d = -2$$

$$\therefore M^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix}$$

The result is easily verified by showing that

$$\begin{pmatrix} 2 & -1 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We use a similar procedure to find the inverse of

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We let

$$M^{-1} = \begin{pmatrix} v & w \\ x & y \end{pmatrix}$$

Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v & w \\ x & y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} av + bx & aw + by \\ cv + dx & cw + dy \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

By equating corresponding entries, we have

$$av + bx = 1 \quad aw + by = 0$$

$$cv + dx = 0 \quad cw + dy = 1$$

If $ad - bc \neq 0$, we can solve these two systems (see Exercise 5-3, question 3).

$$v = \frac{d}{ad - bc} \quad w = \frac{-b}{ad - bc}$$

$$x = \frac{-c}{ad - bc} \quad y = \frac{a}{ad - bc}$$

$$M^{-1} = \begin{pmatrix} \frac{d}{ad - bc} & \frac{-b}{ad - bc} \\ \frac{-c}{ad - bc} & \frac{a}{ad - bc} \end{pmatrix}$$

Since each denominator is $\det M$,
we have, if $\det M \neq 0$

$$M^{-1} = \frac{1}{\det M} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

It can be verified that $MM^{-1} = I$ as well as $M^{-1}M = I$.
You can find the inverse of a 2×2 matrix M provided that its determinant is not 0. If $\det M = 0$, M has no inverse.

EXAMPLE 3. If $M = \begin{pmatrix} 4 & -3 \\ -2 & 2 \end{pmatrix}$, find M^{-1} .

Solution

$$\det M = 4(2) - (-3)(-2) = 2$$

$$M^{-1} = \frac{1}{2} \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & \frac{3}{2} \\ 1 & 2 \end{pmatrix}$$

EXERCISE 5-6

B 1. Find the determinant of each of the following matrices.

- (a) $\begin{pmatrix} 5 & 1 \\ 2 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & -1 \\ 3 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 4 & 2 \\ 6 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & -4 \\ -1 & 6 \end{pmatrix}$
 (e) $\begin{pmatrix} -3 & 4 \\ 2 & 0 \end{pmatrix}$ (f) $\begin{pmatrix} -6 & 2 \\ \frac{1}{2} & 3 \end{pmatrix}$ (g) $\begin{pmatrix} 10 & 5 \\ -3 & -2 \end{pmatrix}$ (h) $\begin{pmatrix} 7 & -6 \\ -1 & -8 \end{pmatrix}$

2. Find the inverse, if one exists, of each of the following matrices.

- (a) $\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 5 & 3 \\ 1 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 4 & 6 \\ -2 & -2 \end{pmatrix}$ (d) $\begin{pmatrix} 5 & 0 \\ 1 & 2 \end{pmatrix}$
 (e) $\begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$ (f) $\begin{pmatrix} -10 & 4 \\ -5 & 2 \end{pmatrix}$ (g) $\begin{pmatrix} 0 & 3 \\ 2 & -4 \end{pmatrix}$ (h) $\begin{pmatrix} 1 & 4 \\ -2 & -6 \end{pmatrix}$

3. If $A = \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}$, find

- (a) A^{-1}
 (b) $(A)(A^{-1})$
 (c) $(A^{-1})(A)$

4. If $B = \begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix}$, find

- (a) B^{-1}
 (b) $(B)(B^{-1})$
 (c) $(B^{-1})(B)$

5.7 SOLVING LINEAR SYSTEMS USING MATRICES

The linear system

$$2x + y = 5$$

$$x + 4y = 6$$

can be represented as a matrix equation

$$\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

The matrix $\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$ is called the *coefficient matrix*

The inverse of the coefficient matrix is $\begin{pmatrix} \frac{4}{7} & -\frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{pmatrix}$

If we multiply both sides of the matrix equation by this inverse we have

$$\begin{pmatrix} \frac{4}{7} & -\frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{4}{7} & -\frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

and the solution is $x=2$ and $y=1$.

EXAMPLE 1. Solve $x-4y=8$
 $2x-5y=13$

Solution

i) write the system as a matrix equation

$$\begin{pmatrix} 1 & -4 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 13 \end{pmatrix}$$

ii) Determine the inverse of the coefficient matrix

$$\begin{pmatrix} 1 & -4 \\ 2 & -5 \end{pmatrix}^{-1} = \frac{1}{3} \begin{pmatrix} -5 & 4 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{5}{3} & \frac{4}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

iii) Multiply both sides of the matrix equation by the inverse.

$$\begin{pmatrix} -\frac{5}{3} & \frac{4}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & -4 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{5}{3} & \frac{4}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 8 \\ 13 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

$\therefore x=4$ and $y=-1$

For higher order systems of linear equations it is more practical to use the augmented matrix and the row manipulation method of solution.

The system of equations

$$\begin{aligned} 2x + y - z &= -1 \\ 3x - y + 2z &= 8 \\ 2x + 2y - 3z &= -6 \end{aligned}$$

has an equivalent matrix equation.

$$\begin{pmatrix} 2 & 1 & -1 \\ 3 & -1 & 2 \\ 2 & 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 8 \\ -6 \end{pmatrix}$$

Riddle of the Sphinx

What animal goes on four feet in the morning, on two feet at noon, and on three feet in the evening?

The coefficient matrix

$$\begin{pmatrix} 2 & 1 & -1 \\ 3 & -1 & 2 \\ 2 & 2 & -3 \end{pmatrix}$$

and the column matrix

$$\begin{pmatrix} -1 \\ 8 \\ -6 \end{pmatrix}$$

of numbers on the right side of the equations can be combined to form the *augmented matrix*

$$\begin{pmatrix} 2 & 1 & -1 & -1 \\ 3 & -1 & 2 & 8 \\ 2 & 2 & -3 & -6 \end{pmatrix}$$

The augmented matrix is simply a notation that avoids repeatedly writing x 's, y 's, z 's and $=$'s in equivalent systems.

The goal in solving a system of equations by matrices is to operate on the rows until the matrix has the following form.

$$\begin{pmatrix} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{pmatrix}$$

This is called the *row reduced echelon form* of a matrix. The solution (c_1, c_2, c_3) can be read directly from the matrix. The reduction of a given matrix to its row echelon form is accomplished using three types of operations on the rows of the matrix. They are called *elementary row operations* and are:

1. the interchange of any two distinct rows.
2. the multiplication of all elements of a row by the same non-zero number.
3. the addition of the same multiple of the elements of one row to the corresponding elements of another row.

EXAMPLE 2. Solve the following using the row manipulation method.

$$2x + y - z = -1$$

$$3x - y + 2z = 8$$

$$2x + 2y - 3z = -6$$

Solution

Express the system in matrix form.

$$\begin{pmatrix} 2 & 1 & -1 & -1 \\ 3 & -1 & 2 & 8 \\ 2 & 2 & -3 & -6 \end{pmatrix}$$

Interchange the first and second rows.

$$\begin{pmatrix} 3 & -1 & 2 & 8 \\ 2 & 1 & -1 & -1 \\ 2 & 2 & -3 & -6 \end{pmatrix}$$

Add the second row to the first and replace the first.

$$\begin{pmatrix} 5 & 0 & 1 & 7 \\ 2 & 1 & -1 & -1 \\ 2 & 2 & -3 & -6 \end{pmatrix}$$

Divide the first row by 5

$$\begin{pmatrix} 1 & 0 & \frac{1}{5} & \frac{7}{5} \\ 2 & 1 & -1 & -1 \\ 2 & 2 & -3 & -6 \end{pmatrix}$$

Subtract the second row from the third and replace the second.

$$\begin{pmatrix} 1 & 0 & \frac{1}{5} & \frac{7}{5} \\ 0 & 1 & -2 & -5 \\ 2 & 2 & -3 & -6 \end{pmatrix}$$

Multiply the first row by -2 and add to the third.

$$\begin{pmatrix} 1 & 0 & \frac{1}{5} & \frac{7}{5} \\ 0 & 1 & -2 & -5 \\ 0 & 2 & -\frac{17}{5} & -\frac{44}{5} \end{pmatrix}$$

Multiply the second row by -2 and add to the third.

$$\begin{pmatrix} 1 & 0 & \frac{1}{5} & \frac{7}{5} \\ 0 & 1 & -2 & -5 \\ 0 & 0 & \frac{3}{5} & \frac{6}{5} \end{pmatrix}$$

Divide the third row by $\frac{3}{5}$

$$\begin{pmatrix} 1 & 0 & \frac{1}{5} & \frac{7}{5} \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

Multiply the third row by $-\frac{1}{5}$ and add to the first.

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

Multiply the third row by 2 and add to the second.

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\therefore x=1, y=-1 \text{ and } z=2$$

The 12 L jug is filled with wine. The 7 L and 5 L jugs are empty. Divide the wine into two equal parts by decanting.



EXERCISE 5-7

B

1. Solve each of the following by the method of Example 1.

- | | | |
|-----------------|----------------|----------------|
| (a) $x+y=3$ | (b) $x+2y=5$ | (c) $2x+3y=-1$ |
| $2x+3y=8$ | $2x-y=5$ | $x+2y=0$ |
| (d) $2x-3y=0$ | (e) $4x+3y=-7$ | (f) $3x=5y-1$ |
| $3x-2y=-13$ | $2x+y=-2$ | $2x=3y-1$ |
| (g) $5x-2y-2=0$ | (h) $2x+2y=5$ | (i) $4x-y=-5$ |
| $6x-5y+8=0$ | $3x-4y=11$ | $3x+2y=-12$ |

2. Solve each of the following by the method of Example 2.

- | | | |
|------------------|-----------------|-----------------|
| (a) $x+y+z=4$ | (b) $x+2y-z=4$ | (c) $2x+y+2z=7$ |
| $x-y+z=2$ | $2x-y+z=1$ | $x+2y-2z=-1$ |
| $2x+y-z=4$ | $x+y+2z=5$ | $x-2y+z=1$ |
| (d) $2x+y+z-2=0$ | (e) $3x+y+2z=7$ | (f) $3x+2y+z=5$ |
| $x-2y+2z+1=0$ | $2x-2y-z=-3$ | $2x-y+3z=8$ |
| $2x-y-z+6=0$ | $x+2y-2z=3$ | $2x+y-2z=4$ |

5.8 APPLICATIONS OF MATRICES

In the previous section it was shown how matrices can be used to solve a system of linear equations. Matrices are also very useful in solving problems in areas such as physics, engineering, psychology, sociology, economics and business.

The following examples illustrate some applications of matrices.

EXAMPLE 1. *The Mirco Instrument Company distributes three types of calculators—T61, T62 and T63. The following table indicates the number of each type ordered by four different stores.*

	T61	T62	T63
Store 1	10	20	20
Store 2	30	20	30
Store 3	20	30	10
Store 4	30	10	10

Each T61, T62 and T63 costs \$10, \$20 and \$80 respectively. Determine Mirco's total income from each store and the total income from all stores together.

Solution

Let A represent the amounts ordered by each store.

$$A = \begin{pmatrix} 10 & 20 & 20 \\ 30 & 20 & 30 \\ 20 & 30 & 10 \\ 30 & 10 & 10 \end{pmatrix}$$

Let C represent the cost of each calculator.

$$C = \begin{pmatrix} 10 \\ 20 \\ 80 \end{pmatrix}$$

The product matrix represents the total income from each store,

$$A \cdot C = \begin{pmatrix} 10 & 20 & 20 \\ 30 & 20 & 30 \\ 20 & 30 & 10 \\ 30 & 10 & 10 \end{pmatrix} \begin{pmatrix} 10 \\ 20 \\ 80 \end{pmatrix} = \begin{pmatrix} 2100 \\ 3100 \\ 1600 \\ 1300 \end{pmatrix}$$

Store 1 paid \$2100; store 2, \$3100; store 3, \$1600; store 4, \$1300.

The total income from all four stores, \$8100, is obtained by adding the entries in the product matrix.

EXAMPLE 2. *In times of world crisis, transmitting messages by code becomes essential. The development of "unbreakable" codes has been a major concern of world leaders. We shall now illustrate how matrices can be used to obtain a code that is relatively difficult to break.*

- Step 1: Let 1 represent A, 2 represent B, . . . , 26 represent Z. (Punctuation marks and spaces between words will be disregarded.)
- Step 2: Break the message up into groups of two letters each. Thus

SEND SHIPS NOW

becomes

19 5/ 14 4/ 19 8/ 9 16/ 19 14/ 15 23

- Step 3: Arrange each group of two numbers as a column matrix so that the message now reads

$$\begin{pmatrix} 19 \\ 5 \end{pmatrix} \begin{pmatrix} 14 \\ 4 \end{pmatrix} \begin{pmatrix} 19 \\ 8 \end{pmatrix} \begin{pmatrix} 9 \\ 16 \end{pmatrix} \begin{pmatrix} 19 \\ 14 \end{pmatrix} \begin{pmatrix} 15 \\ 23 \end{pmatrix}$$

- Step 4: Choose any 2×2 matrix that has an inverse and multiply each of the column matrices by this matrix. If we choose $\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$ as the matrix, multiplication yields.

$$\begin{pmatrix} 62 \\ 43 \end{pmatrix} \begin{pmatrix} 46 \\ 32 \end{pmatrix} \begin{pmatrix} 65 \\ 46 \end{pmatrix} \begin{pmatrix} 43 \\ 34 \end{pmatrix} \begin{pmatrix} 71 \\ 52 \end{pmatrix} \begin{pmatrix} 68 \\ 53 \end{pmatrix}$$

- Step 5: Transmit the coded message horizontally as follows.

62 43 46 32 65 46 43 34 71 52 68 53

When the secret agent receives the message he converts it into a set of 2×1 column matrices. He then multiplies each matrix by the inverse of $\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$ which is $\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$. The message is then made readable by converting 1 to A, 2 to B etc.

Three men and their wives had to cross a river by boat. There was only one boat available, and that had only room for two people. Since the husbands were jealous, no woman could be left with a man unless her husband was present. How did they cross the river?

EXERCISE 5-8

1. The Binkley Produce Company supplies oranges, lemons and limes to supermarkets. The following table indicates the number of cases of each ordered by four different stores.

	Oranges	Lemons	Limes
Store A	70	10	5
Store B	60	10	10
Store C	70	5	7
Store D	50	15	9

Oranges cost \$20/case; lemons, \$17/case and limes \$15/case.

- (a) Write the total order as a matrix.
 (b) Write the cost/case as a column matrix.
 (c) Using matrix multiplication, find the amount owing to the produce company by each store.
 (d) Find the total amount owing to the company.
2. The following table indicates the number of cases of deodorant, bath powder and toothpaste ordered from a pharmaceutical supply company by five different stores.

	Deodorant	Bath powder	Toothpaste
Store 1	3	2	5
Store 2	4	2	6
Store 3	5	3	4
Store 4	2	2	2
Store 5	2	1	3

Deodorant costs \$90/case; bath powder, \$50/case and toothpaste \$70/case.

- (a) Write the order as a matrix.
 (b) Write the cost/case as a column matrix.
 (c) Use matrix multiplication to determine the total income from each store.
 (d) Determine the total income from all stores together.
3. The following table indicates the number and type of vehicle using a parking lot during a one week period.

	Cars	Motorcycles	Buses
Monday	90	10	2
Tuesday	95	12	1
Wednesday	101	11	0
Thursday	115	9	1
Friday	123	14	2
Saturday	107	3	1
Sunday	46	2	0

Each car is charged \$5/day; motorcycles, \$3/day and buses \$15/day.

- (a) Express the parking lot space rental as a matrix.
 (b) Write the cost/vehicle as a column matrix.
 (c) Use matrix multiplication to determine the total income per day.
 (d) Determine the total income for the week.

4. Use the method of Example 2 and the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ to encode the message

NEED MORE CASH

5. You are a secret agent. Your code book informs you that the matrix for the day is $\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$.

You receive the following message

12 29 35 85 41 102 34 83 29 72

Decode it.

6. Use the matrix $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ to encode the message

I HAVE MICROFILM

Check your answer by decoding it.

7. As a secret agent you receive the following message.

40 33 113 93 40 35 125 104

The matrix for the day is $\begin{pmatrix} 5 & 1 \\ 4 & 1 \end{pmatrix}$

Decode the message.

5.9 SYSTEMS OF LINEAR INEQUALITIES IN TWO VARIABLES

We have seen that the graph of every linear equation is a straight line. Figure 5-2 shows the graph of $2x + y = 4$. The line divides the plane into two regions, one above the line and the other below the line.

If we choose some point in the *half-plane* below the line, say $(-1, 2)$, and substitute its coordinates in the equation $2x + y = 4$ we get

$$2(-1) + (2) \neq 4$$

or

$$0 < 4$$

For this ordered pair, $2x + y < 4$. Other points in this region will give the same result.

Similarly, for points in the half-plane above the line $2x + y > 4$.

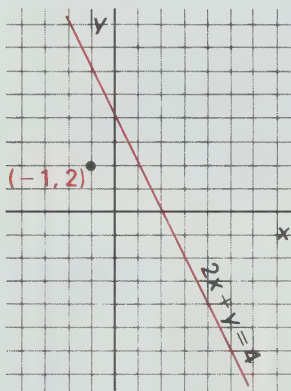
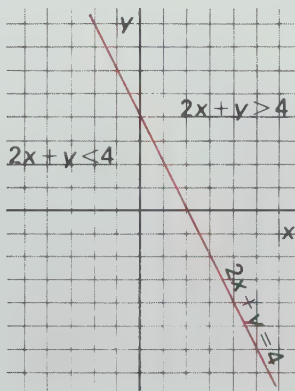


Figure 5-2

In summary, the straight line $2x + y = 4$ divides the plane into three distinct regions, namely

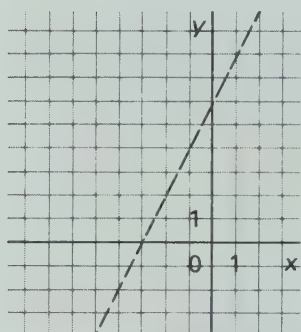
- (i) $2x + y = 4$
- (ii) $2x + y < 4$
- (iii) $2x + y > 4$



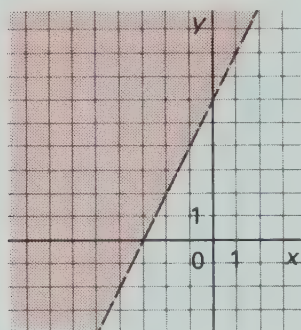
EXAMPLE 1. Draw the graph of

$$2x - y + 6 < 0$$

Solution



Draw the line $2x - y + 6 = 0$. Since the line is not part of the solution, draw a broken line.



Choose a convenient test point, in this case $(0, 0)$, to determine whether the points in this region satisfy the given inequality.

$$2x - y + 6 < 0$$

$$2(0) - (0) + 6 < 0$$

$$6 < 0$$

which is *not* true

Therefore we conclude that the points in the half-plane containing $(0, 0)$ are not part of the solution. The other half-plane must be the solution.

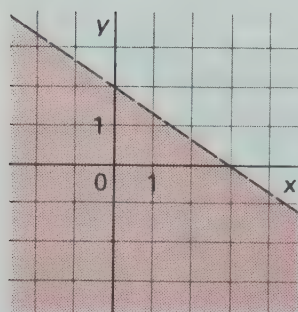
If you express a linear inequality in the form $y > mx + b$ or $y < mx + b$ then you need not use a test point. For $y > mx + b$, the required region or half-plane is above the line. For $y < mx + b$, the region is below the line.

EXAMPLE 2. Draw the graphs of

(a) $2x + 3y - 6 < 0$

(b) $3x - y \leq 4$

Solution



(a) $2x + 3y - 6 < 0$

$$3y < -2x + 6$$

$$y < -\frac{2}{3}x + 2$$

Graph the line $y = -\frac{2}{3}x + 2$ and show it as a broken line. Shade the half-plane below the line.

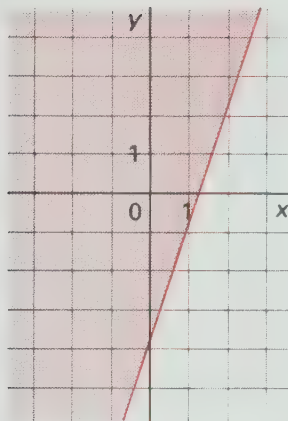
(b)

$$3x - y \leq 4$$

$$-y \leq -3x + 4$$

$$y \geq 3x - 4$$

Graph the line $y = 3x - 4$ and show it as a solid line. Shade the half-plane above the line.



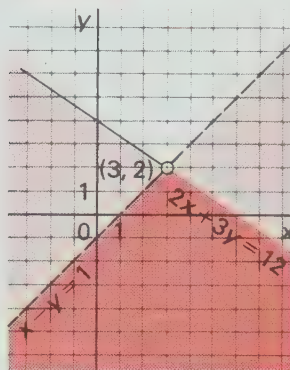
EXAMPLE 3. Solve $\begin{cases} 2x + 3y \leq 12 \\ x - y > 1 \end{cases}$ by graphing.

$$\{(x, y) \mid 2x + 3y \leq 12 \text{ and } x - y > 1\}$$

Solution

Graph each relation on the same set of axes. The region where the shadings overlap is the solution set of the system.

Why is the point $(3, 2)$ not included in the solution set?



EXERCISE 5-9

A 1. Is the given point in the solution set of the given relation?

(a) $(1, 3); 3x + y \leq 7$

(b) $(5, 2); x - y > 4$

(c) $(3, 2); 2x + 3y > 11$

(d) $(-1, 3); x - y < -4$

(e) $(0, -4); 3x - 4y > 18$

(f) $(0, 0); x - 5y + 6 > 0$

(g) $(-2, -3); x + y < -4$

(h) $(1, -2); x - y - 3 \leq 0$

(i) $(2, 4); 3x + 2y - 15 > 0$

(j) $(2, 3); y > x + 2$

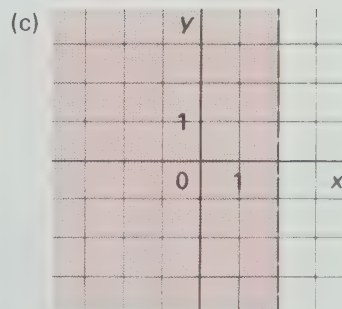
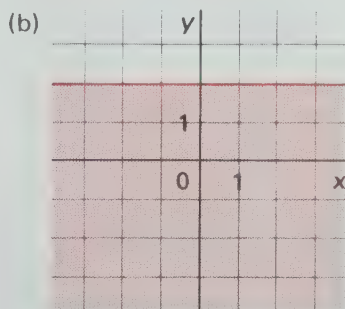
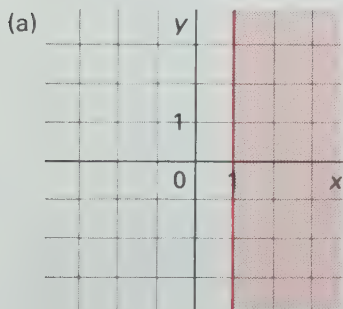
(k) $(-1, -3); y < -x - 3$

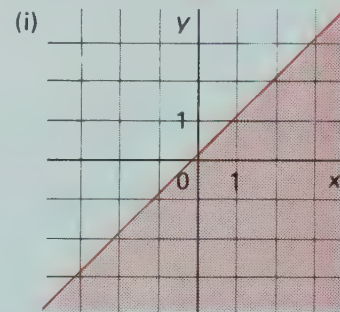
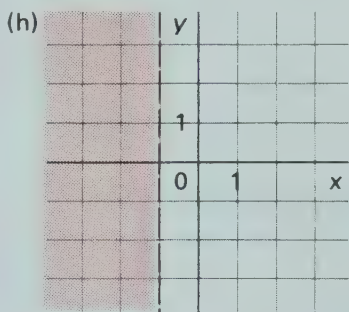
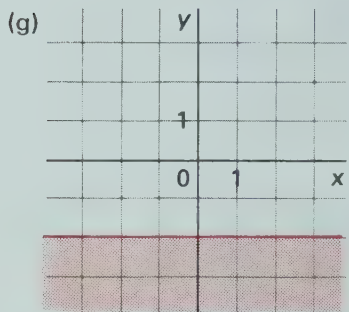
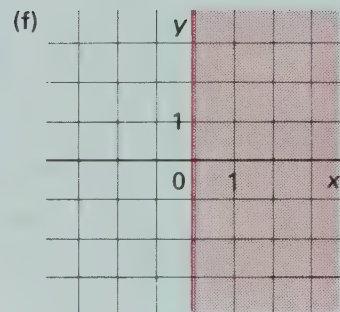
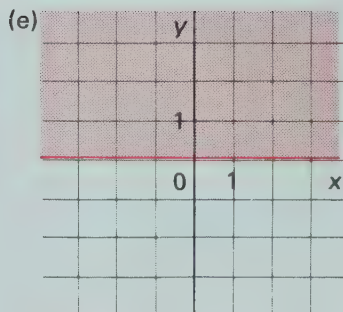
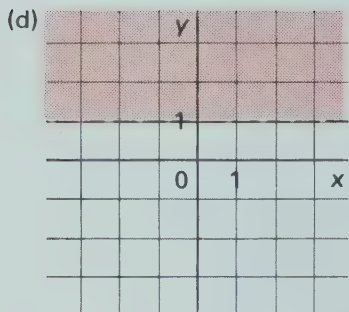
(l) $(3, 4); x \geq 4$

(m) $(3, -4); y \geq -4$

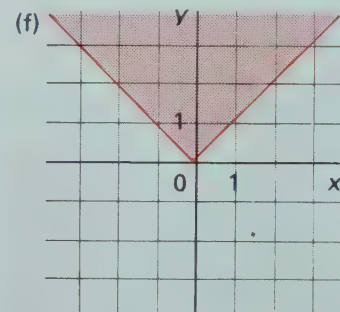
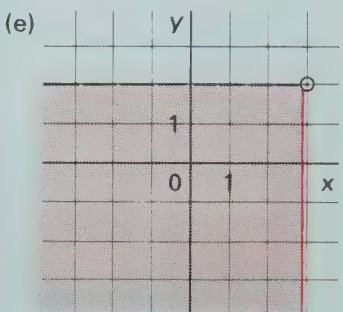
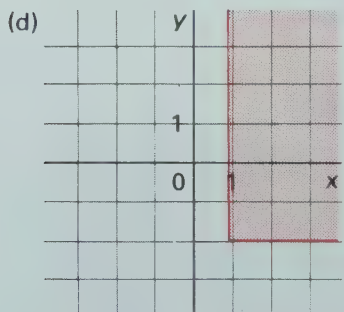
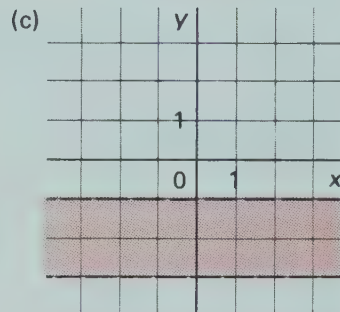
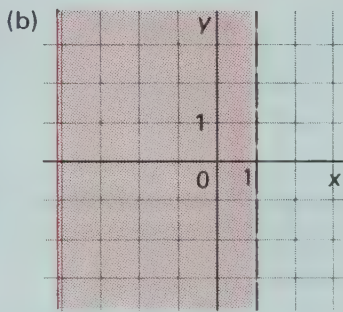
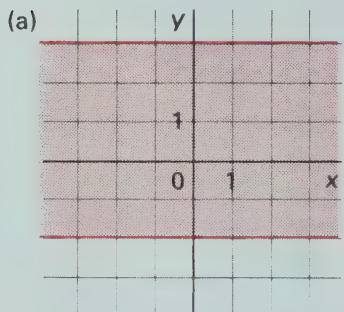
(n) $(4, 3); y - x < 0$

2. State an inequality whose solution set is the given graph.





3. State a system of linear inequalities whose solution set is the given graph.



4. Graph the solution set of each inequality.

(a) $x + y > 4$

(b) $2x + y \leq 6$

(c) $x - 2y \geq 8$

(d) $3x - 2y < 6$

(e) $2x + 5y - 10 > 0$

(f) $y > 2x - 4$

(g) $y \leq 3 - 2x$

(h) $y \geq -\frac{1}{2}x - 3$

(i) $y > x$

- (j) $y \leq -x$ (k) $x \geq -2$ (l) $y < 3$
 (m) $x \geq 0$ (n) $y \geq 0$ (o) $-x < y$
 (p) $y - 2x < 2$ (q) $y - 3x > -3$ (r) $5x - 10y > 20$
 (s) $-2x + 3y < 12$ (t) $3x - 4y \geq -2$ (u) $-2x - 5y \leq -10$

5. Graph the solution set of each system of inequalities.

- (a) $x + y \leq 6$ and $x - y \geq 2$ (b) $x - y > 4$ and $x + y > 10$
 (c) $x + y > 8$ and $x - y \leq 2$ (d) $2x + 3y \geq 12$ and $x - y < 1$
 (e) $3x - y < 6$ and $2x + y > 4$ (f) $x + 3y \leq 12$ and $x - 2y \leq 2$
 (g) $3x + 5y > 15$ and $2x - 7y \leq 14$ (h) $4x + 5y \leq 20$ and $3x + y \leq 6$

6. Graph the solution set of each system of inequalities.

- (a) $y \geq x - 6$ and $y \leq -x + 2$ (b) $y < -x + 3$ and $y > x - 1$
 (c) $y > 3x - 1$ and $y > -2x + 4$ (d) $y \leq 2x - 1$ and $y \geq -x + 5$
 (e) $y < 3x - 2$ and $y > 2x + 1$ (f) $y \geq 3x - 1$ and $y < 2x - 2$
 (g) $y > \frac{1}{2}x + 3$ and $y \geq x + 2$ (h) $y < 2x + 7$ and $y \geq -3x - 3$

7. Graph the solution set of each of the following in $R \times R$. Recall that a continued inequality such as $-1 \leq x \leq 2$ is equivalent to the compound inequality " $x \geq -1$ and $x \leq 2$ ".

- (a) $-2 \leq x \leq 3$ (b) $1 \leq y \leq 4$ (c) $-3 < x \leq 5$
 (d) $-3 < y < -1$ (e) $-2 < x \leq 0$ (f) $0 \leq y < 4$
 (g) $|x| > 2$ (h) $|y| < 1$ (i) $|x| \leq 3$
 (j) $|2x + 1| \leq 5$ (k) $|2x + y| \leq 2$ (l) $|3x + 2y| \leq 6$

8. Graph the following.

- (a) $\{(x, y) | 2x + 3y \leq 6 \text{ and } x \geq 0 \text{ and } y \geq 0\}$
 (b) $\{(x, y) | 2x + y \geq 4 \text{ and } x \geq 0 \text{ and } y \geq 0\}$
 (c) $\{(x, y) | 2x + y \leq 8 \text{ and } 2x + 3y \leq 12 \text{ and } x \geq 0 \text{ and } y \geq 0\}$
 (d) $\{(x, y) | x \leq 4 \text{ and } x \geq -1 \text{ and } y \geq 1\}$
 (e) $\{(x, y) | 2x + y \geq 6 \text{ and } x + 2y \geq 6 \text{ and } x \geq 0 \text{ and } y \geq 0\}$
 (f) $\{(x, y) | x \leq 3 \text{ and } x \geq -2 \text{ and } y \leq 4 \text{ and } y \geq -1\}$

9. An orbiting space station has two types of fuel on board, F_1 and F_2 . For an emergency return to earth, 10 000 kg of F_1 and 20 000 kg of F_2 are required. The fuel tanks can hold a total of 90 000 kg of fuel. Draw a figure showing all the allowable combinations of the two fuels.

10. On a certain military base the total number of sergeants and corporals must be at least 12 but may not exceed 20. A minimum of 5 corporals and 3 sergeants are required. Draw a figure showing all the allowable combinations of the two ranks.

11. The food package for a space station must contain at least 20 kg of ingredient A and 40 kg of ingredient B. The combined amounts of A and B must be at least 80 kg but may not exceed 160 kg. Draw a figure showing all allowable combinations of the two ingredients.

WORD LADDER

Start with the word "bread" and change one letter at a time to form a new word until you reach "toast". The best solution has the fewest steps.

b r e a d

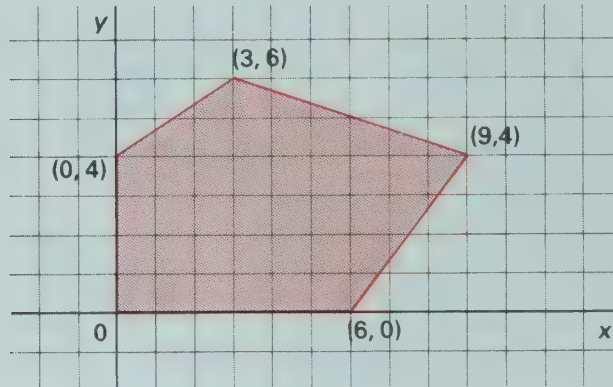
t o a s t

5.10 THE VERTEX THEOREM FOR REGIONS

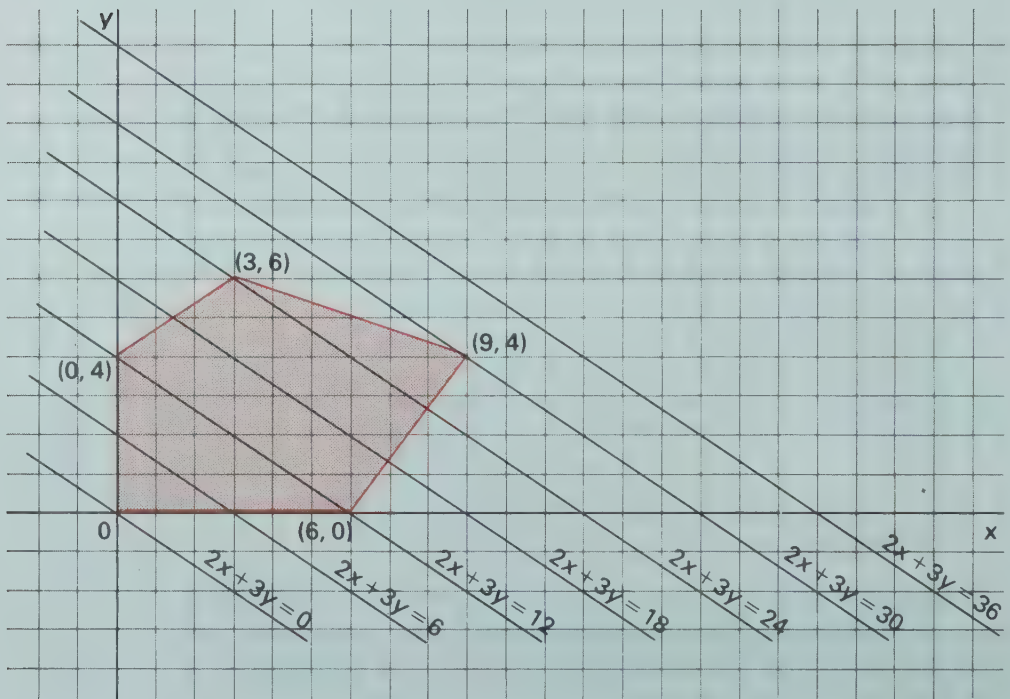
A system of linear inequalities is often used to decide which of several business alternatives is least expensive or most profitable. This branch of mathematics is called *Linear Programming*. Before studying this topic we must look at a theorem we shall need.

If a linear expression $z = ax + by + c$ is to be evaluated for all points of a convex polygonal region, then the maximum value of z , if there is one, will occur at a vertex of the region. Similarly, the minimum value of z , if there is one, will also occur at a vertex of the region.

Suppose that $z = 2x + 3y$ is to be evaluated at the points of polygonal region shown below.



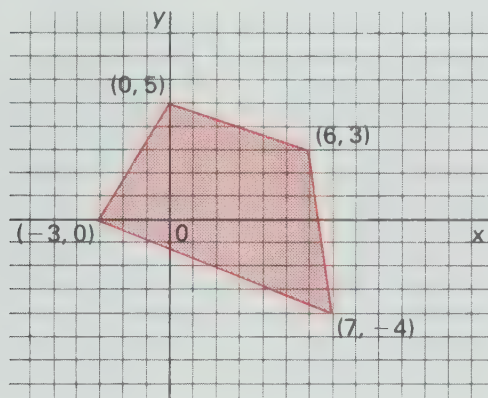
As the diagram shown below illustrates, the value of $z = 2x + 3y$ remains constant along any line with slope $-\frac{2}{3}$.



The values of $2x+3y$ are greater the farther the line is away from the origin. For the points of the shaded region, the maximum value of $2x+3y$ occurs at a vertex, in this case $(9,4)$. Similarly, the minimum value of $2x+3y$ for points of the shaded region occurs at a vertex, $(0,0)$.

The maximum or minimum values may also occur at other points in the region, but all we need test are the vertices.

EXAMPLE 1. For the given region and the expression $z=3x+4y$, find a point at which z has a maximum value. Then find a point at which z has a minimum value.



Solution

The maximum and minimum values of z , if they exist, will occur at a vertex.

$$\begin{aligned}\text{at } (0, 5), \quad z &= 3(0) + 4(5) \\ &= 0 + 20 \\ &= 20\end{aligned}$$

$$\begin{aligned}\text{at } (-3, 0), \quad z &= 3(-3) + 4(0) \\ &= -9 + 0 \\ &= -9\end{aligned}$$

$$\begin{aligned}\text{at } (7, -4), \quad z &= 3(7) + 4(-4) \\ &= 21 - 16 \\ &= 5\end{aligned}$$

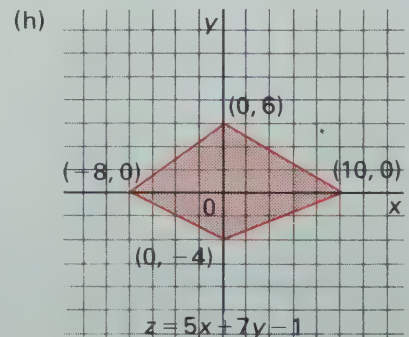
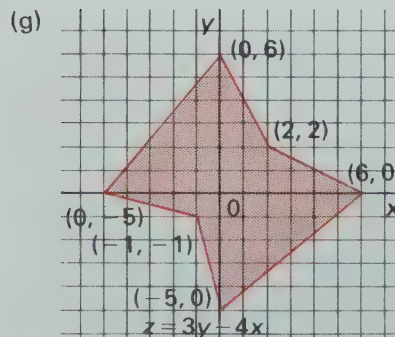
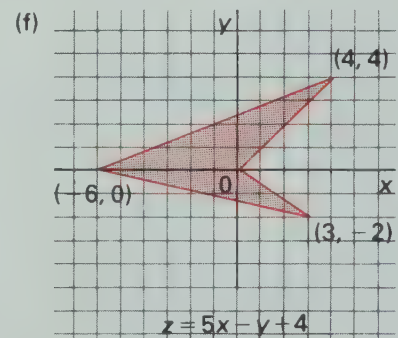
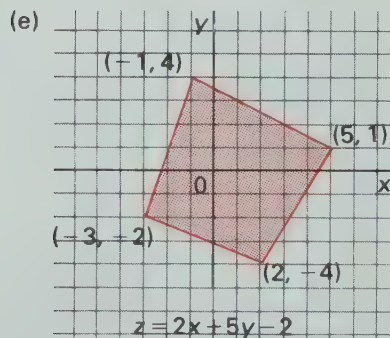
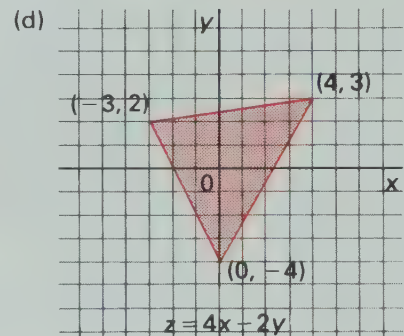
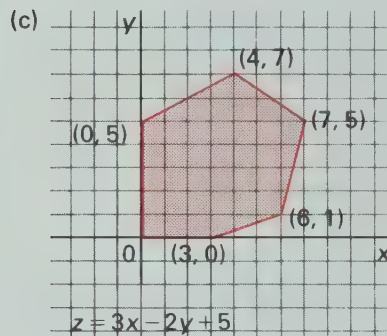
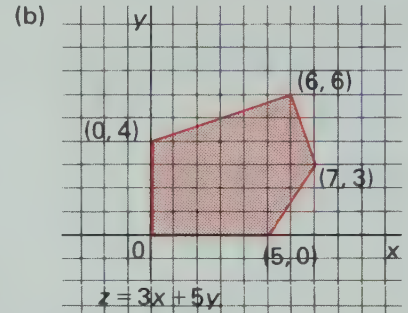
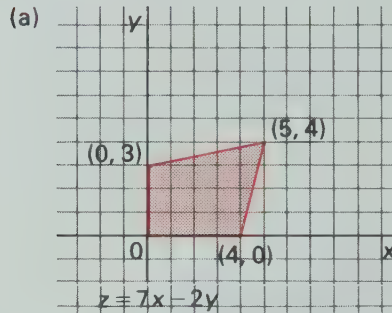
$$\begin{aligned}\text{at } (6, 3), \quad z &= 3(6) + 4(3) \\ &= 18 + 12 \\ &= 30\end{aligned}$$

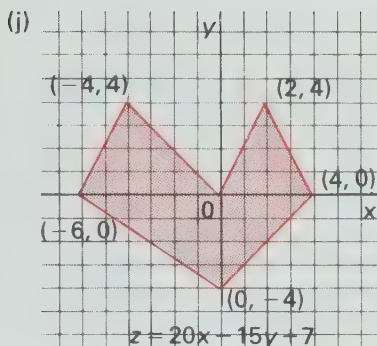
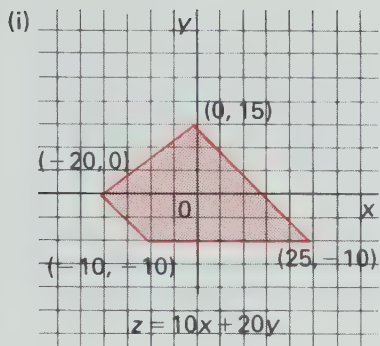
z is maximum at $(6, 3)$

z is minimum at $(-3, 0)$

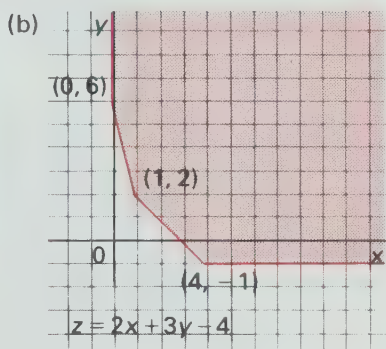
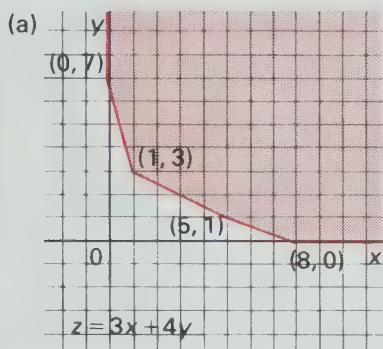
EXERCISE 5-10

- B** 1. For each region and corresponding equation, find a point at which z has a maximum value and a point at which z has a minimum value.





2. For each region and corresponding equation, find a point at which z has a minimum value.



3. Draw the graph of the region bounded by $2x + y \leq 8$, $x \geq 0$ and $y \geq 0$. If $z = 3x + 4y$, determine a point in the region at which z has a maximum value and a point at which z has a minimum value.

4. Draw the graph of the region bounded by $x + y \leq 10$, $x + 2y \leq 16$, $x \geq 0$ and $y \geq 0$. If $z = 4x + 2y - 1$, determine a point in the region at which z has a maximum value and a point at which z has a minimum value.

5. Draw the graph of the region bounded by $y \leq -2x + 6$, $y \leq -x + 5$, $x \geq 0$ and $y \geq 0$. If $z = 3x - 2y + 4$, determine a point in the region at which z has a maximum value and a point at which z has a minimum value.

5.11 LINEAR PROGRAMMING

Linear Programming is a technique used to solve some problems that arise in business or industry. In general, the problems may be classified into two categories: those where the profit is to be maximized and those where the expenses are to be minimized.

The word *programming* does not refer to a computer. It means that the solution gives a *programme* to follow for best results.

The two examples that follow will illustrate each type of linear programming problem.

EXAMPLE 1. Maximizing a Profit

A company has access to two computers, A and B, which it uses to solve water pollution and air pollution problems. Computer A is limited to 8 min/h and computer B to 18 min/h for solving pollution problems. To solve a water pollution problem takes 1 min on A plus 2 min on B. To solve an air pollution problem takes 1 min on A plus 3 min on B. Solving a water pollution problem yields a profit of \$400 and an air pollution problem a profit of \$500. How many problems of each type should be solved in a one hour period to maximize the company's profit?

Solution

	Water pollution	Air pollution	Time available
computer A	1 min	1 min	8 min/h
computer B	2 min	3 min	18 min/h

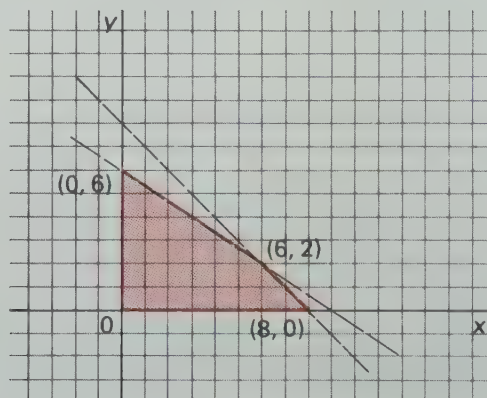
Let x represent the number of water pollution problems solved.

Let y represent the number of air pollution problems solved.

This gives the following inequalities.

$$\begin{array}{lcl} \left. \begin{array}{l} x \geq 0 \\ y \geq 0 \end{array} \right\} & \text{You cannot solve a negative} & \\ & \text{number of problems.} & \\ \left. \begin{array}{l} 1x + 1y \leq 8 \end{array} \right\} & \text{1 min of A is needed to solve a water} & \\ & \text{pollution problem and 1 min of A is needed} & \\ & \text{to solve an air pollution problem. Total time} & \\ & \text{must be less than or equal to 8 min.} & \\ \left. \begin{array}{l} 2x + 3y \leq 18 \end{array} \right\} & \text{2 min of B is needed to solve a water pollution} & \\ & \text{problem and 3 min of B to solve an air pollution} & \\ & \text{problem. Total time must be less than or equal} & \\ & \text{to 18 min.} & \end{array}$$

Graph the region that satisfies all the inequalities.



The profit in dollars for solving x water pollution problems and y air pollution problems is

$$P = 400x + 500y.$$

The profit is a maximum at a vertex of the region, so each vertex is tested.

At (x, y)	$P = 400x + 500y$
At $(0, 0)$	$P = 400(0) + 500(0) = 0$
At $(8, 0)$	$P = 400(8) + 500(0) = 3200$
At $(6, 2)$	$P = 400(6) + 500(2) = 3400$
At $(0, 6)$	$P = 400(0) + 500(6) = 3000$

The maximum profit occurs when $x = 6$ and $y = 2$ or when 6 water pollution problems are solved per hour and 2 air pollution problems.

Multiplication

NORA
L
ARON

EXAMPLE 2. Minimizing a Cost

A smelting company receives an order for 7 t of iron, 19 t of lead and 8 t of copper. The order can be filled by melting down either of two ores, A or B . Each railroad car of ore A contains 3 t of iron, 3 t of lead and 1 t of copper. Each car of ore B contains 1 t of iron, 4 t of lead and 3 t of copper. If it costs \$6000 to buy and process each carload of ore A and \$5000 for each carload of ore B , how many carloads of each should be bought to fill the order at minimum cost?

Solution

	One carload of A	One carload of B	Number of tonnes to fill order
Tonnes of Iron	3	1	7
Tonnes of lead	3	4	19
Tonnes of copper	1	3	8

Let x represent the number of carloads of ore A to buy. Let y represent the number of carloads of ore B to buy.

Tonnes of iron $3x + y \geq 7$

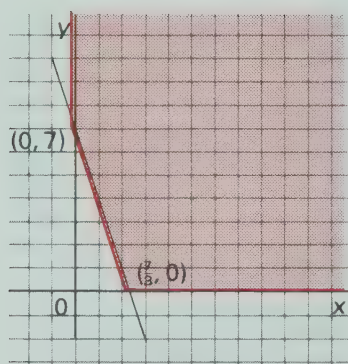
Tonnes of lead $3x + 4y \geq 19$

Tonnes of copper $x + 3y \geq 8$

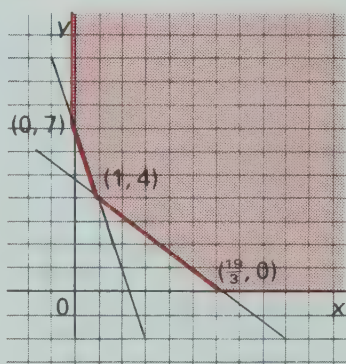
$$x \geq 0$$

$$y \geq 0$$

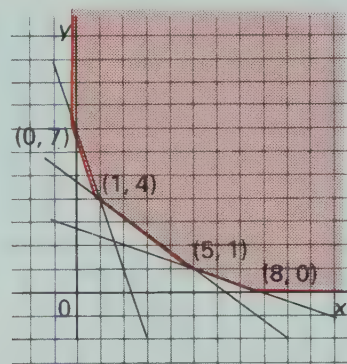
We now graph the region that satisfies all the inequalities.



Graph $3x + y \geq 7$, $x \geq 0$, $y \geq 0$



Add $3x + 4y \geq 19$



Add $x + 3y \geq 8$

The final region shows that there are infinitely many ways to fill the order. We want the cheapest.

The cost in dollars for x carloads of ore A and y carloads of ore B is

$$c = 6000x + 5000y$$

The cost is a minimum at a vertex of the region so each vertex is tested.

At (x, y)	$c = 6000x + 5000y$
At $(0, 7)$	$c = 6000(0) + 5000(7) = 35\,000$
At $(1, 4)$	$c = 6000(1) + 5000(4) = 26\,000$
At $(5, 1)$	$c = 6000(5) + 5000(1) = 35\,000$
At $(8, 0)$	$c = 6000(8) + 5000(0) = 48\,000$

The minimum cost occurs at $(1, 4)$ so the smelting company should buy 1 carload of ore A and 4 carloads of ore B .

EXERCISE 5-11

- B** 1. A sports equipment company makes two types of footballs—Amateur and Professional. Each Amateur football requires one minute at the cutting machine and two minutes at the sewing machine. Each Professional football requires one minute at the cutting machine and three minutes at the sewing machine. Since both machines are required for other jobs, the cutting machine is available for a maximum of 100 min/d and the sewing machine 240 min/d.

Assume that the company can sell all the footballs it makes. If the profit on each Amateur football is \$2.10 and on each Professional football \$3.00, how many of each type should be made each day in order to maximize the profit?

2. A manufacturing company makes two types of car radios—AM/FM and AM. Each radio must be processed by three machines—A, B and C. To produce an AM/FM radio takes one hour at machine A, one hour at B and three hours at C. To produce an AM radio takes two hours at machine A, one hour at B and one hour at C. In a given day, machines A, B and C can work a maximum of 16, 9 and 21 h respectively.

If the profit on an AM/FM radio is \$50.00 and \$40.00 on an AM radio, how many radios of each type should be produced per day in order to maximize the profit?

3. A parking lot has an area of 1440 m^2 . A car requires 8 m^2 and a bus 40 m^2 of space. The attendant can handle 100 vehicles. If a car is charged \$3 and a bus \$9, how many of each should be accepted in order to maximize income?

4. A manufacturing company makes two types of calculators—Super I and Super II. Each calculator must be processed by three machines—A, B and C. The following table shows the time required at each machine and the amount of machine time available per hour.

	Super I	Super II	Minutes Available per Hour.
Machine A	1 min	3 min	42 min
Machine B	3 min	2 min	42 min
Machine C	3 min	1 min	36 min

If the profit on a Super I is \$20 and \$25 on a Super II, how many of each should be made per hour in order to maximize the profit?

5. A painting contractor has 28 units of yellow dye and 18 units of red dye. He plans to mix as many litres as possible of colour X and colour Y. Each litre of colour X requires 4 units of yellow dye and 1 unit of red dye. Each litre of colour Y requires 1 unit of yellow dye and 4 units of red dye. How many litres of each colour should he mix?

6. A contractor wishes to install as many bathtubs as possible in an apartment complex during the first week of September. Bathtubs A and B cost the same and either type may be installed. During the week the carpenter can work 28 hours and the plumber 32 hours. Installing bathtub A takes the carpenter 1 hour and the plumber 4 hours. Bathtub B takes the carpenter 3 hours and the plumber 2 hours.

To install as many tubs as possible, how many of each type should be installed?

7. A smelting company receives an order for 20 t of lead, 12 t of copper and 20 t of iron. The order can be filled by melting down either of two ores, X or Y. Each railroad car of X ore contains 5 t of lead, 1 t of copper and 1 t of iron. Each car of Y ore contains 1 t of lead, 1 t of copper and 3 t of iron. If it costs \$5000 to buy and process each carload of ore X and \$4000 for each carload of ore Y, how many carloads of each should be bought to fill the order at minimum cost?

8. Every day a patient needs his diet supplemented with a minimum of 6 mg of Vitamin A, 11 mg of Vitamin B and 125 mg of Vitamin C. There are two brands of vitamin pills available, Brand X and Brand Y. A Brand X pill supplies 2 mg of Vitamin A, 3 mg of Vitamin B and 25 mg of Vitamin C. A Brand Y pill supplies 1 mg of Vitamin A, 2 mg of Vitamin B and 50 mg of Vitamin C.

A Brand X pill costs 10¢ and a Brand Y pill costs 8¢. How many pills of each brand should be taken each day in order to satisfy the minimum daily need most economically?

9. The following table gives the amount of each of Vitamins A, B and C that a Brand X and Brand Y pill supply. The minimum daily requirement of a patient is also shown.

	Brand X	Brand Y	Minimum Daily Requirement
Vitamin A	2 mg	1 mg	5 mg
Vitamin B	3 mg	3 mg	12 mg
Vitamin C	25 mg	50 mg	125 mg

Each Brand X pill costs 15¢ and each Brand Y pill costs 12¢. How many pills of each brand should be taken each day to minimize the cost?

- C** 10. A farmer has 250 ha available for planting peas and beans. He has 20 d available for planting. Peas can be planted at the rate of 10 ha/d and beans at 15 ha/d. If peas produce a profit of \$90/ha and beans \$80/ha, how many hectares of each should be planted in order to maximize the profit?

REVIEW EXERCISE

1. Solve the following.

- (a) $2x + 3y = 13$ (b) $2x - 5y = -1$ (c) $3x + 5y = 1$
 $3x - y = 3$ $x - 4y = -2$ $4x - 3y = -18$
 (d) $5x - 2y = -10$ (e) $4x + 5y = -13$ (f) $3x - 6y = -7$
 $3x - 7y = 23$ $6x + 4y = -9$ $5x + 9y = -18$

2. Solve.

- (a) $\frac{x}{2} - \frac{y}{3} = 6$ (b) $0.3x + 0.4y = -3.2$ (c) $6x - 5y = 2.7$,
 $\frac{x}{4} - \frac{y}{6} = 3$ $0.5x - 0.6y = 1$ $5x - 4y = 2.2$
 (d) $2(x - 1) - (y + 3) = -4$ (e) $\frac{x+2}{2} - \frac{y-1}{5} = -2$
 $3(x+4) + 2(y-1) = 1$ $\frac{x-3}{7} - \frac{y+2}{2} = -5$

3. Find two numbers whose sum is 418 and whose difference is 68.
4. Nine times the larger of two numbers plus 4 times the smaller is 135. Six times the larger less 5 times the smaller is 21. Find the numbers.
5. A chemist wants to get 200 L of 37% alcohol solution by volume by mixing 40% alcohol solution with 30% alcohol solution. How many litres of each should be used?
6. During September, stock X gained or lost x dollars in value per share and stock Y , y dollars in value per share. On stocks X and Y , two shareholders realized the following loss and gain during September.

Shareholder	Shares of X	Shares of Y	Monthly Gain or Loss
Bradley	150	70	-\$170
O'Shea	60	50	\$20

Find x and y .

7. With a certain tail wind a jet aircraft takes 7 h to travel 2800 km. Flying against the same wind, the plane makes the return trip in 8 h. Find the wind speed.

8. Solve the following systems for x and y in terms of the other variables in the equations.

(a) $ax - by = a^2 + b^2$

(b) $x - y = a - b$

$x - y = 2b$

$ax - by = 2(a^2 - b^2)$

9. Graph the solution set of each system of inequalities.

(a) $x + y \leq 4$ and $x - y \geq 2$

(b) $3x + 2y \geq 12$ and $x - y < -1$

(c) $y > 3x - 2$ and $y < -2x + 3$

(d) $y < 4x - 3$ and $y \geq 2x + 1$

10. A small manufacturing company makes \$75 profit on each colour TV set it produces and \$50 profit on each black-and-white TV set. Each TV set must be processed by three machines. The following table shows the time required at each machine and the amount of machine time available per day.

	Colour TV	Black-and-White TV	Total hours available per day.
Machine A	1	2	16
Machine B	1	1	9
Machine C	4	1	24

How many colour TVs and how many black-and-white TVs should be produced per day in order to maximize the profit?

11. A farmer requires 10 t of nitrate and 9 t of phosphate to fertilize his fields. He can buy fertilizer A, which is 8% nitrate and 6% phosphate, or fertilizer B, which is 10% nitrate and 10% phosphate. Fertilizer A costs \$70/t and fertilizer B costs \$100/t. How many tonnes of each should be bought to satisfy the requirements at lowest cost?

12. Solve the following systems of equations.

(a) $3x + y - 2z = 15$

$2x - 3y + z = -2$

$4x + 2y + 3z = 10$

(c) $x + y = 1$

$y + z = -3$

$x - z = 4$

(b) $2x + y - 3z = -10$

$3x + 2y + 2z = -2$

$x - 3y - 4z = -10$

(d) $2w + x + y - z = 6$

$w - 2x + 2y - 2z = 2$

$3w - x + 3y + z = 0$

$2w - 3x - 2y - z = 5$

13. The following table represents the number of traffic summonses issued for various traffic violations in Culver City for a three month period.

	Failing to yield	Failing to stop	Illegal U-turn
May	107	135	48
June	158	173	31
July	135	181	16

The fines for these offenses are \$30, \$40 and \$20 respectively.

(a) Express the number of summonses as a matrix.

(b) Express the fines as a column matrix.

(c) Determine the amount of money collected each month.

(d) Find the total amount collected for all three months.

14. Use the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ to encode the message

MISSION COMPLETED

Check your answer by decoding it.

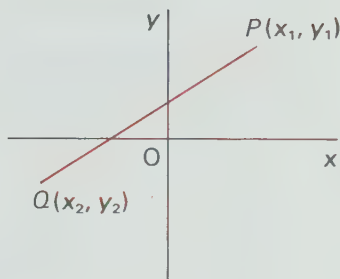
REVIEW AND PREVIEW TO CHAPTER 6

EXERCISE 1

DISTANCE BETWEEN TWO POINTS

The distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is found by the formula

$$\begin{aligned} PQ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(\Delta x)^2 + (\Delta y)^2} \end{aligned}$$



1. Find the distance between each pair of points.

- | | |
|------------------------------|------------------------------|
| (a) (5, 3), (1, 1) | (b) (4, 8), (-3, 1) |
| (c) (-6, 2), (8, 0) | (d) (-3, -1), (-5, 7) |
| (e) (4, -8), (-5, 6) | (f) (-10, 7), (-4, -13) |
| (g) (0, -4), (-5, 6) | (h) (3, -11), (3, 6) |
| (i) (-8, 0), (0, -6) | (j) (2, -5), (-6, -5) |
| (k) (3.5, 0.5), (-7.5, 2.5) | (l) (0.8, -0.3), (1.1, -2.4) |
| (m) (-3.2, 1.6), (7.8, -1.4) | (n) (1.5, -2.8), (1.3, -0.3) |
| (o) (h, k), (h + 3, k - 5) | (p) (a, 0), (0, b) |

2. Use the distance formula to show that the triangle with vertices $A(-3, 1)$, $B(1, 7)$ and $C(5, 1)$ is isosceles.

3. Give the coordinates of 3 points each equidistant from $P(-1, 6)$ and $Q(5, 6)$.

4. Show that the points $A(-1, 3)$, $B(3, 11)$ and $C(5, 15)$ are collinear by showing that $AB + BC = AC$.

5. Use the distance formula to determine which of the following sets of points are collinear.

- | | |
|--------------------------------|----------------------------------|
| (a) (-3, -2), (1, 1), (9, 7) | (b) (-8, 7), (-4, 2), (5, -3) |
| (c) (-5, 10), (-1, 4), (3, -1) | (d) (-4, 2), (-2, -3), (-10, 17) |

6. If P is a point on the x -axis which is 10 units from $(3, 8)$, find all the possible coordinates of P .

7. If Q is a point on the x -axis which is 13 units from $(-1, 5)$, find all the possible coordinates of Q .

8. If R is a point on the y -axis which is 5 units from $(-3, -7)$, find all the possible coordinates of R .

9. Show that the points $(5, -1)$, $(2, 8)$ and $(-2, 0)$ lie on a circle whose centre is $(2, 3)$.

10. Find a point on the y -axis which is equidistant from $(5, -5)$ and $(1, 1)$.

Analytic Geometry

Nature has... some sort of arithmetical-geometrical coordinate system, because nature has all kinds of models.

Buckminster Fuller

6.1 MIDPOINT OF A LINE SEGMENT

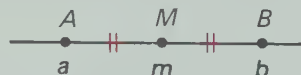
On a number line, the coordinate of the midpoint of a line segment is the average of the coordinates of the end points. For a line segment with end points $A(2)$ and $B(12)$,



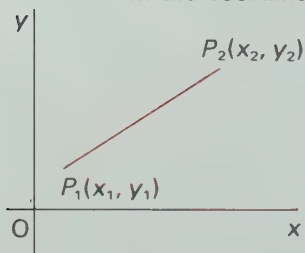
the midpoint M has coordinate $\frac{12+2}{2} = 7$.

This may be proved as follows. Consider the line segment with endpoints $A(a)$ and $B(b)$. If $M(m)$ is the midpoint, then

$$\begin{aligned} MA &= BM \\ \text{or } m - a &= b - m \\ 2m &= a + b \\ \text{and } m &= \frac{a + b}{2} \end{aligned}$$



In order to determine the formula for the midpoint of a line segment in the coordinate plane a similar procedure can be used.



EXAMPLE 1. Find the coordinates of the midpoint of the line segment joining $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

Solution

Let $P(x, y)$ be the midpoint of P_1P_2 . Draw the rise and run of PP_1 and P_2P .

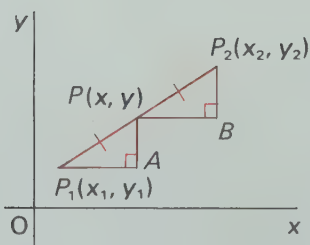
Since $P_1A \parallel PB$, $\angle PP_1A = \angle P_2PB$

$$\angle A = \angle B = 90^\circ$$

$$PP_1 = P_2P$$

$$\therefore \triangle PP_1A \cong \triangle P_2PB \text{ (ASA)}$$

$$\therefore AP_1 = BP$$



$$\therefore x - x_1 = x_2 - x$$

$$2x = x_1 + x_2$$

$$x = \frac{x_1 + x_2}{2}$$

Similarly $PA = P_2B$

and

$$y - y_1 = y_2 - y$$

$$2y = y_1 + y_2$$

$$y = \frac{y_1 + y_2}{2}$$

If (x_1, y_1) and (x_2, y_2) are any two points in the coordinate plane, then the midpoint of the line segment joining them is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

EXAMPLE 2. If $(-2, 8)$ and $(6, -3)$ are the endpoints of a line segment, find its midpoint.

Solution

$$\begin{aligned} x &= \frac{x_1 + x_2}{2} \\ &= \frac{-2 + 6}{2} \\ &= 2 \end{aligned}$$

$$\begin{aligned} y &= \frac{y_1 + y_2}{2} \\ &= \frac{8 + (-3)}{2} \\ &= \frac{5}{2} \end{aligned}$$

$\therefore$ the midpoint is $(2, \frac{5}{2})$.

What is the largest number of non-overlapping triangles that can be produced by drawing seven straight lines?

EXERCISE 6-1

B 1. Find the midpoint of the line segment joining the given pair of points.

- | | |
|------------------------------|---------------------------------|
| (a) $(2, 2), (4, 8)$ | (b) $(3, 10), (7, 12)$ |
| (c) $(-2, 2), (-6, -8)$ | (d) $(-7, 4), (-1, 9)$ |
| (e) $(-3, -5), (-6, 4)$ | (f) $(10, -5), (0, -8)$ |
| (g) $(0, 0), (-8, 10)$ | (h) $(0, -5), (6, 0)$ |
| (i) $(4.2, 0.4), (-1.6, -2)$ | (j) $(3.7, -5.1), (-6.3, -4.5)$ |

2. One endpoint of a line segment is $(-4, 3)$. The midpoint is $(-3, 6)$. Find the other endpoint.

3. If the midpoint of a line segment is $(-3, -5)$ and one end-point is $(5, -6)$, find the other endpoint.

4. The endpoints of a diameter of a circle are $(-5, 6)$ and $(11, -12)$. Find the coordinates of the centre of the circle.

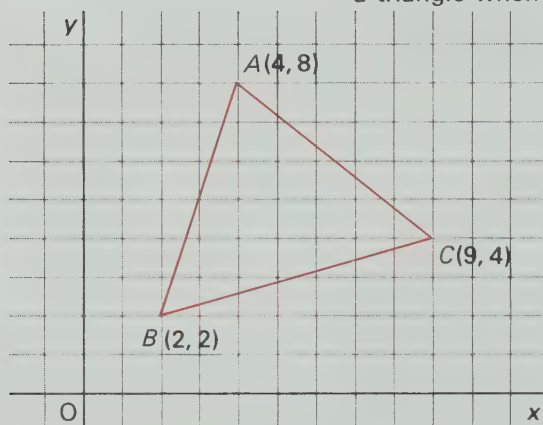
5. Find the points of division that would divide the line segment with endpoints $(-8, 8)$ and $(4, -12)$ into 4 equal parts.

6. Find the midpoint of the line segment joining $(\sqrt{2}, \sqrt{3})$ and $(\sqrt{18}, \sqrt{75})$.

7. If the line segment joining $(-4, y)$ to $(x, -3)$ is bisected at $(1, -1)$, find the values of x and y .
8. Determine the equation of the perpendicular bisector of the line segment with endpoints $(-4, 3)$ and $(2, 7)$.
9. The vertices of $\triangle ABC$ are $A(5, 6)$, $B(1, 2)$ and $C(9, 4)$. Show that the line segment joining the midpoint of AB to the midpoint of AC is
 - (a) parallel to BC
 - (b) one-half the length of BC .
10. Quadrilateral $ABCD$ has vertices $A(2, 9)$, $B(-2, 3)$, $C(8, 1)$ and $D(6, 7)$. Find the midpoints of the sides of the quadrilateral and show that these midpoints are the vertices of a parallelogram.
- C** 11. Find the endpoints of two line segments with the midpoint $(3, -4)$.
12. Find the two points that trisect the line segment with endpoints $(8, 12)$ and $(2, 3)$.

6.2 AREA OF A TRIANGLE

In this section we shall develop a formula for determining the area of a triangle when the coordinates of each vertex are known.



EXAMPLE 1. Determine the area of $\triangle ABC$ with vertices $A(4, 8)$, $B(2, 2)$ and $C(9, 4)$.

Solution

Construct rectangle $DBEF$ as shown in Fig. 6-1 and determine the coordinates of vertices D , E and F .

The area of $\triangle ABC$ may be found by subtracting the areas of $\triangle DAB$, $\triangle FAC$ and $\triangle BEC$ from the area of rectangle $DBEF$.

Area rectangle $DBEF = lw = 7 \cdot 6 = 42$ square units.

Area $\triangle DAB = \frac{1}{2}bh = \frac{1}{2} \cdot 2 \cdot 6 = 6$ square units.

Area $\triangle FAC = \frac{1}{2}bh = \frac{1}{2} \cdot 5 \cdot 4 = 10$ square units.

Area $\triangle BEC = \frac{1}{2}bh = \frac{1}{2} \cdot 7 \cdot 2 = 7$ square units.

$\therefore$ Area $\triangle ABC$
 $=$ area rectangle $DBEF - \text{area}(\triangle DAB + \triangle FAC + \triangle BEC)$
 $= 42 - (6 + 10 + 7)$
 $= 19$ square units.

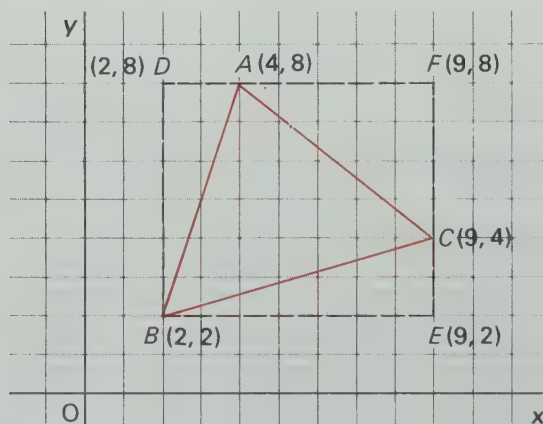


Figure 6-1

We shall now use the method employed in Example 1 to develop a general formula for the area of a triangle given the coordinates of the vertices.

EXAMPLE 2. Determine the area of $\triangle ABC$ with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$

Solution

Construct rectangle $ADEF$ and determine the coordinates of the vertices D , E and F .

$$\text{Area rectangle } ADEF = (x_1 - x_2)(y_1 - y_3)$$

$$\text{Area } \triangle ADB = \frac{1}{2}(x_1 - x_2)(y_1 - y_2)$$

$$\text{Area } \triangle BEC = \frac{1}{2}(x_3 - x_2)(y_2 - y_3)$$

$$\text{Area } \triangle ACF = \frac{1}{2}(x_1 - x_3)(y_1 - y_3)$$

Area $\triangle ABC$

$$= \text{area rectangle } ADEF - \text{area } (\triangle ADB + \triangle BEC + \triangle ACF)$$

$$= (x_1 - x_2)(y_1 - y_3) - \left[\frac{1}{2}(x_1 - x_2)(y_1 - y_2) + \frac{1}{2}(x_3 - x_2)(y_2 - y_3) + \frac{1}{2}(x_1 - x_3)(y_1 - y_3) \right]$$

$$\begin{aligned} &= (x_1 - x_2)(y_1 - y_3) - \frac{1}{2}(x_1 - x_2)(y_1 - y_2) - \frac{1}{2}(x_3 - x_2)(y_2 - y_3) - \frac{1}{2}(x_1 - x_3)(y_1 - y_3) \\ &= x_1y_1 - x_1y_3 - x_2y_1 + x_2y_3 - \frac{1}{2}x_1y_1 + \frac{1}{2}x_1y_2 + \frac{1}{2}x_2y_1 - \frac{1}{2}x_2y_2 - \frac{1}{2}x_3y_2 + \frac{1}{2}x_3y_3 \\ &\quad + \frac{1}{2}x_2y_2 - \frac{1}{2}x_2y_3 - \frac{1}{2}x_1y_1 + \frac{1}{2}x_1y_3 + \frac{1}{2}x_3y_1 - \frac{1}{2}x_3y_3 \\ &= -\frac{1}{2}x_1y_3 - \frac{1}{2}x_2y_1 + \frac{1}{2}x_2y_3 + \frac{1}{2}x_1y_2 - \frac{1}{2}x_3y_2 + \frac{1}{2}x_3y_1 \\ &= \frac{1}{2}x_1y_2 + \frac{1}{2}x_2y_3 + \frac{1}{2}x_3y_1 - \frac{1}{2}x_1y_3 - \frac{1}{2}x_3y_2 - \frac{1}{2}x_2y_1 \\ &= \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_3y_2 - x_2y_1) \\ &= \frac{1}{2}[(x_1y_2 + x_2y_3 + x_3y_1) - (x_1y_3 + x_3y_2 + x_2y_1)] \end{aligned}$$

Since the vertices were taken in a counterclockwise direction the result is positive. To ensure that this expression results in a positive quantity at all times, we write the formula using absolute value bars.

$$\triangle ABC = \frac{1}{2} |(x_1y_2 + x_2y_3 + x_3y_1) - (x_1y_3 + x_3y_2 + x_2y_1)|$$

This formula is remembered in the following manner. List the coordinates of the vertices and repeat the first pair. Then determine the sum of the "down products" and the sum of the "up products" as shown.

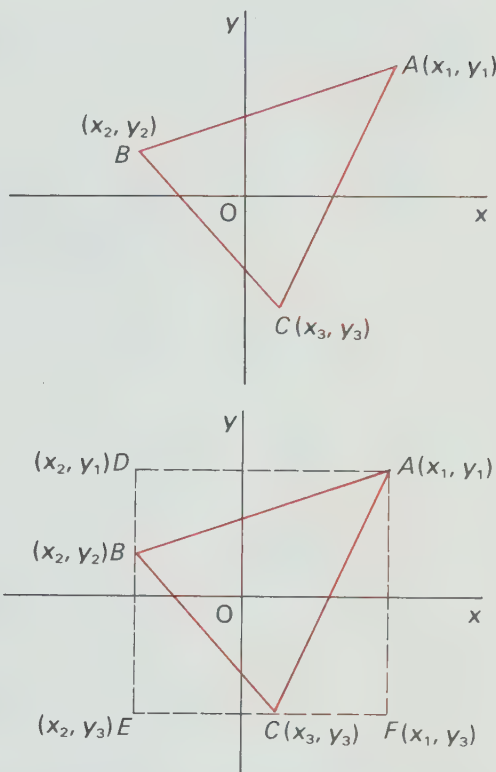
coordinates listed	down products	up products
(x_1, y_1)	$x_1 \rightarrow y_1$	$x_1 \rightarrow y_1$
(x_2, y_2)	$x_2 \rightarrow y_2$	$x_2 \rightarrow y_2$
(x_3, y_3)	$x_3 \rightarrow y_3$	$x_3 \rightarrow y_3$
(x_1, y_1)	$x_1 \rightarrow y_1$	$x_1 \rightarrow y_1$

The sum of the "down products" is $x_1y_2 + x_2y_3 + x_3y_1$

The sum of the "up products" is $x_1y_3 + x_3y_2 + x_2y_1$

This area of $\triangle ABC = \frac{1}{2} |(\text{sum of down products}) - (\text{sum of up products})|$

$$\triangle ABC = \frac{1}{2} |(x_1y_2 + x_2y_3 + x_3y_1) - (x_1y_3 + x_3y_2 + x_2y_1)|$$



EXAMPLE 3. Use the formula to determine the area of $\triangle RST$ with vertices $R(5, 6)$, $S(-3, 2)$ and $T(-1, 10)$

Solution

$$\begin{aligned}\triangle RST &= \frac{1}{2} |(10 - 30 - 6) - (50 - 2 - 18)| \\ &= \frac{1}{2} |-26 - 30| \\ &= \frac{1}{2} |-56| \\ &= 28\end{aligned}$$

The area of $\triangle RST$ is 28 square units.

EXAMPLE 4. Determine the area of quadrilateral $ABCD$ with vertices $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$.

Solution

Join AO , BO , CO , and DO .

The area of quadrilateral $ABCD$ is equal to the sum of the areas of $\triangle ABO$, $\triangle BCO$, $\triangle CDO$ and $\triangle DAO$. We take the vertices in a counterclockwise direction to ensure a positive result.

Area $ABCD$

$$\begin{aligned}&= \triangle ABO + \triangle BCO + \triangle CDO + \triangle DAO \\ &= \frac{1}{2}(x_1 y_2 - x_2 y_1) + \frac{1}{2}(x_2 y_3 - x_3 y_2) + \frac{1}{2}(x_3 y_4 - x_4 y_3) + \frac{1}{2}(x_4 y_1 - x_1 y_4) \\ &= \frac{1}{2}(x_1 y_2 - x_2 y_1 + x_2 y_3 - x_3 y_2 + x_3 y_4 - x_4 y_3 + x_4 y_1 - x_1 y_4) \\ &= \frac{1}{2}[(x_1 y_2 + x_2 y_3 + x_3 y_4 + x_4 y_1) - (x_1 y_4 + x_4 y_3 + x_3 y_2 + x_2 y_1)]\end{aligned}$$

This result is also found by listing the vertices in order, repeating the first and

$$\text{Area} = \frac{1}{2}[(\text{sum of down products}) - (\text{sum of up products})]$$

This method may be used to determine the area of any polygon provided that the vertices of the polygon are listed in order—either clockwise or counterclockwise. To ensure a positive result at all times we use absolute value bars.

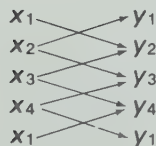
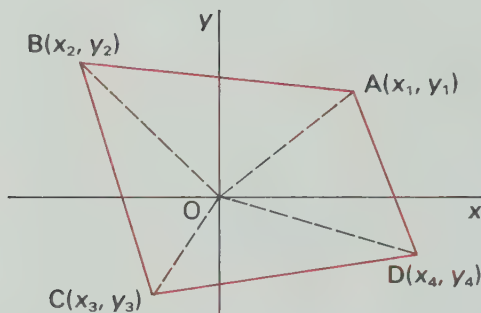
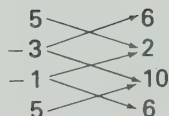
$$\text{Area} = \frac{1}{2} |(\text{sum of down products}) - (\text{sum of up products})|$$

EXERCISE 6-2

B 1. Given the vertices, determine the area of each of the following triangles.

- | | |
|---|--|
| (a) $(5, 6)$, $(1, 2)$, $(7, 3)$ | (b) $(10, 8)$, $(0, 3)$, $(6, 0)$ |
| (c) $(4, 3)$, $(-3, 1)$, $(0, -4)$ | (d) $(-1, 1)$, $(-6, 4)$, $(-4, -6)$ |
| (e) $(2, -1)$, $(-4, -8)$, $(4, -10)$ | (f) $(-2, 1)$, $(0, -4)$, $(5, 6)$ |
| (g) $(-4, 7)$, $(-7, -1)$, $(-2, -2)$ | (h) $(0, 6)$, $(-6, 2)$, $(6, 0)$ |
| (i) $(-6, -4)$, $(-4, 2)$, $(6, -2)$ | (j) $(2, -8)$, $(-4, 2)$, $(6, 6)$ |

2. By calculating an area, show that the points $A(1, 3)$, $B(-5, -9)$, and $C(6, 13)$ are collinear.



3. Use areas to determine which of the following sets of points are collinear.

(a) $A(6, 4)$, $B(-4, -5)$, $C(2, 0)$

(b) $R(0, 3)$, $S(9, 9)$, $T(3, 5)$

(c) $P(1, -1)$, $Q(-3, -4)$, $R(5, 2)$

4. Determine the area of each polygon having the given vertices.

(a) $(4, 6)$, $(-2, 4)$, $(-4, -2)$, $(6, -2)$

(b) $(5, 0)$, $(-3, 8)$, $(-5, -4)$, $(3, -2)$

(c) $(0, 6)$, $(-7, 4)$, $(-5, -4)$, $(3, -4)$, $(5, 2)$

(d) $(6, 5)$, $(-2, -3)$, $(4, -5)$, $(-4, 7)$, $(8, 1)$

5. The area of a triangle with vertices $(-2, 2)$, $(4, 6)$ and $(0, y)$ is 14 square units. Determine the value of y .

6. If the area of a triangle with vertices $(4, 2)$, $(-2, 4)$ and $(x, 0)$ is 8 square units, find the value of x .

7. $\triangle ABC$ has vertices $A(6, 4)$, $B(2, 1)$ and $C(8, 0)$. Determine the length of the perpendicular from C to AB .

C 8. If the area of the triangle with vertices $(-5, 3)$, $(4, 3)$ and (x, y) is 9 square units, determine the value of y .

9. Determine the equations of the lines that (x, y) satisfies if the triangle with vertices $(4, -1)$, $(1, 1)$ and (x, y) has an area of 14 square units.

10. If $A(x, y)$ is equidistant from $B(6, 4)$ and $C(4, -2)$ and the area of $\triangle ABC$ is 10 square units, find (x, y) .

WORD LADDER

Start with the word "rest" and change one letter at a time to form a new word until you reach "sofa". The best solution has the fewest steps.

rest

sofa

6.3 DISTANCE FROM A POINT TO A LINE

In this section we shall develop a formula for determining the distance from a point to a line.

EXAMPLE 1. Determine the distance from the point $P(1, -2)$ to the line defined by $2x + 3y - 12 = 0$.

Solution

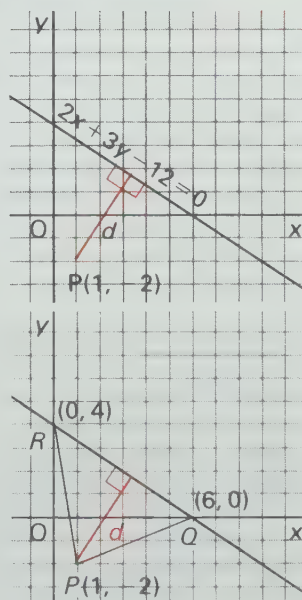
The line $2x + 3y - 12 = 0$ crosses the x -axis at $Q(6, 0)$ and the y -axis at $R(0, 4)$. Join PQ and PR .

The distance d from P to $2x + 3y - 12 = 0$ is an altitude of $\triangle PQR$.

$$\text{Area } \triangle PQR = \frac{1}{2}bh$$

$$\triangle PQR = \frac{1}{2} \cdot QR \cdot d$$

$$\therefore d = \frac{2\triangle PQR}{QR}$$



Since the 3 vertices of PQR are known we can determine the area of PQR by the method of section 6.2.

$$\begin{aligned}\text{Area } \triangle PQR &= \frac{1}{2} |(0+0+24)-(0-12+4)| \\ &= \frac{1}{2} |(24)-(-8)| \\ &= \frac{1}{2} |32| \\ &= 16 \text{ square units}\end{aligned}$$

We can determine the length of QR .

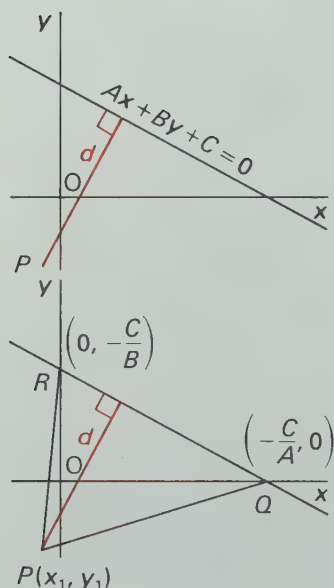
$$\begin{aligned}QR &= \sqrt{(0-6)^2 + (4-0)^2} \\ &= \sqrt{36+16} \\ &= \sqrt{52} \\ &= 2\sqrt{13}\end{aligned}$$

We can now find d .

$$\begin{aligned}d &= \frac{2\Delta PQR}{QR} \\ &= \frac{2(16)}{2\sqrt{13}} \\ &= \frac{16}{\sqrt{13}} \\ &= \frac{16\sqrt{13}}{13}\end{aligned}$$

the distance from $P(1, -2)$ to the line $2x+3y-12=0$ is $\frac{16\sqrt{13}}{13}$ units.

We shall now use the method of Example 1 to find the general formula for determining the distance from a point to a line.



EXAMPLE 2. Find the distance from the point $P(x_1, y_1)$ to the line $Ax+By+C=0$.

Solution

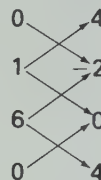
The line $Ax+By+C=0$ crosses the x -axis at $Q\left(-\frac{C}{A}, 0\right)$ and the y -axis at $R\left(0, -\frac{C}{B}\right)$. (When $A=0$ or $B=0$ the line is parallel to an axis. The proof that the formula remains the same is left to the student.)

Join PQ and PR .

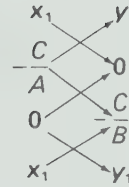
$$\text{Area } \triangle PQR = \frac{1}{2} \cdot QR \cdot d$$

$$\therefore d = \frac{2\Delta PQR}{QR}$$

The vertices of $\triangle PQR$ are (x_1, y_1) , $\left(-\frac{C}{A}, 0\right)$ and $\left(0, -\frac{C}{B}\right)$ so we can determine the area of $\triangle PQR$ by the method of section 6.2.



$$\begin{aligned}
 \text{Area } \triangle PQR &= \frac{1}{2} \left| \left(\frac{C^2}{AB} \right) - \left(-\frac{Cx_1}{B} - \frac{Cy_1}{A} \right) \right| \\
 &= \frac{1}{2} \left| \frac{C^2}{AB} + \frac{Cx_1}{B} + \frac{Cy_1}{A} \right| \\
 &= \frac{1}{2} \left| \frac{ACx_1 + BCy_1 + C^2}{AB} \right| \\
 &= \frac{C}{2AB} |Ax_1 + By_1 + C|
 \end{aligned}$$



We now determine the length of QR .

$$\begin{aligned}
 QR &= \sqrt{\left(0 + \frac{C}{A}\right)^2 + \left(-\frac{C}{B} - 0\right)^2} \\
 &= \sqrt{\frac{C^2}{A^2} + \frac{C^2}{B^2}} \\
 &= \sqrt{\frac{B^2 C^2 + A^2 C^2}{A^2 B^2}} \\
 &= \sqrt{\frac{C^2 (A^2 + B^2)}{A^2 B^2}} \\
 &= \frac{C}{AB} \sqrt{A^2 + B^2} \\
 d &= \frac{2\triangle PQR}{QR} \\
 &= \frac{2 \cdot \frac{C}{2AB} |Ax_1 + By_1 + C|}{\frac{C}{AB} \sqrt{A^2 + B^2}} \\
 &= \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}
 \end{aligned}$$

EXAMPLE 3. Determine the distance from the point $(-1, -4)$ to the line defined by $3x - y - 14 = 0$.

Solution

$$A = 3, B = -1, C = -14, x_1 = -1 \text{ and } y_1 = -4$$

$$\begin{aligned}
 d &= \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} \\
 &= \frac{|3(-1) - 1(-4) - 14|}{\sqrt{3^2 + (-1)^2}} \\
 &= \frac{|-3 + 4 - 14|}{\sqrt{9 + 1}}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{|-13|}{\sqrt{10}} \\
 &= \frac{13}{\sqrt{10}} \\
 &= \frac{13\sqrt{10}}{10}
 \end{aligned}$$

The distance from $(-1, -4)$ to $3x - y - 14 = 0$ is $\frac{13\sqrt{10}}{10}$ units.

EXERCISE 6-3

- B** 1. In each of the following find the distance from the origin to the given line.
- | | |
|-----------------------|------------------------|
| (a) $3x - 4y + 6 = 0$ | (b) $5x - 12y - 7 = 0$ |
| (c) $6x + 8y = 5$ | (d) $8x = 15y + 17$ |
| (e) $x - 3 = 0$ | (f) $2x - 3y = 4$ |
2. In each of the following find the distance from the given point to the given line.
- | | |
|---|------------------------------------|
| (a) $(1, 1)$, $3x + 4y + 2 = 0$ | (b) $(-3, -2)$, $5x - y + 8 = 0$ |
| (c) $(-3, 5)$, $4x - 5y = 7$ | (d) $(-5, 0)$, $7x = 3y - 5$ |
| (e) $(1, -2)$, $5y = 8 - 3x$ | (f) $(-3, -5)$, $7x - 2y = -11$ |
| (g) $(2, -2)$, $\frac{x}{2} + \frac{y}{3} = 1$ | (h) $(5, 5)$, $y = \frac{x+2}{4}$ |
3. Find the distance from $(-2, -2)$ to the line joining $(5, 2)$ and $(-1, 4)$.
4. Determine the distance between the given pair of parallel lines.
- | |
|---|
| (a) $y = 3x + 4$, $y = 3x - 7$ |
| (b) $4x + 3y - 10 = 0$, $4x + 3y + 10 = 0$ |
| (c) $5x + 12y = 26$, $5x + 12y = -2$ |
5. What are the lengths of the altitudes of the triangle whose vertices are $(1, -1)$, $(0, 5)$, and $(-3, 0)$?
6. Find the distance from the point of intersection of $2x + 3y = 10$ and $3x - y = 4$ to the line $5x - 6y = 1$.
7. Find the lengths of the altitudes of the triangle whose sides are defined by $2x - 3y + 6 = 0$, $x = 3$ and $2x + 3y - 6 = 0$.
- C** 8. Find the equations of the lines that (x, y) satisfies if (x, y) is 4 units from $3x + 4y + 6 = 0$.
9. Find the equation of the bisector of the acute angle formed by $12x - 5y + 7 = 0$ and $3x - 4y - 12 = 0$.

6.4 APPLICATIONS

In this section techniques studied in previous sections will be used to illustrate some theorems in geometry.

In most cases a diagram will be helpful.

EXAMPLE 1. Trapezoid $ABCD$ has vertices $A(0, 4)$, $B(-2, 2)$, $C(6, -2)$ and $D(4, 2)$ and $AD \parallel BC$. Prove that the line joining the midpoints of the non-parallel sides

(a) is parallel to the parallel sides

(b) has a length equal to half the sum of the parallel sides.

Solution

Let M be the midpoint of AB and N the midpoint of DC .

M has coordinates $\left(\frac{0-2}{2}, \frac{4+2}{2}\right)$ or $(-1, 3)$

N has coordinates $\left(\frac{4+6}{2}, \frac{2-2}{2}\right)$ or $(5, 0)$

$$(a) \quad m_{MN} = \frac{\Delta y}{\Delta x} = \frac{0-3}{5-(-1)} = -\frac{1}{2}$$

$$m_{AD} = \frac{2-4}{4-0} = -\frac{1}{2}$$

$\therefore MN$ is parallel to AD and BC

$$(b) \quad AD = \sqrt{(4-0)^2 + (2-4)^2} = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$$

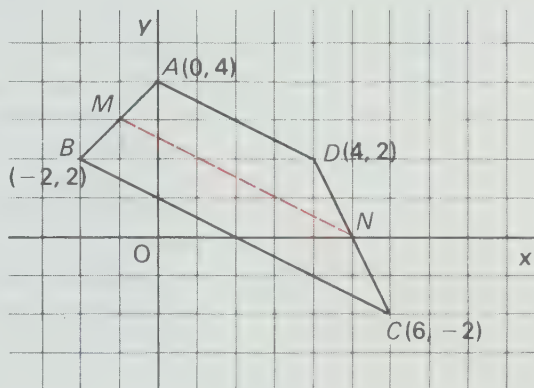
$$BC = \sqrt{(6+2)^2 + (-2-2)^2} = \sqrt{64+16} = \sqrt{80} = 4\sqrt{5}$$

$$\frac{1}{2}(AD + BC) = \frac{1}{2}(2\sqrt{5} + 4\sqrt{5}) = 3\sqrt{5}$$

$$MN = \sqrt{(5+1)^2 + (0-3)^2} = \sqrt{36+9} = \sqrt{45} = 3\sqrt{5}$$

$\therefore MN = \frac{1}{2}(AD + BC)$

MN is called the median of the trapezoid.



EXERCISE 6-4

B 1. $\triangle ABC$ has vertices $A(7, 6)$, $B(-3, -2)$ and $C(11, -4)$.

(a) Find D , the midpoint of AC . (b) Draw BD , a median of $\triangle ABC$.

(c) Find the area of $\triangle ABD$ and $\triangle BDC$ and show that a median of the triangle bisects the area.

2. Quadrilateral $ABCD$ has vertices $A(-6, 5)$, $B(-4, -3)$, $C(6, -5)$ and $D(8, 7)$.

(a) Find the midpoints of the sides of the quadrilateral.

(b) Prove that the midpoints of the sides are the vertices of a parallelogram.

3. (a) Find the vertices of the triangle whose sides lie on the lines $x - 3y + 8 = 0$, $7x - y - 24 = 0$ and $x + y = 0$.

(b) Find the area of the triangle.

4. $\triangle ABC$ has vertices $A(-10, 7)$, $B(2, 1)$ and $C(4, 5)$.

(a) Show that $\triangle ABC$ is right angled.

(b) Find the coordinates of D , the midpoint of the hypotenuse.

(c) Show that D is equidistant from the three vertices of the triangle.

5. Parallelogram has vertices $A(-4, 2)$, $B(-6, -4)$, $C(4, 2)$ and $D(6, 8)$.

(a) Draw diagonal BD .

(b) Find the area of $\triangle ABD$ and $\triangle BDC$ and show that the diagonal bisects the area of the parallelogram.

6. Find the fourth vertex of a parallelogram with three vertices $(3, 5)$, $(-2, 1)$ and $(6, 2)$. (There are three answers.)
7. Parallelogram $ABCD$ has vertices $A(-1, 1)$, $B(-3, -3)$, $C(9, 3)$ and $D(11, 7)$.
 - (a) Find the equation of diagonal AC .
 - (b) Find the equation of diagonal BD .
 - (c) Find the point of intersection E of AC and BD .
 - (d) Show that the diagonals of the parallelogram bisect each other.
8. $\triangle ABC$ has vertices $A(1, 9)$, $B(-3, -3)$ and $C(9, 3)$.
 - (a) Find the midpoint D of AB .
 - (b) Find the equation of the line through D and parallel to BC .
 - (c) Find the equation of AC .
 - (d) Find the point of intersection of the lines from (b) and (c) and show that this point is the midpoint of AC .
9. $\triangle ABC$ has vertices $A(7, 4)$, $B(-3, 2)$ and $C(5, -4)$. Compare the area of $\triangle ABC$ to the area of the triangle formed by joining the midpoints of the sides of $\triangle ABC$.
10. Quadrilateral $ABCD$ has vertices $A(5, 3)$, $B(-3, 7)$, $C(-5, -1)$ and $D(3, -5)$.
 - (a) Find the equations of the lines joining the midpoints of the opposite sides of the quadrilateral.
 - (b) Find the point of intersection of the lines found in (a).
 - (c) Show that the lines joining the midpoints of the opposite sides of the quadrilateral bisect each other.
11. Show that the points $A(3, 5)$, $B(1, -1)$ and $C(-2, -10)$ are collinear by showing that
 - (a) $m_{AB} = m_{BC}$
 - (b) $AC = AB + BC$
 - (c) the area of $\triangle ABC$ is zero
 - (d) B satisfies the equation of AB
 - (e) the distance from B to line AC is zero.
12. $\triangle ABC$ has vertices $A(4, 5)$, $B(-2, 3)$ and $C(2, -3)$.
 - (a) Find the equations of the three medians of $\triangle ABC$.
 - (b) Show that the three medians of the triangle are concurrent.
13. $\triangle ABC$ has vertices $A(-3, 4)$, $B(1, -4)$ and $C(5, 2)$.
 - (a) Determine the equations of the right bisectors of the three sides of the triangle.
 - (b) Show that the three right bisectors are concurrent.
14. $\triangle ABC$ has vertices $A(3, 4)$, $B(-3, 2)$ and $C(1, -2)$.
 - (a) Find the equation of the altitude from A to BC .
 - (b) Find the equation of the altitude from B to AC .
 - (c) Find the equation of the altitude from C to AB .
 - (d) Show that the three altitudes of the triangle are concurrent.
- C 15. The midpoints of the sides of a triangle are $(6, 2)$, $(3, 3)$ and $(4, -3)$. Find the coordinates of the vertices of the triangle.
16. $\triangle ABC$ has vertices $A(6, 4)$, $B(-2, 2)$ and $C(2, -4)$.
 - (a) Determine the equations of the altitudes and find their point of intersection, the orthocentre.

Three sportscasters have predicted winners for tonight's games as follows.

Sportscaster A:

Toronto, Winnipeg, Edmonton, Regina

Sportscaster B:

Montreal, Calgary, Winnipeg, Toronto

Sportscaster C:

Calgary, Winnipeg, Edmonton, Ottawa

No one picked British Columbia. Who plays whom?

- (b) Determine the equations of the medians and find their point of intersection, the centroid.
- (c) Determine the equations of the right bisectors of the three sides and find their point of intersection, the circumcentre.
- (d) Show that the orthocentre, centroid and circumcentre are collinear.

6.5 THE CIRCLE

A circle is the set of points in a plane that are a fixed distance from a fixed point. The fixed distance is the radius of the circle and the fixed point is its centre.

With the help of the distance formula we can determine the equation of a circle.

EXAMPLE 1. Find an equation of the circle having centre $O(0, 0)$ and radius 4.

Solution

Let $P(x, y)$ be any point on the circle. P is on the circle if and only if $PO = 4$, that is:

$$\begin{aligned}\sqrt{(x-0)^2 + (y-0)^2} &= 4 \\ (x-0)^2 + (y-0)^2 &= 16 \\ x^2 + y^2 &= 16\end{aligned}$$

Since the above steps can be reversed, a point belongs to the circle if and only if its coordinates satisfy $x^2 + y^2 = 16$.

$\therefore$ an equation of the circle is $x^2 + y^2 = 16$.

EXAMPLE 2. Find an equation of the circle with centre $O(0, 0)$ and radius r .

Solution

Let $P(x, y)$ be any point on the circle. Then $PO = r$.

$$\begin{aligned}\sqrt{(x-0)^2 + (y-0)^2} &= r \\ (x-0)^2 + (y-0)^2 &= r^2 \quad (\text{both sides are positive}) \\ x^2 + y^2 &= r^2\end{aligned}$$

Conversely if $P(x, y)$ is any point whose coordinates satisfy

$$x^2 + y^2 = r^2$$

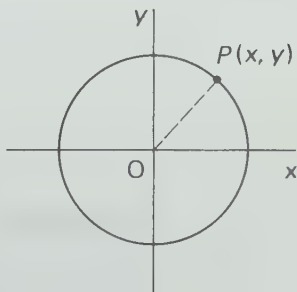
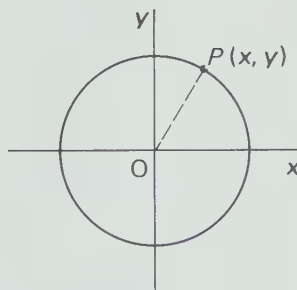
then

$$\sqrt{x^2 + y^2} = r$$

$$\therefore \sqrt{(x-0)^2 + (y-0)^2} = r$$

$$\therefore PO = r$$

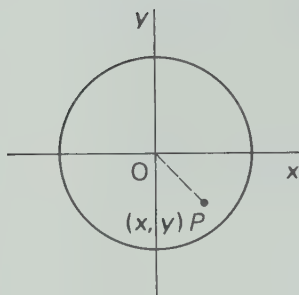
and $P(x, y)$ is a point of the circle.



An equation of the circle with centre $(0, 0)$ and radius r is

$$x^2 + y^2 = r^2$$

EXAMPLE 3. Determine an inequality that defines the interior region of a circle with centre $O(0, 0)$ and radius r .



Solution

Let $P(x, y)$ be any point in the interior region of the circle. Then $PO < r$, that is:

$$\sqrt{(x-0)^2 + (y-0)^2} < r$$

then

$$(x-0)^2 + (y-0)^2 < r^2$$

$$\therefore x^2 + y^2 < r^2$$

Conversely if $P(x, y)$ is any point whose coordinates satisfy

$$x^2 + y^2 < r^2$$

then

$$\sqrt{x^2 + y^2} < r$$

$$\therefore \sqrt{(x-0)^2 + (y-0)^2} < r$$

$$\therefore PO < r$$

and $P(x, y)$ lies in the interior region of the circle.

An inequality that defines the interior region of a circle with centre $(0, 0)$ and radius r is

$$x^2 + y^2 < r^2$$

Similarly it may be shown that

An inequality that defines the exterior region of a circle with centre $(0, 0)$ and radius r is

$$x^2 + y^2 > r^2$$

EXERCISE 6-5

A 1. State an equation of a circle with centre $(0, 0)$ and the given radius

- (a) 5 (b) 9 (c) $\sqrt{3}$ (d) $2\sqrt{5}$ (e) m

2. For each equation, give the radius of the circle.

- (a) $x^2 + y^2 = 36$ (b) $x^2 + y^2 = 64$ (c) $x^2 + y^2 = 144$
 (d) $x^2 + y^2 - 16 = 0$ (e) $x^2 + y^2 = 24$ (f) $x^2 + y^2 = 1$

3. State an inequality that defines the interior region of a circle with centre $(0, 0)$ and the given radius.

- (a) 7 (b) 2 (c) $\sqrt{5}$ (d) $3\sqrt{2}$ (e) a

4. State an inequality that defines the exterior region of a circle with centre $(0, 0)$ and the given radius.

- (a) 3 (b) 10 (c) $\sqrt{7}$ (d) $4\sqrt{3}$ (e) b

B 5. For each of the following state an equation of the circle with centre $(0, 0)$ and

- (a) passing through $(3, 4)$
 (b) passing through $(-4, 2)$
 (c) having an x -intercept of 6
 (d) having a y -intercept of 2

(e) passing through $(-\sqrt{2}, 2\sqrt{5})$

6. Draw the graphs of the following, $x, y \in R$.

- (a) $\{(x, y) \mid x^2 + y^2 < 9\}$
 (b) $\{(x, y) \mid x^2 + y^2 > 16\}$
 (c) $\{(x, y) \mid x^2 + y^2 \geq 36\}$ (d) $\{(x, y) \mid x^2 + y^2 \leq 4\}$
 (e) $\{(x, y) \mid x^2 + y^2 = 25\}$ (f) $\{(x, y) \mid x^2 + y^2 = 1\}$

7. Draw the graphs of the following, $x, y \in R$.

- (a) $\{(x, y) \mid x^2 + y^2 \leq 25 \text{ and } x^2 + y^2 \geq 16\}$
 (b) $\{(x, y) \mid x^2 + y^2 < 36 \text{ and } x \geq 0\}$
 (c) $\{(x, y) \mid x^2 + y^2 > 4 \text{ and } x > 0 \text{ and } y > 0\}$
 (d) $\{(x, y) \mid x^2 + y^2 \geq 9 \text{ and } x \leq 0 \text{ and } y \leq 0\}$
 (e) $\{(x, y) \mid x^2 + y^2 \leq 16 \text{ and } y \geq x\}$
 (f) $\{(x, y) \mid x^2 + y^2 < 49 \text{ and } y < x\}$

8. Find an equation of a circle with centre $(0, 0)$ and passing through the point of intersection of $3x - 2y = 2$ and $4x + 3y = -20$.

9. Determine which of the following define a circle with centre $(0, 0)$ and state the radius.

- (a) $x^2 + y^2 + 16 = 0$ (b) $x^2 - 20 = y^2$ (c) $4x^2 + 4y^2 = 36$
 (d) $2x^2 + 3y^2 = 24$ (e) $x^2 - y^2 = 1$ (f) $100 - x^2 - y^2 = 0$

C 10. Use the method of Example 1 to determine an equation of the circle with centre $(2, 1)$ and radius 5.

11. Use the method of Example 2 to determine an equation of the circle with centre (h, k) and radius r .

REVIEW EXERCISE

B 1. Find the distance between each pair of points.

- (a) $(7, 11), (5, 2)$ (b) $(-3, 4), (-5, 6)$
 (c) $(-2, 0), (4, -6)$ (d) $(8, -7), (6, -4)$
 (e) $(-3, -1), (-4, -2)$ (f) $(5, -1), (-5, 2)$

2. Find the midpoint of the line segment joining the given pair of points.

- (a) $(8, 6), (2, 2)$ (b) $(3, 1), (7, 11)$
 (c) $(-3, 6), (7, -8)$ (d) $(-2, -5), (-8, 7)$
 (e) $(3, -4), (8, -5)$ (f) $(7, -1), (-6, -4)$

3. Given the vertices, determine the area of each of the following triangles.

- (a) $(8, -2), (6, -4), (10, -6)$ (b) $(5, -3), (-6, 0), (-8, -6)$
(c) $(7, 3), (1, 1), (-1, -1)$ (d) $(6, 0), (0, -8), (10, -10)$
(e) $(-3, -4), (10, -2), (6, 8)$ (f) $(10, 6), (0, 8), (-4, -7)$

4. In each of the following find the distance from the given point to the given line.

- (a) $(2, 3), 3x - 4y - 1 = 0$ (b) $(-1, -3), 12x - 5y - 2 = 0$
(c) $(-3, 5), 4x - y = 6$ (d) $(4, -3), 3x + 2y = 7$
(e) $(6, 0), 4x = 5y + 9$ (f) $(4, -6), y = 3x + 4$

5. For each of the following state an equation of the circle with centre $(0, 0)$ and

- (a) passing through $(-3, -4)$
(b) passing through $(5, -12)$
(c) passing through $(7, 3)$
(d) having an x -intercept of -7
(e) having a y -intercept of 4 .

6. Draw the graphs of the following, $x, y \in R$

- (a) $\{(x, y) \mid x^2 + y^2 = 16\}$ (b) $\{(x, y) \mid x^2 + y^2 < 25\}$
(c) $\{(x, y) \mid x^2 + y^2 \geq 49\}$ (d) $\{(x, y) \mid x^2 + y^2 \leq 1\}$

7. Determine the area of the quadrilateral with vertices $(4, 2), (-2, 3), (-4, -5)$ and $(5, -3)$.

8. Determine the lengths of the altitudes of the triangle whose vertices are $(2, 2), (-4, 4)$ and $(0, -6)$.

9. Find the vertices of the triangle whose sides lie on the lines $5x + y - 16 = 0, 3x + 5y + 8 = 0$ and $x - 2y + 10 = 0$.

10. Determine which of the following sets of points are collinear.

- (a) $(4, 5), (-1, -5), (-5, -12)$
(b) $(4, -11), (1, -2), (-3, 10)$
(c) $(6, 0), (-4, -5), (10, 2)$
(d) $(7, 3), (-3, -3), (3, 1)$

11. Quadrilateral $ABCD$ has vertices $A(4, 2), B(-4, 0), C(-6, -6)$ and $D(6, -8)$. Show that the midpoints of the sides of the quadrilateral are the vertices of a parallelogram.

12. $\triangle ABC$ has vertices $A(-3, -1), B(7, 5)$ and $C(3, -3)$. Show that the median from A to BC bisects the area of the triangle.

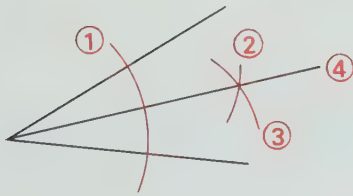
REVIEW AND PREVIEW TO CHAPTER 7

EXERCISE 1

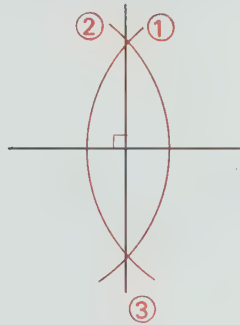
CONSTRUCTIONS

The following constructions are performed using ruler and compass only.

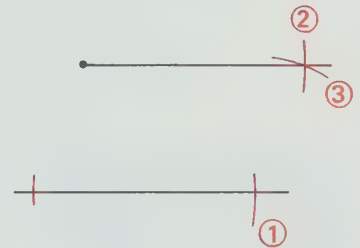
Angle Bisector



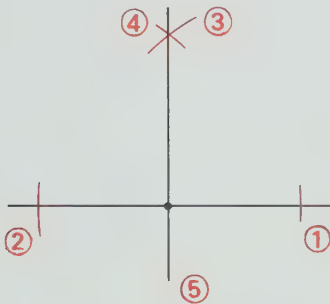
Right Bisector



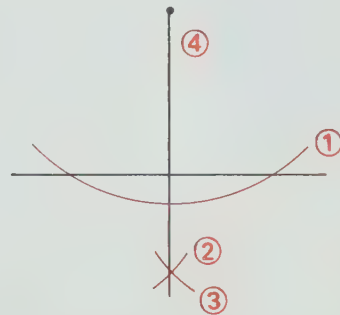
Parallel Line



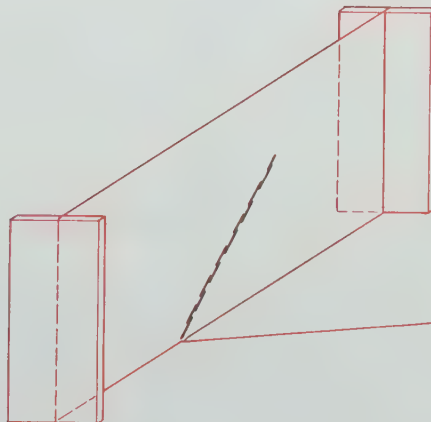
Perpendicular (at)



Perpendicular (from)



These constructions may also be performed using a semi-transparent mirror.



1. Construct and label angles whose measures are

(a) 90°

(b) 45°

(c) $22\frac{1}{2}^\circ$

(d) 135°

(e) 225°

(f) 315°

2. (a) Bisect the angles of (i) an acute triangle

(ii) an obtuse triangle.

(b) Use the point of intersection to draw the inscribed circle.

3. (a) Bisect the sides of (i) an acute triangle

(ii) an obtuse triangle.

(b) Use the point of intersection to draw the circumcircle.

4. (a) Draw the altitudes of (i) an acute triangle

(ii) an obtuse triangle.

(b) Do the altitudes cross at one point?

5. (a) Draw the medians of (i) an acute triangle, and

(ii) an obtuse triangle.

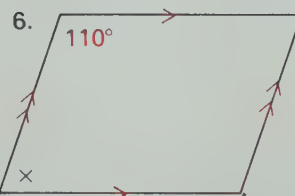
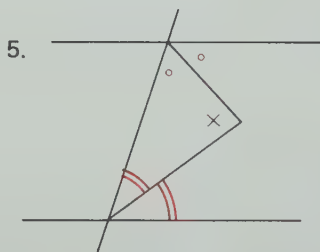
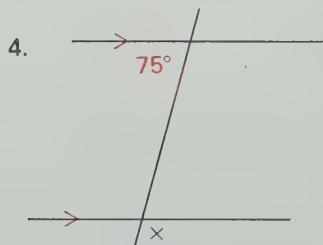
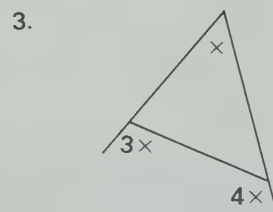
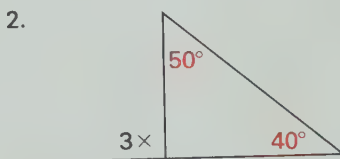
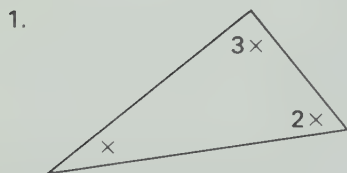
(b) Do the medians cross at one point?

6. In questions 2 to 5 above, locate the following centres of triangles: incentre, circumcentre, orthocentre, centroid. Give a definition for each centre and identify the centre located in each triangle above.

EXERCISE 2

ANGLES

Find the value of x in each of the following.



Congruence and Parallelism

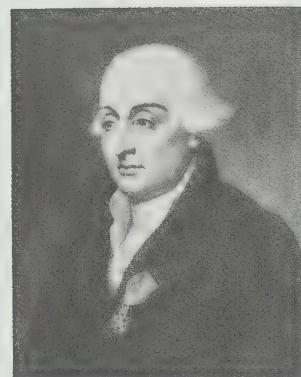
Reductio ad absurdum is one of a mathematician's finest weapons. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game.

G. H. Hardy

JOSEPH-LOUIS LAGRANGE (1736–1813)

Lagrange was born in Italy of mixed French and Italian blood. He was a child prodigy in mathematics, and in fact he became a professor of mathematics at the age of sixteen! By the time he was in his early twenties he was recognized as the equal of any mathematician in Europe, and today many people regard him as the leading mathematician of the eighteenth century. Frederick the Great of Prussia invited Lagrange to become director of the mathematical division of the Berlin Academy, and after Frederick's death King Louis XVI of France invited Lagrange to Paris as a member of the French Academy of Sciences. Even after the French Revolution, Lagrange's expertise was sought after by the rulers. Napoleon said, "Lagrange is the lofty pyramid of the mathematical sciences." Between military campaigns Napoleon often talked with Lagrange about philosophy and how mathematics could be useful to the state.

One of Lagrange's accomplishments as a member of the Committee on Weights and Measures was his part in perfecting the metric system. Some people had proposed the number 12 as the base for a metric system, but Lagrange insisted on 10.



Geometry is one of the oldest branches of mathematics. When people first became concerned with forms and shapes of things, geometry was born. In synthetic geometry we use the rules of arithmetic and algebra along with undefined terms, definitions and postulates in order to prove theorems. These are then used to prove deductions.

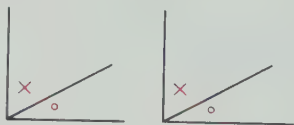
7.1 ANGLE AND CONGRUENCE THEOREMS

In previous work we began geometry by performing some basic constructions, and then followed with the deduction of other facts. In this section we will use some of these to expand our knowledge of geometry. The following is a list of some important theorems and postulates.

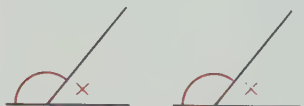


ANGLE THEOREMS

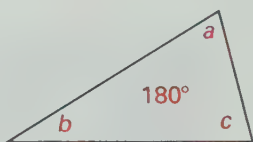
(OAT) Opposite Angle Theorem. If two lines intersect then the opposite angles are equal.



(CAT) Complementary Angle Theorem. If two angles are equal, then their complements are equal.

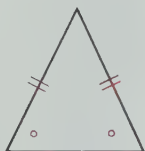


(SAT) Supplementary Angle Theorem. If two angles are equal, then their supplements are equal.

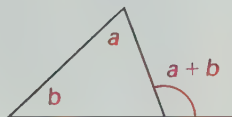


(SATT) Sum of the Angles of a Triangle Theorem. The sum of the interior angles of a triangle is 180° .

When two angles of one triangle are respectively equal to two angles of another triangle, the third angles are equal.

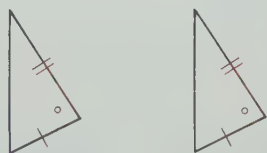


(ITT) Isosceles Triangle Theorem. The angles opposite the equal sides are equal.



(EAT) Exterior Angle Theorem. An exterior angle of a triangle is equal to the sum of the interior and non-adjacent angles.

POSTULATES



(SAS) Side-Angle-Side Congruence. If two sides and the contained angle of one triangle are respectively equal to two sides and the contained angle of another triangle, then the triangles are congruent.



(SSS) Side-Side-Side Congruence. If three sides of one triangle are respectively equal to three sides of another triangle, then the triangles are congruent.



(ASA) Angle-Side-Angle Congruence. If two angles and the contained side of one triangle are respectively equal to two angles and the contained side of another triangle, then the triangles are congruent.

EXAMPLE 1. AB and CD are diameters of a circle with centre O .
Prove $AC = DB$.

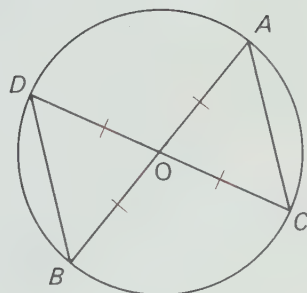
Solution

Given: Circle centre O , with diameters AB and CD .

Required: To prove $AC = DB$.

Proof

Statements	Reasons
In $\triangle AOC$ and $\triangle DOB$, $AO = DO$	radii
AB and DC intersect at O	given
$\angle AOC = \angle DOB$	OAT
$CO = BO$	radii
$\therefore \triangle AOC \cong \triangle DOB$	SAS
$\therefore AC = DB$	



EXAMPLE 2. In $\triangle ACD$, $AC = AD$ and DC is produced to B and CD is produced to E so that $CB = DE$. Prove that $\triangle ABD \cong \triangle AEC$ and $AB = AE$.

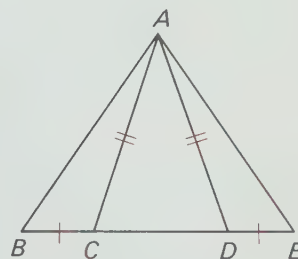
Solution

Given: $\triangle ACD$, $AC = AD$, DC produced to B and CD produced to E with $CB = DE$.

Required: To prove (i) $\triangle ABD \cong \triangle AEC$ (ii) $AB = AC$

Proof

Statements	Reasons
In $\triangle ACD$, $AC = AD$	given
$\therefore \angle ACD = \angle ADC$	ITT
$BC = ED$	given
$BC + CD = ED + DC$	addition
$BD = EC$	
In $\triangle ABD$ and $\triangle AEC$,	
$AD = AC$	given
$\angle ADB = \angle ACE$	proved above
$DB = CE$	proved above
$\therefore \triangle ABD \cong \triangle AEC$	SAS
$\therefore AB = AC$	

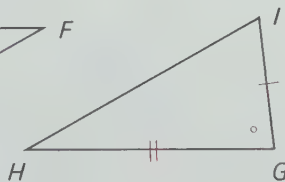
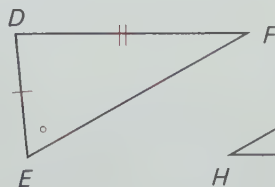
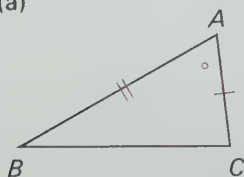


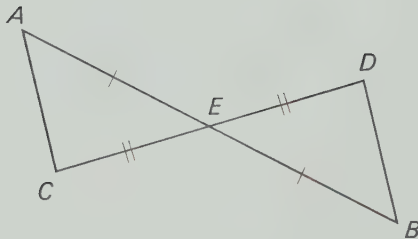
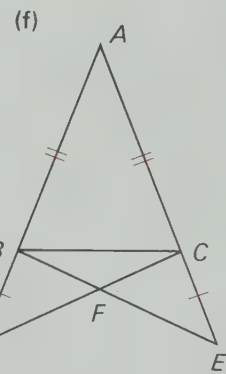
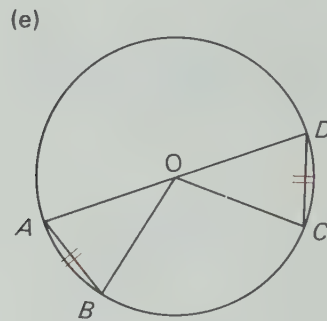
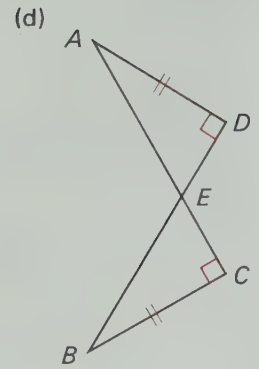
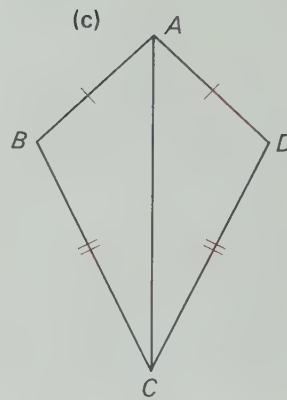
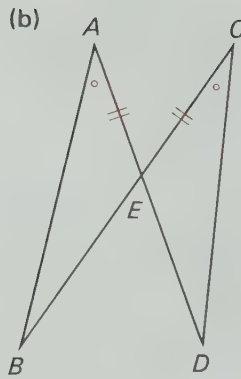
What is the only word in the English language that contains the letter sequence GNT?

EXERCISE 7-1

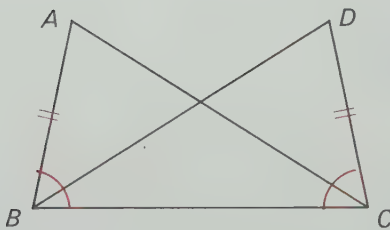
1. Name a pair of congruent triangles in each case and state the reason.

(a)

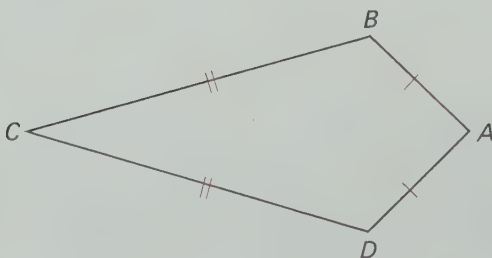




- 2.** Given: AB and CD bisect each other at E .
Required: To prove $AC = BD$.

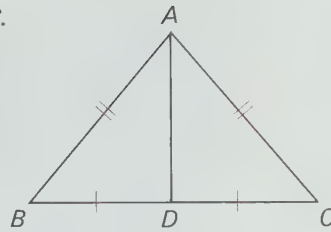


- 3.** Given: $\triangle ABC$ and $\triangle DCB$,
 $AB = DC$, $\angle ABC = \angle DCB$.
Required: To prove $\angle BAC = \angle CDB$.

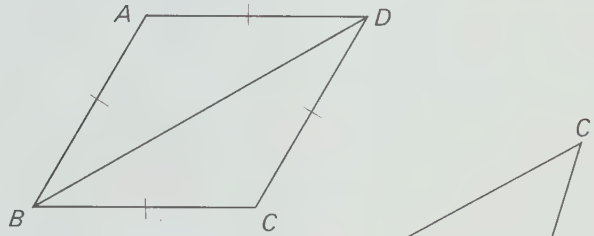


- 4.** Given: Quadrilateral $ABCD$
 $AB = AD$, $BC = DC$.
Required: To prove $\angle CBA = \angle CDA$.

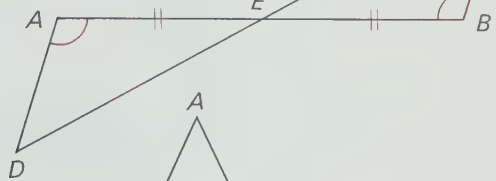
5. Given: $\triangle ABC$, $AB = AC$, D is the midpoint of BC .
Required: To prove $\angle ADB = \angle ADC = 90^\circ$.



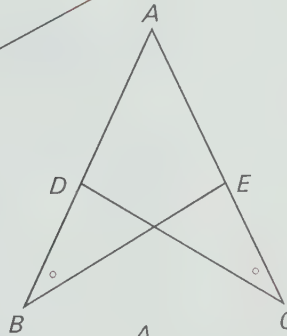
6. Given: Rhombus $ABCD$.
Required: To prove $\angle BAD = \angle BCD$.



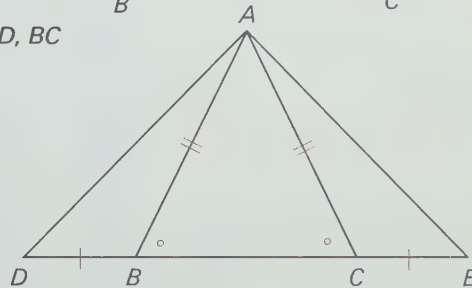
7. Given: AB and CD intersect at E ,
 $AE = BE$, $\angle DAE = \angle CBE$
Required: To prove $AD = BC$.



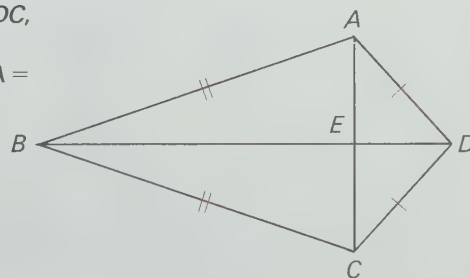
8. Given: $\triangle ABC$, $AB = AC$, D is on AB , E is on AC ,
 $\angle ABE = \angle ACD$.
Required: To prove $BE = CD$.



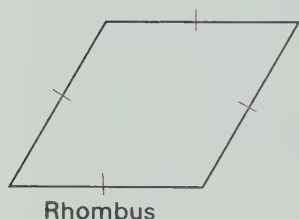
9. Given: $\triangle ABC$, $AB = AC$, CB is produced to D , BC is produced to E , $BD = CE$.
Required: To prove $\angle DAB = \angle EAC$.



10. Given: Quadrilateral $ABCD$, $AB = CB$, $AD = DC$,
 AC and BD intersect at E .
Required: To prove (i) $AE = EC$ (ii) $\angle BEA = \angle DEC = 90^\circ$.



The distance from a point to a line is the perpendicular distance.



C Prove the following statements, giving the “Given, Required, diagram and Proof.”

11. Any point on the bisector of an angle is equidistant from the arms of the angle.

12. Any point on the right bisector of a line segment is equidistant from the end points of the line segment.

13. If a quadrilateral has both pairs of opposite sides equal, then the opposite angles are equal.

14. The bisector of the vertical angle of an isosceles triangle is the right bisector of the base.

15. The diagonals of a rhombus right bisect each other.

7.2 INEQUALITY RELATIONS OF A TRIANGLE

In this section we shall see *converses* of statements and we shall use *indirect* proofs to prove some triangle inequalities.

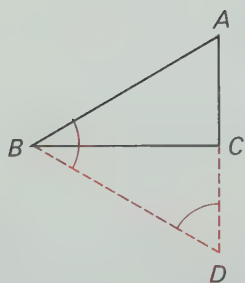
(IRT-1) *Inequality Relation of a Triangle—Part 1.*

If one side of a triangle is greater than a second side, then the angle opposite the greater side is greater than the angle opposite the smaller side.

Given: $\triangle ABC$, $AB > AC$

Required: To prove $\angle ACB > \angle ABC$.

Proof



Produce AC to D so that

$$AD = AB$$

construction

$$\angle ABD = \angle ADB$$

ITT

$$\text{Since } \angle ABD = \angle ABC + \angle CBD$$

$$\therefore \angle ABD > \angle ABC$$

$$\angle ABD = \angle ADB$$

proved above

$$\therefore \angle ADB > \angle ABC$$

In $\triangle BCD$, $\angle ACB$ is an exterior angle.

$$\angle ACB = \angle BDC + \angle CBD \quad \text{EAT}$$

$$\therefore \angle ACB > \angle BDC$$

$$\therefore \angle ACB > \angle ADB$$

$$\angle ADB = \angle BDC$$

$$\angle ADB > \angle ABC$$

proved above

$$\therefore \angle ACB > \angle ABC$$

transitive property of order.

The converse of a simple statement is formed by interchanging the “if... , then...” parts of the statement. This results in interchanging the *Given* and *Required* in the deductive proof of a statement. One

must be careful not to conclude that if a statement is true, its converse is necessarily true. For example, while the statement “If a triangle has two equal sides, then the angles opposite these sides are equal” and its converse “If a triangle has two equal angles, then the sides opposite these angles are equal” are both true, there are cases where the converse is not true. The converse statement of “If two angles are right angles, then they are equal” is “If two angles are equal then they are right angles” which of course is not true. We shall now state the converse of IRT-1 as a second theorem of triangle inequality and prove the theorem using an indirect proof.

(IRT-2) *Inequality Relation of a Triangle—Part 2.*

If one angle of a triangle is greater than a second angle, then the side opposite the greater angle is greater than the side opposite the smaller angle.

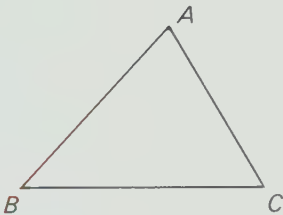
Given: $\triangle ABC$, $\angle ACB > \angle ABC$
 Required: To prove $AB > AC$.
 Proof: (INDIRECT)

Statements	Reasons
There are three possibilities for AB and AC :	trichotomy property
(1) $AB < AC$	
(2) $AB = AC$	
(3) $AB > AC$	
Assuming (1) $AB < AC$	
$\therefore \angle ACB < \angle ABC$	IRT-1
But $\angle ACB > \angle ABC$	given
Hence $\angle ACB < \angle ABC$ is false and so	
(1) $AB < AC$ is impossible.	
Assuming (2) $AB = AC$	
$\therefore \angle ACB = \angle ABC$	ITT
But $\angle ACB > \angle ABC$	given
Hence $\angle ACB = \angle ABC$ is false and so	
(2) $AB = AC$ is impossible.	
The only remaining possibility	
is (3) $AB > AC$	
which is what we wanted to prove.	

When an “if . . . , then . . .” statement and its converse are both true, we can combine the two statements as a single *dual* statement using the connective “...if and only if...”. We now combine IRT-1 and IRT-2 as a single dual theorem as follows.

(IRT) *Inequality Relation of a Triangle.*

One angle of a triangle is larger than another angle if and only if it is opposite the longer side.



WORD LADDER

Start with the word “tiger” and change one letter at a time to form a new word until you reach “roses”. The best solution has the fewest steps.

t i g e r

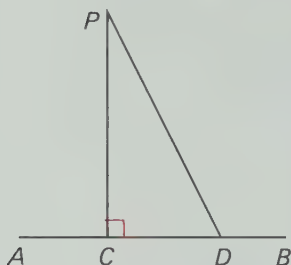
 r o s e s

When a dual statement is to be proved one of the simple statements and its converse must be proved.

The following are the steps for writing an indirect proof:

1. State all possibilities (This usually involves [=, or $\neq$] or [$<$, =, or $>$]).
2. Assume a negation of what you want to prove is true.
3. Reason correctly from the assumption until a contradiction of a known postulate, theorem or given fact is reached.
4. State that what was assumed to be true in step 2 is false. Hence it follows that one of the other possibilities is true.
5. Eventually the one remaining possibility is the desired conclusion.

Indirect proof is also called *proof by contradiction* or *reductio ad absurdum*.



EXAMPLE 1. Prove the following statement using an indirect proof. The shortest distance from a point to a line is the perpendicular distance.

Solution

Given: Line AB and P not on AB . C is on AB and $PC \perp AB$.

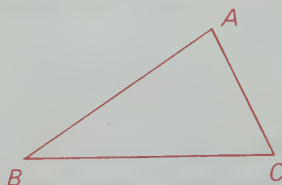
Required: To prove PC is the shortest distance to AB .

Proof:

Statements	Reasons
PC is the shortest distance to AB or it is not the shortest distance.	
Assume PC is not the shortest distance so that there exists D on AB where $PD < PC$	assumption
$\therefore \angle PCD < \angle PDC$	IRT
Since $PC \perp AB$	given
$\angle PCA = 90^\circ = \angle PCD$	definition of perpendicular
$\therefore \angle PCA < \angle PDC$	
In $\triangle PCD$, $\angle PCA = \angle PDC + \angle CPA$	EAT
$\angle PCA > \angle PDC$	
Hence $\angle PCA < \angle PDC$ is false and PC is the shortest distance.	

To complete this section, we can prove an additional inequality of the triangle:

The sum of any two sides of a triangle is greater than the third side.



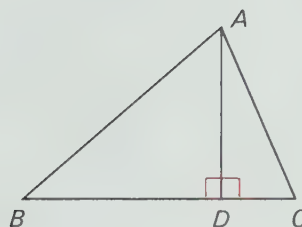
$$\begin{aligned}
 AB + BC &> AC \\
 AC + CB &> AB \\
 BA + AC &> BC
 \end{aligned}$$

If a ball of yarn 16 cm in diameter will make one pair of gloves, how many pairs will a ball 32 cm in diameter make?

Given: $\triangle ABC$

Required: To prove $AB + BC > AC$
 $AC + CB > AB$
 $BA + AC > BC$

Proof



Draw $AD \perp BC$

In $\triangle ABD$, $\angle ADB > \angle BAD$

$BA > BD$

IRT

In $\triangle ADC$, $\angle ADC > \angle DAC$

$AC > DC$

IRT

$BA > BD$

proved above

$BA + AC > BD + DC$

addition

Since $BD + DC = BC$

$BA + AC > BC$

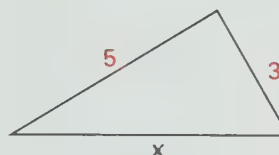
substitution

Similarly, we can prove

$AB + BC > AC$

$AC + CB > AB$

If a triangle has two sides with lengths 3 units and 5 units then we can find that the length of the third side lies between 2 and 8 in the following manner.



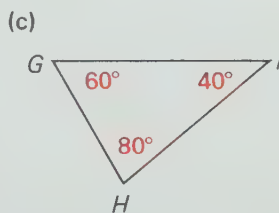
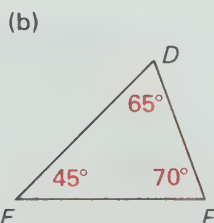
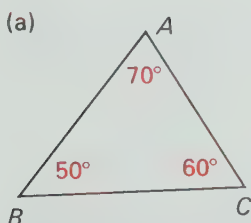
$$5 - 3 < x < 5 + 3$$

$$2 < x < 8$$

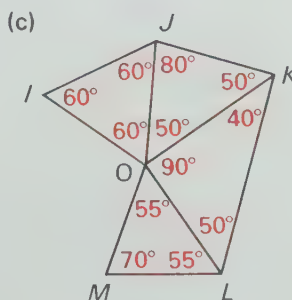
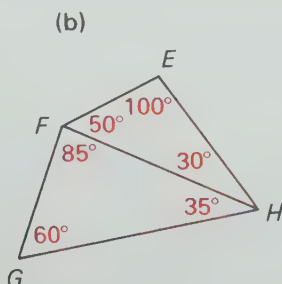
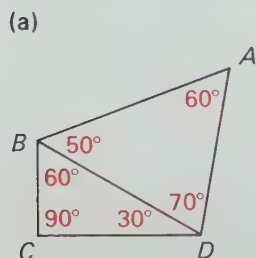
EXERCISE 7-2

A

1. Name the line segments in each figure in order of increasing length.



2. State the longest side in each of the following.



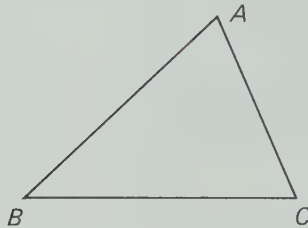
3. State whether it is possible to draw a triangle with the following sides:

- (a) 2, 5, 6 (b) 3, 4, 8 (c) 3.5, 2, 5 (d) 2, 3, 6
 (e) $\sqrt{2}$, $\sqrt{3}$, 2 (f) 3, 3, 4 (g) 2, 5, 2 (h) 1, 4, 4

4. The lengths of two of the sides of a triangle are given. Between what two lengths will the third side lie?

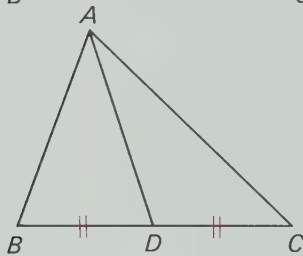
- (a) 3, 5 (b) 7, 15 (c) 5, 5 (d) 11, 12
 (e) 8, 8 (f) 8, 11 (g) 9, 19 (h) a , b , $a > b$

B Give an indirect proof for each of the following.



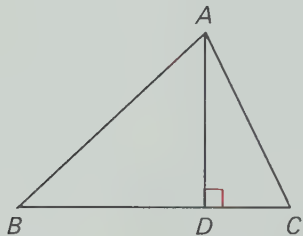
5. Given: $\triangle ABC$, $AB \neq AC$

Required: To prove $\angle ABC \neq \angle ACB$.



6. Given: In $\triangle ABC$, AD is a median but not an altitude.

Required: To prove $\angle ABC \neq \angle ACB$.



7. Given: $\triangle ABC$ is scalene and AD is an altitude.

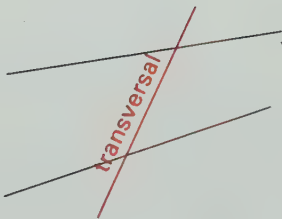
Required: To prove AD is not a median.

C 8. The bisector of any angle of a scalene triangle is not perpendicular to the opposite side.

9. Only one perpendicular can be drawn from a point to a line.

7.3 PARALLEL LINES

Two lines are parallel if and only if they lie in the same plane and do not intersect. A transversal is a line that intersects two or more distinct lines. When a transversal meets two parallel lines certain angle relationships are true. The *Transversal Parallel Theorem* (TPT) is a dual theorem, and requires a double proof—the proof of the simple statement, and its converse.



(TPT-1) Transversal Parallel Theorem—Part 1

If a transversal meets two straight lines in the same plane making the alternate angles equal, then the lines are parallel.

Given: Two lines AB and CD with transversal $EFGH$, $\angle AFG = \angle FGD$.

Required: To prove $AB \parallel CD$.

Proof

Statements	Reasons
Either (1) $AB \nparallel CD$ and they intersect or	
(2) $AB \parallel CD$	
Assume (1) $AB \nparallel CD$	
Let AB and CD intersect at X	
(The proof is similar if the lines intersect on the other side of EH)	

In $\triangle FGX$, $\angle AFG = \angle FGX + \angle FXG$ EAT

$$\therefore \angle AFG > \angle FGX$$

$$\angle FGX = \angle FGD$$

same angle

$$\therefore \angle AFG > \angle FGD$$

$$\text{But } \angle AFG = \angle FGD$$

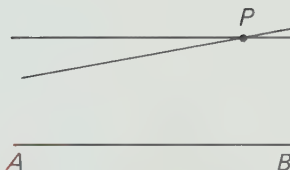
given

Hence (1) $AB \nparallel CD$ is false, and $AB \parallel CD$.

In order to prove the converse of (TPT-1), we first state the *parallel line postulate*.

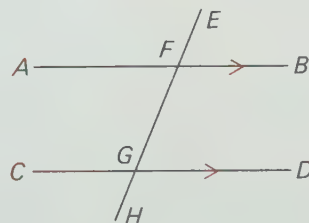
(PLP) Parallel Line Postulate

Through a given point, there is at most one line parallel to a given line.



(TPT-2) Transversal Parallel Theorem—Part 2

If a transversal meets two parallel lines, then the alternate angles are equal.

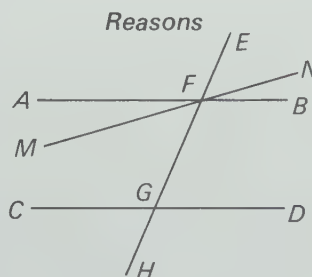


Given: $AB \parallel CD$, transversal EH cutting AB at F , CD at G .

Required: To prove $\angle AFG = \angle FGD$.

Proof

Statements	Reasons
Either (1) $\angle AFG \neq \angle FGD$, or	
(2) $\angle AFG = \angle FGD$	
Assume (1) $\angle AFG \neq \angle FGD$.	



Therefore there must exist a line MN through F such that $\angle MFG = \angle FGD$ ($\angle MFG$ and $\angle FGD$ are alternate angles)

$$\therefore MN \parallel CD$$

$$\text{But } AB \parallel CD$$

$$\text{Hence } MN \parallel CD$$

and $\angle AFG \neq \angle FGD$ is false, and
 $\angle AFG = \angle FGD$.

TPT-1

given

PLP

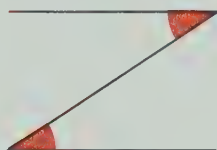
Addition

BEAVER

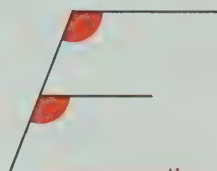
TI GER

RABBIT

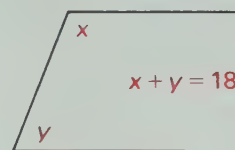
It is left to the student to extend TPT-1 and TPT-2 to corresponding angles and also to the interior angles on the same side of the transversal. The three cases are remembered in the Z, F, and C patterns given below.



alternate
angles



corresponding
angles

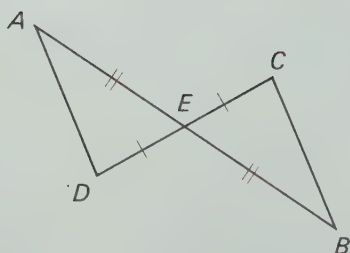


interior angles
on the same side

Noting that TPT-2 is the converse of TPT-1, we bring together the ideas of the alternate, corresponding and interior angles in the following theorem.

(TPT) *Transversal Parallel Theorem*

When a transversal intersects two straight lines, the lines are parallel if and only if (1) the alternate angles are equal or (2) the corresponding angles are equal or (3) the interior angles on the same side of the transversal are supplementary.



EXAMPLE 1. Two line segments AB and CD intersect and bisect each other at E . Prove $AD \parallel BC$.

Solution

Given: AB and CD intersect at E .

$$AE = BE, \quad DE = CE.$$

Required: To prove $AD \parallel BC$.

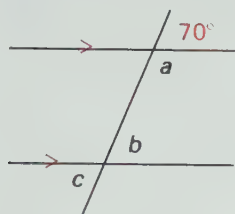
Proof

Statements	Reasons
In $\triangle AED$ and $\triangle BEC$,	
$AE = BE$	given
$\angle AED = \angle BEC$	OAT
$ED = EC$	given
$\therefore \triangle AED \cong \triangle BEC$	SAS
$\therefore \angle DAE = \angle CBE$	
$\therefore AD \parallel CB$	TPT

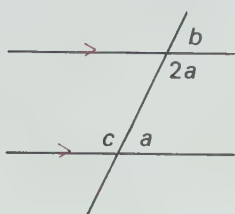
EXERCISE 7-3

A 1. Find the values of a , b , and c :

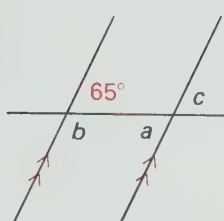
(a)



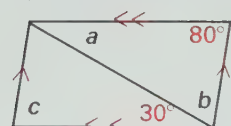
(b)



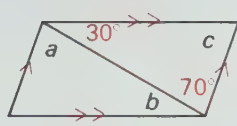
(c)



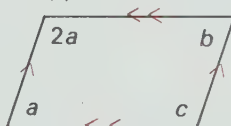
(d)



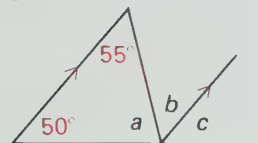
(e)



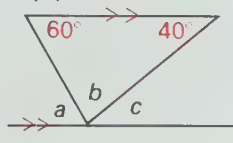
(f)



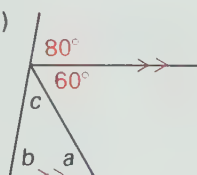
(g)



(h)

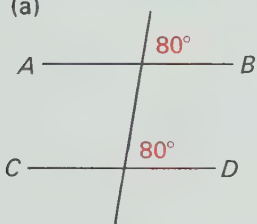


(i)

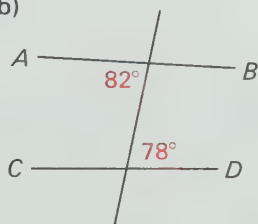


2. In each of the following, state with reasons whether you can conclude $AB \parallel CD$.

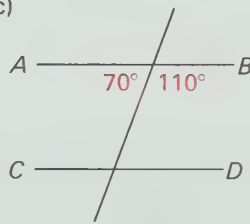
(a)



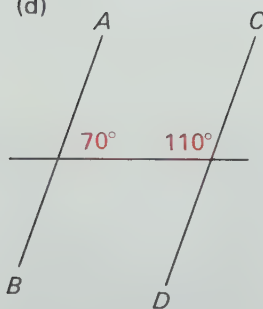
(b)



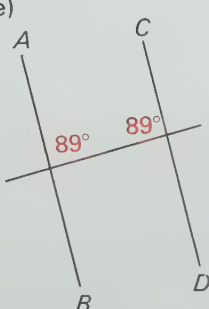
(c)



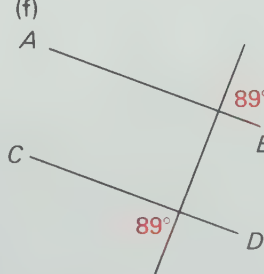
(d)

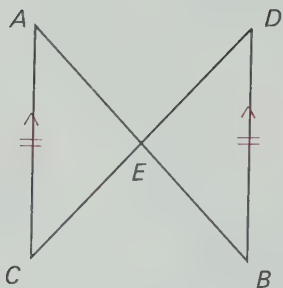


(e)



(f)

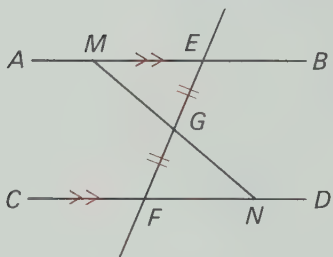




B Give complete proofs for the following:

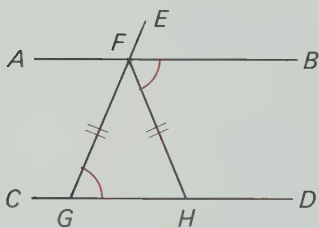
3. Given: AB and CD intersect at E . $AC = DB$, $AC \parallel DB$.

Required: To prove $\triangle AEC \cong \triangle DEB$.



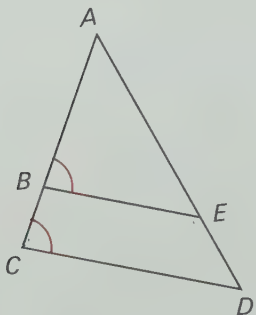
4. Given: $AB \parallel CD$, G is the midpoint of transversal EF . MN is a transversal passing through G .

Required: To prove $MG = NG$.



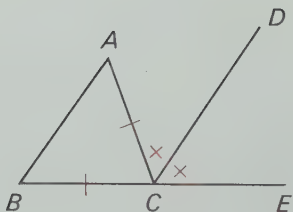
5. Given: $GF = FH$, $\angle BFH = \angle FGH$.

Required: To prove $AB \parallel CD$.



6. Given: $\triangle ACD$, $\angle ABE = \angle ACD$.

Required: To prove $\angle AEB = \angle ADC$.

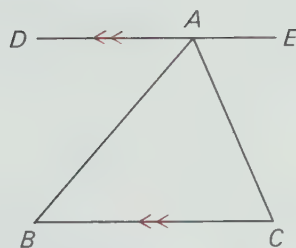


7. Given: $\triangle ABC$, exterior angle $\angle ACE$, $\angle ACD = \angle DCE$, $AC = BC$.

Required: To prove $AB \parallel DC$.

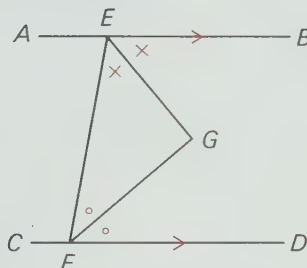
8. Given: $\triangle ABC$, line DE passing through A , $DE \parallel BC$.

Required: To prove $\angle ABC + \angle BAC + \angle BCA = 180^\circ$.



9. Given: $AB \parallel CD$, EF a transversal, EG bisects $\angle BEF$, FG bisects $\angle EFD$.

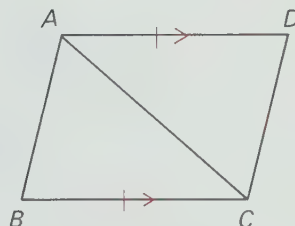
Required: To prove $\angle EGF = 90^\circ$.



10. Given: Quadrilateral $ABCD$, $AD = BC$, $AD \parallel BC$.

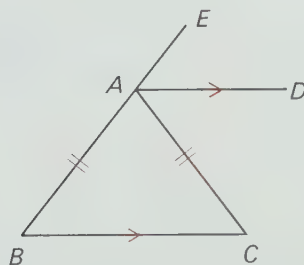
Required: To prove (1) $AB \parallel DC$

(2) $AB = DC$.



11. Given: $\triangle ABC$, $AB = BC$, $AD \parallel BC$.

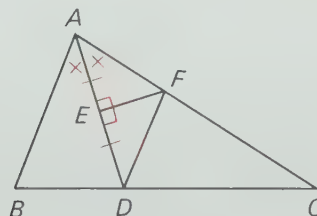
Required: To prove AD bisects $\angle EAC$.



12. Given: $\triangle ABC$, AD bisects $\angle BAC$

$AE = ED$, $EF \perp AD$.

Required: To prove $AB \parallel FD$.



C Prove the following:

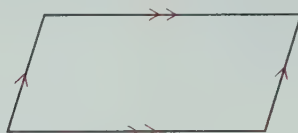
13. Two lines perpendicular to the same line are parallel to each other.

14. Two lines parallel to the same line are parallel to each other.

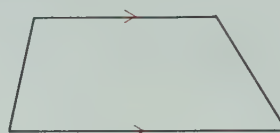
15. If two parallel lines are cut by a transversal, then the bisectors of any two corresponding angles are parallel.

7.4 PARALLELOGRAMS AND TRAPEZOIDS

In this section we shall consider the relationships resulting from the parallel lines in two special quadrilaterals:



Parallelogram



Trapezoid

Parallelogram: a quadrilateral with opposite sides parallel.

Trapezoid: a quadrilateral with one pair of sides parallel.

(PT) *Parallelogram Theorem*

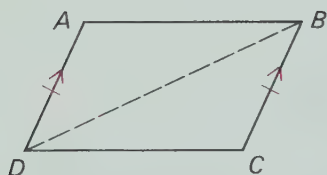
In any parallelogram,

- (i) the opposite angles are equal
- (ii) the opposite sides are equal
- (iii) the diagonals bisect each other.
- (iv) each diagonal bisects the parallelogram.

The proofs for the parts of this theorem are left to the student.
See Exercise 7-4, question 1.

(QPT) *Quadrilateral-Parallelogram Theorem*

If one pair of opposite sides of a quadrilateral are equal and parallel, then the figure is a parallelogram.



Given: Quadrilateral $ABCD$, $AB = DC$, $AB \parallel DC$.

Required: To prove $ABCD$ is a parallelogram.

Proof

Statements	Reasons
Join BD .	
In $\triangle ABD$ and $\triangle CDB$,	
$AB \parallel DC$,	given
$\angle ABD = \angle CDB$	TPT
$AB = CD$	given
$BD = BD$	common
$\therefore \triangle ABD \cong \triangle CDB$	SAS
$\therefore \angle ADB = \angle CBD$	
$\therefore AD \parallel BC$	TPT
$AB \parallel DC$	given
$\therefore ABCD$ is a parallelogram.	definition

This theorem is used as an authority to prove the Two-Midpoint Theorem (TMT) which follows.

(TMT) Two-Midpoint Theorem

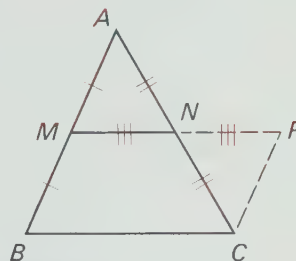
The line joining the midpoints of two sides of any triangle is parallel to the third side and equal to one-half of it.

Solution

Given: $\triangle ABC$, with M and N the midpoints of AB and AC respectively.

Required: To prove $MN \parallel BC$, $MN = \frac{1}{2}BC$.

Proof:



Statements	Reasons
Extend MN to P so that	
$NP = MN$	
In $\triangle MAN$ and $\triangle PCN$	
$AN = CN$	given
$\angle MNA = \angle PNC$	OAT
$MN = PN$	construction
$\therefore \triangle MAN \cong \triangle PCN$	SAS
$\therefore \angle MAN = \angle PCN$	corresponding angles
$\therefore AB \parallel PC$	TPT
and $AM = PC$	corresponding sides
$AM = MB$	definition of midpoint
$MB = PC$	transitive property
$MB \parallel PC$	proved above
$\therefore MBCP$ is a parallelogram	QPT
$\therefore MN \parallel BC$	
$MN = \frac{1}{2}MP$	construction
$\therefore MN = \frac{1}{2}BC$	PT

(MPT) Midpoint Parallel Theorem

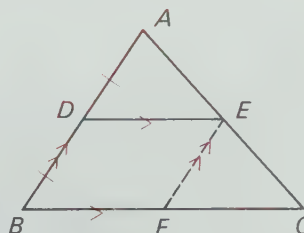
If a line bisects one side of a triangle and is parallel to a second side, then it bisects the third side.

Given: $\triangle ABC$, $AD = DB$, $DE \parallel BC$

Required: To prove $AE = EC$

Proof

Statements	Reasons
Draw $EF \parallel DB$	construction
$DE \parallel BF$	given
$\therefore DBFE$ is a parallelogram	definition
$\therefore DB = EF$	PT
$DB = AD$	given



In $\triangle ADE$ and $\triangle EFC$,

$$AD = EF$$

transitive property

$$AB \parallel EF$$

construction

$$\therefore \angle EAD = \angle CEF$$

TPT

$$DE \parallel BC$$

given

$$\therefore \angle AED = \angle ECF$$

TPT

$$\angle ADE = \angle EFC$$

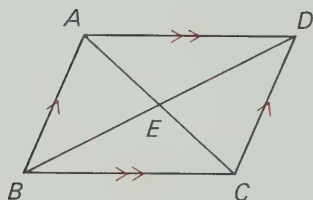
SATT

$$\therefore \triangle ADE \cong \triangle EFC$$

ASA

$$\therefore AE = EC$$

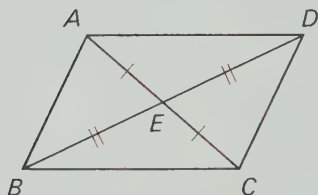
EXERCISE 7-4



- 1.** Given: $\parallel^{\text{gm}} ABCD$ with diagonals intersecting at E .

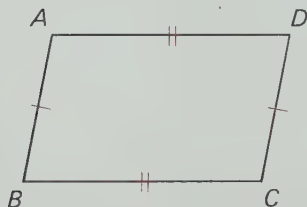
Required: To prove

- (a) $AB = DC$, $AD = BC$
- (b) $\angle ABC = \angle CDA$, $\angle BAD = \angle BCD$
- (c) $AE = EC$, $BE = ED$
- (d) $\triangle ABC \cong \triangle CDA$, $\triangle BAD \cong \triangle DCB$



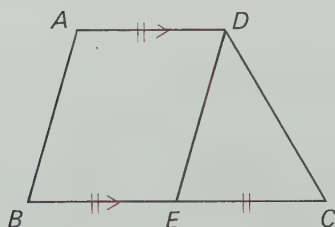
- 2.** Given: Quadrilateral $ABCD$, where the diagonals intersect at E , making $AE = CE$, $BE = DE$.

Required: To prove $ABCD$ is a parallelogram.



- 3.** Given: Quadrilateral $ABCD$, where $AD = BC$ and $AB = DC$.

Required: To prove $ABCD$ is a parallelogram.

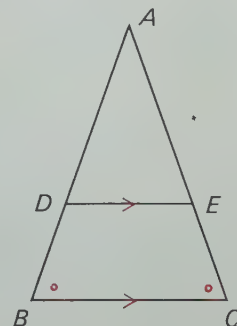


- 4.** Given: Trapezoid $ABCD$, $AD \parallel BC$, $AD = BE = EC$.

Required: To prove $ABED$ is a parallelogram.

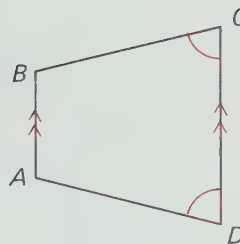
- 5.** Given: $\triangle ABC$, $\angle ABC = \angle ACB$, $DE \parallel BC$.

Required: To prove $AD = AE$.



6. Given: Trapezoid $ABCD$, where $AB \parallel DC$ and $\angle ADC = \angle BCD$.

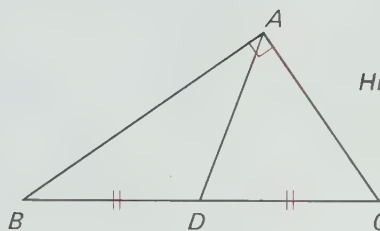
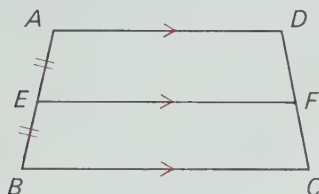
Required: To prove $AD = BC$.



7. Given: Trapezoid $ABCD$, $AE = EB$, $EF \parallel BC \parallel AD$.

Required: To prove $DF = FC$.

(Hint: Join AC)



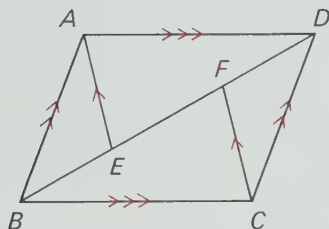
Hint Draw $DE \parallel CA$

8. Given: $\triangle ABC$, $BD = DC$, $BA \perp AC$.

Required: To prove $BD = AD$.

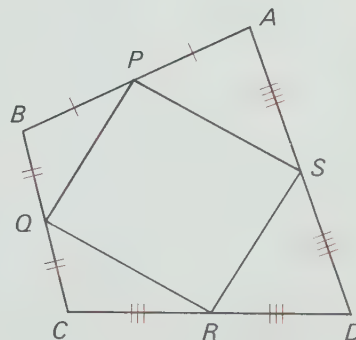
9. Given: Parallelogram $ABCD$, E and F are points on BD such that $AE \parallel FC$.

Required: To prove $BE = DF$.



10. Given: Quadrilateral $ABCD$, $AP = PB$, $BQ = QC$, $CR = RD$, $AS = SD$.

Required: To prove $PQRS$ is a parallelogram.



C Prove the following.

11. The diagonals of a rhombus bisect each other at right angles.

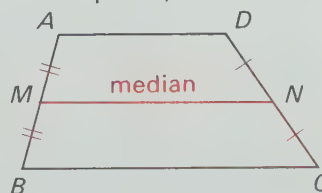
12. The quadrilateral formed by joining the midpoints of the sides of a parallelogram is also a parallelogram.

13. The diagonals of a rectangle are equal.

14. (MPT) If a line bisects one side of a triangle and is parallel to a second side, then it bisects the third side. (Use an indirect proof.)

15. (MTT) Median of a Trapezoid Theorem.

The median of a trapezoid is (i) parallel to the bases and (ii) has a length equal to one-half the sum of the lengths of the bases.

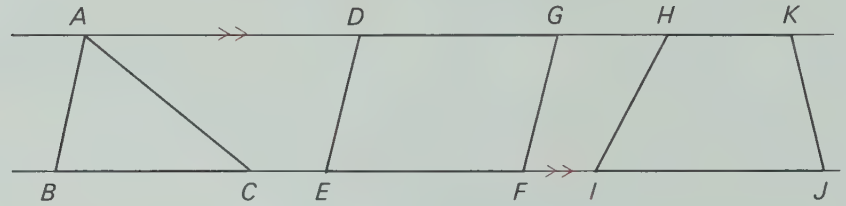


7.5 FIGURES BETWEEN THE SAME PARALLELS

When we write $\triangle ABC \cong \triangle DEF$, we mean that the triangles are congruent—equal in all respects including area. If we wish to state that two triangles are equal only in area, then the equal sign is used. $\triangle ABC = \triangle DEF$ means that the triangles ABC and DEF are equal in area.

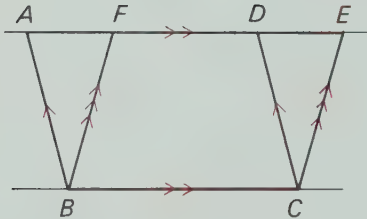
In this section, we shall be working with figures *between the same parallels*. In Figure 7-1, $AK \parallel BJ$, so that $\triangle ABC$, parallelogram $DEFG$, and trapezoid $HIJK$ lie between the same parallels.

Figure 7-1



(PAT) Parallelogram Area Theorem

If two parallelograms have the same base and lie between the same parallel lines, then the parallelograms are equal in area.



Given: $AE \parallel BC$, $\parallel^{gm} ABCD$ and $\parallel^{gm} FBCE$.

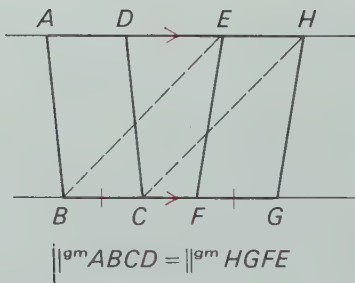
Required: To prove $\parallel^{gm} ABCD = \parallel^{gm} FBCE$.

Proof

Statements	Reasons
In $\triangle ABF$ and $\triangle DCE$,	
$AB = DC$	PT
$\angle BAF = \angle CDE$	TPT
$\angle AFB = \angle DEC$	TPT
$\angle ABF = \angle DCE$	SATT
$\triangle ABF \cong \triangle DCE$	ASA
$\triangle ABF = \triangle DCE$	congruent

$$\triangle ABF + \text{Figure } FBCE = \triangle DCE + \text{Figure } FBCE$$

$$\parallel^{gm} ABCD = \parallel^{gm} FBCE$$



$$\parallel^{gm} ABCD = \parallel^{gm} HGFE$$

Corollary: If two parallelograms are on equal bases and lie between the same parallel lines, then they are equal in area.

(TAT) Triangle Area Theorem

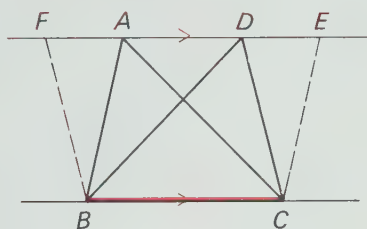
If two triangles are on the same base and lie between the same parallel lines, then the triangles are equal in area.

Given: $\triangle ABC$ and $\triangle DBC$, $AD \parallel BC$.

Required: To prove $\triangle ABC = \triangle DBC$.

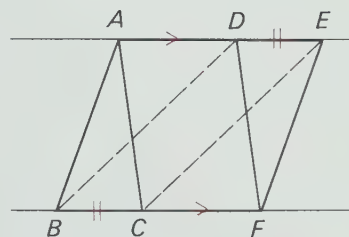
Proof

Statements	Reasons
Draw $FB \parallel DC$	construction
$EC \parallel AB$	construction
$\triangle ABC = \frac{1}{2} \parallel^{\text{gm}} ABCE$	PT
$\triangle DBC = \frac{1}{2} \parallel^{\text{gm}} FBCD$	PT
$BC = BC$ and $FE \parallel BC$	given
$\parallel^{\text{gm}} ABCE = \parallel^{\text{gm}} FBCD$	PAT
$\therefore \triangle ABC = \triangle DBC$	



Corollaries: 1. If two triangles are on equal bases and lie between the same parallel lines, then the triangles are equal in area.

2. If a triangle and a parallelogram are on equal bases and lie between the same parallel lines, then the area of the triangle is one-half the area of the parallelogram.



EXAMPLE 1. Find a triangle equal in area to a given quadrilateral.

Solution

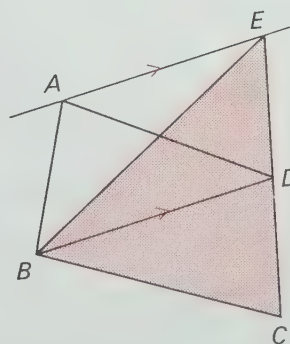
Given: Quadrilateral $ABCD$.

Required: To find a triangle equal in area to $ABCD$.

Construction: Draw a line through A parallel to BD . Extend CD to meet this line at E . $\triangle BCE$ is the required triangle.

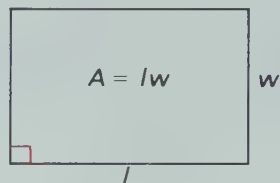
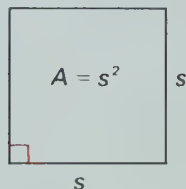
Proof

Statements	Reasons
In $\triangle ABD$ and $\triangle EBD$,	
$AE \parallel BD$	construction (PLP)
$BD = BD$	
$\therefore \triangle ABD = \triangle EBD$	TAT
$\triangle ABD + \triangle BCD = \triangle EBD + \triangle BCD$	
Quadrilateral $ABCD = \triangle BCE$.	

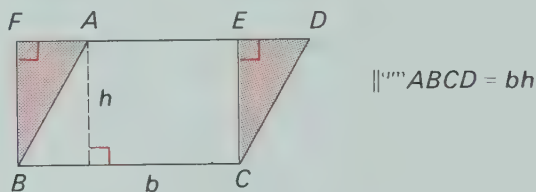


We can now use the theorems of plane geometry along with the rectangle area definition $A = lw$ to re-establish the formulas for the following figures.

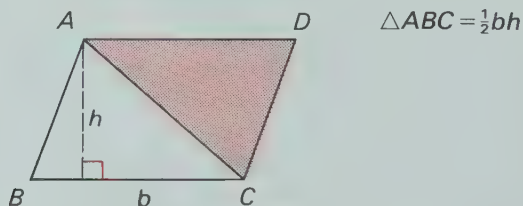
1. Square and General Rectangle



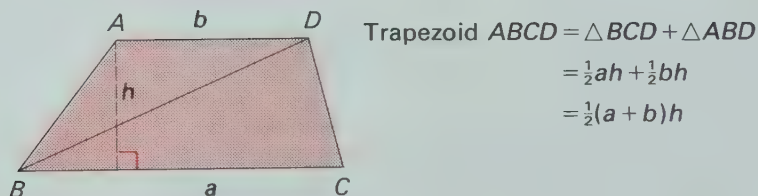
2. Parallelogram: Since $\parallel^{\text{gm}} ABCD = \text{rectangle } FBCE$



3. Triangle: Since $\triangle ABC = \frac{1}{2} \parallel^{\text{gm}} ABCD$

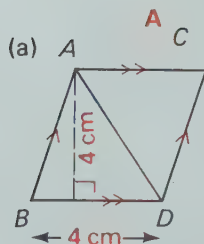


4. Trapezoid

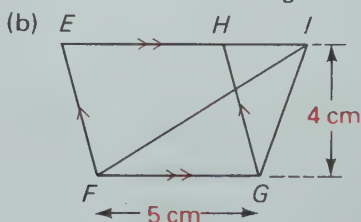


EXERCISE 7-5

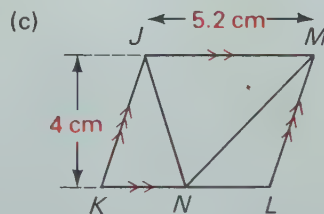
1. Find the area of the following as indicated:



- (i) $\parallel^{\text{gm}} ABCD$
(ii) $\triangle ABC$

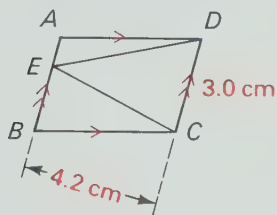


- (i) $\parallel^{\text{gm}} EFGH$
(ii) $\triangle IFG$

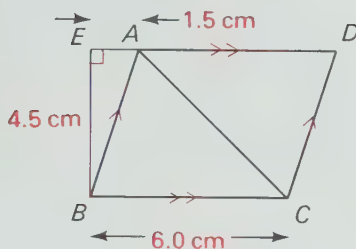


- (i) $\parallel^{\text{gm}} JKLM$
(ii) $\triangle JKN + \triangle MLN$

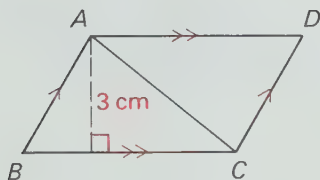
2. Find the area of
 (a) $\triangle ABC$
 (b) Trapezoid $BCDE$



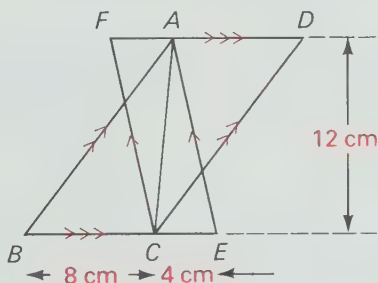
3. Find the area of
 (a) $\triangle EDC$
 (b) $\parallel^{\text{gm}} ABCD$



4. If $\triangle ABC = 12 \text{ cm}^2$, find the
 (a) area of $\parallel^{\text{gm}} ABCD$
 (b) length of BC



5. Find the area of
 (a) $\triangle ACF + \triangle ABC$
 (b) $\parallel^{\text{gm}} FCEA + \parallel^{\text{gm}} ABCD$

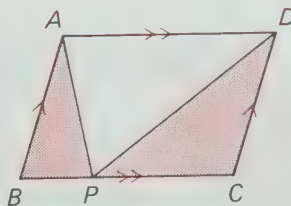


B 6. Prove: If two parallelograms are on equal bases and lie between the same parallel lines, then they are equal in area.

7. Prove: If two triangles are on equal bases and lie between the same parallel lines, then they are equal in area.

8. Given: $\parallel^{\text{gm}} ABCD$ and any point P on BC .
 Required: To prove

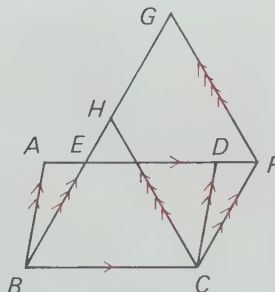
$$\triangle ABP + \triangle DPC = \frac{1}{2} \parallel^{\text{gm}} ABCD.$$

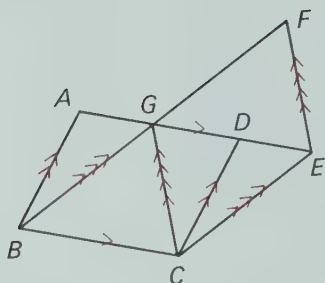


9. Given: $\parallel^{\text{gm}} ABCD$, E is any point on AD , $EF \parallel BC$,
 H and G are on BE produced, $CH \parallel FG$

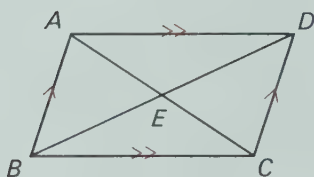
Required: To prove

$$\parallel^{\text{gm}} ABCD = \parallel^{\text{gm}} CFGH$$

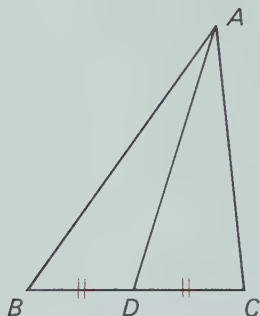




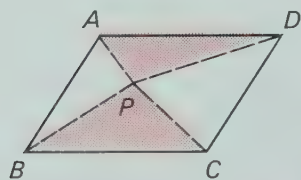
10. Given: $\parallel^{\text{gm}} ABCD$, $\parallel^{\text{gm}} GBCE$, $\parallel^{\text{gm}} GCEF$.
 Required: To prove
 $\parallel^{\text{gm}} ABCD = \parallel^{\text{gm}} GCEF$.



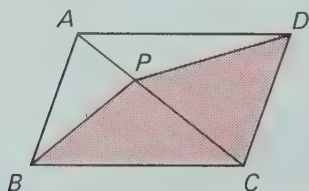
11. Given: $\parallel^{\text{gm}} ABCD$, AC and BD intersect at E .
 Required: To prove $\triangle AEB = \triangle DEC$ (i) using an area theorem (ii) using a congruence theorem.



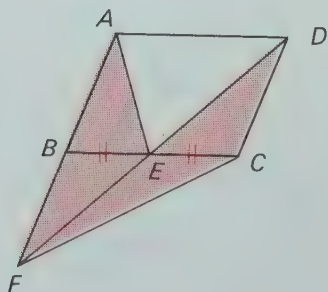
12. The median of a triangle bisects the area.
 Given: $\triangle ABC$, $BD = DC$.
 Required: To prove AD bisects $\triangle ABC$.



13. Given: $\parallel^{\text{gm}} ABCD$ and any point P in the interior.
 Required: To prove
 $\triangle APD + \triangle BPC = \frac{1}{2} \parallel^{\text{gm}} ABCD$.



14. Given: $\parallel^{\text{gm}} ABCD$ and P , any point on AC .
 Required: To prove $\triangle PBC = \triangle PDC$.



15. Given: $\parallel^{\text{gm}} ABCD$, E is the midpoint of BC , DE produced meets AB produced at F .
 Required: To prove $\triangle AEF = \triangle CDF$.

7.6 THE THEOREM OF PYTHAGORAS

Many proofs have been given for the Pythagorean Theorem. The proof which follows employs the method first used by Euclid.

The area of the square on the hypotenuse of a right triangle is equal to the sum of the areas of the squares on the other two sides.

Given: $\triangle ABC$, $\angle BAC = 90^\circ$, squares $BHIC$, $DABE$, $FACG$

Required: To prove square $BHIC$ = square $DABE$ + square $FACG$.

Proof

Statements

Draw line $AJ \parallel CI$ to form rectangle $KJIC$.

Join AI and BG .

BAF is a straight line

$\angle ACG = \angle BCI = 90^\circ$

$\angle ACG + \angle ACB = \angle BCI + \angle ACB$

In $\triangle AIC$ and $\triangle BGC$,

$\angle BCG = \angle ACI$

$GC = AC$

$BC = IC$

$\therefore \triangle AIC \cong \triangle BGC$

$AJ \parallel CI$

$\therefore \triangle AIC = \frac{1}{2}$ rectangle $KJIC$

$BF \parallel GC$

$\therefore \triangle BGC = \frac{1}{2}$ square $FACG$

$\triangle AIC = \triangle BGC$

rectangle $KJIC$ = square $FACG$

similarly we can prove

rectangle $KJHB$ = square $DABE$

rectangle $KJHB$ + rectangle $KJIC$ = square $DABE$ + square $FACG$

square $BHIC$ = square $DABE$ + square $FACG$

When we write BC^2 we mean $BC \times BC$ which is the area of the square on BC , (i.e. $BHIC$) so that we can restate the theorem as follows:

$$BC^2 = AB^2 + AC^2$$

Reasons

180°

square

definition of square

definition of square

SAS

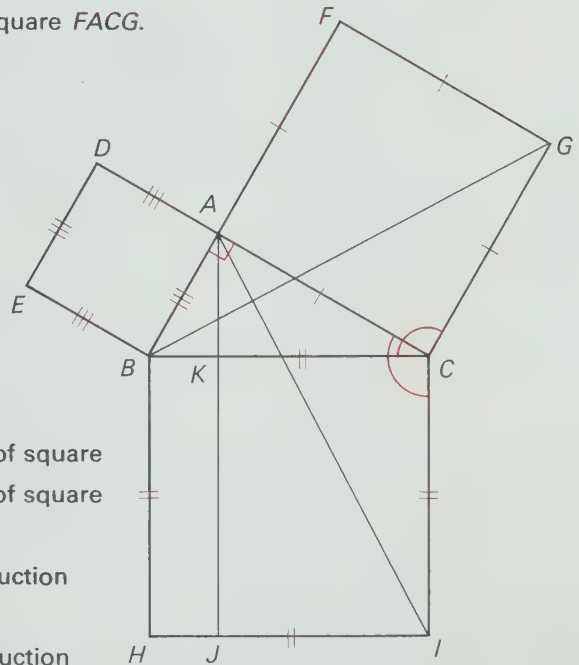
PLP construction

TAT

PLP construction

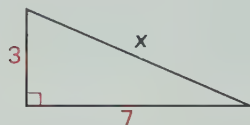
TAT

proved congruent above

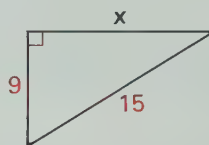


EXAMPLE 1. Calculate the value of x in each of the following using the Pythagorean Theorem.

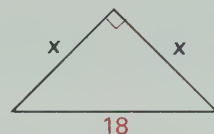
(a)



(b)



(c)



Solution

(a) $x^2 = 3^2 + 7^2$

$$= 9 + 49$$

$$= 58$$

$$x \doteq 7.62$$

(b) $x^2 = 15^2 - 9^2$

$$= 225 - 81$$

$$= 144$$

$$x = 12$$

(c) $x^2 + x^2 = 18^2$

$$2x^2 = 324$$

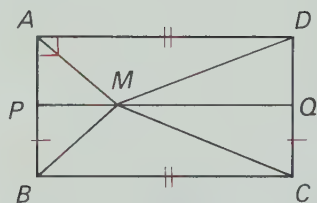
$$x^2 = 162$$

$$x \doteq 12.7$$

EXAMPLE 2. Given: Rectangle $ABCD$ and M in the interior.

Required: To prove $MA^2 + MC^2 = MB^2 + MD^2$.

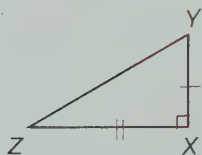
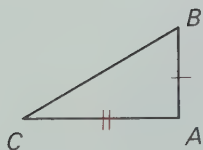
Proof:



Statements	Reasons
Draw $PMQ \parallel AD$	
$\angle DAP = \angle APM = \angle MPB = 90^\circ$	TPT and
$\angle ADQ = \angle DQM = \angle MQC = 90^\circ$	definition of rectangle
In $\triangle APM$, $MA^2 = AP^2 + MP^2$	Pythagorean Theorem
In $\triangle MQC$, $MC^2 = MQ^2 + QC^2$	Pythagorean Theorem
In $\triangle PMB$, $MB^2 = PB^2 + MP^2$	Pythagorean Theorem
In $\triangle DMQ$, $MD^2 = MQ^2 + DQ^2$	Pythagorean Theorem
$MA^2 + MC^2 = (AP^2 + MP^2) + (MQ^2 + QC^2)$	Substitution
$= (DQ^2 + MP^2) + (MQ^2 + PB^2)$	Opposite sides
$= \underbrace{PB^2 + MP^2}_{MB^2} + \underbrace{MQ^2 + DQ^2}_{MD^2}$	Commutative Property
$= MB^2 + MD^2$	Substitution

Converse of the Pythagorean Theorem

If the square of the length of one side of a triangle is equal to the sum of the squares of the lengths of the other two sides, then the angle opposite the longest side is a right angle.



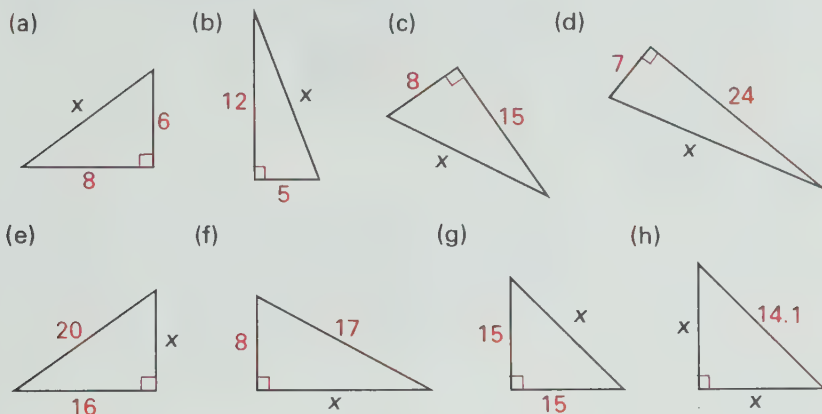
Given: $\triangle ABC$, $BC^2 = AB^2 + AC^2$.

Required: To prove $\angle BAC = 90^\circ$.

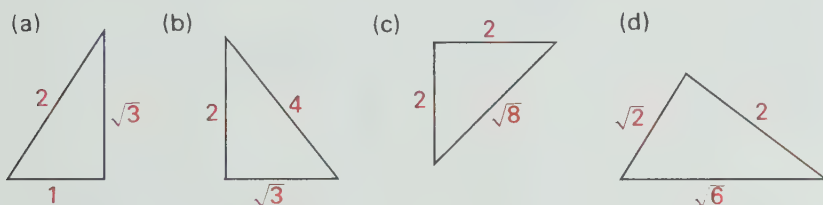
Proof: The proof is left to the student. First construct $\triangle XYZ$ with $YX = BA$, $ZX = CA$, and $\angle ZXY = 90^\circ$.

EXERCISE 7-6

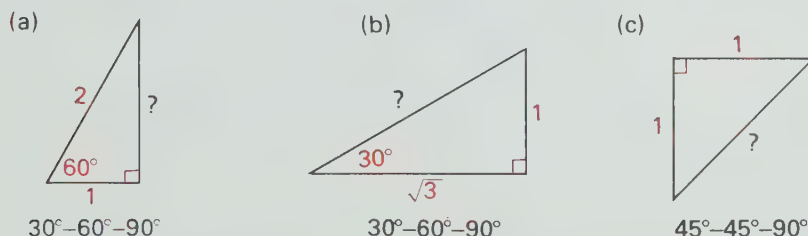
A 1. Find the value of the unknown in each of the following.



2. Determine which of the following are right triangles. (The figures may not be drawn to scale)



B 3. Special triangles: Find the length of the unknown side.



4. Find the length of the hypotenuse of a right triangle where the lengths of the other two sides are:

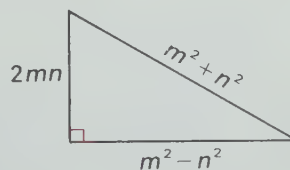
- (a) 6 cm, 8 cm (b) 7 cm, 11 cm (c) 15 cm, 2 cm
(d) 3 cm, 7 cm (e) 6 cm, 9 cm (f) 10 cm, 17 cm

5. Find the area of the right triangle with sides whose lengths are:

- (a) 6, 8, 10, (cm) (b) 5, 12, 13, (m) (c) 8, 15, 17, (cm)
(d) 3, 5, $\sqrt{34}$, (cm) (e) 5, 6, $\sqrt{61}$, (m) (f) 2, 7, $\sqrt{53}$, (m)

6. If a triangle has sides with lengths $m^2 + n^2$, $m^2 - n^2$; and $2mn$, as in the diagram, where $m, n \in \mathbb{N}$ and $m > n$, then the triangle is a right triangle.

(a) Prove the above statement.



n	$\sqrt{n}$	n	$\sqrt{n}$	n	$\sqrt{n}$
1	1.0000	41	6.4031	81	9.0000
2	1.4142	42	6.4807	82	9.0554
3	1.7321	43	6.5574	83	9.1104
4	2.0000	44	6.6333	84	9.1652
5	2.2361	45	6.7082	85	9.2195
6	2.4495	46	6.7823	86	9.2736
7	2.6458	47	6.8557	87	9.3274
8	2.8284	48	6.9282	88	9.3808
9	3.0000	49	7.0000	89	9.4340
10	3.1623	50	7.0711	90	9.4868
11	3.3166	51	7.1414	91	9.5394
12	3.4641	52	7.2111	92	9.5917
13	3.6056	53	7.2801	93	9.6437
14	3.7417	54	7.3485	94	9.6954
15	3.8730	55	7.4162	95	9.7468
16	4.0000	56	7.4833	96	9.7980
17	4.1231	57	7.5498	97	9.8489
18	4.2426	58	7.6158	98	9.8995
19	4.3589	59	7.6812	99	9.9499
20	4.4721	60	7.7460	100	10.0000
21	4.5826	61	7.8103	101	10.0499
22	4.6904	62	7.8740	102	10.0995
23	4.7958	63	7.9373	103	10.1489
24	4.8990	64	8.0000	104	10.1980
25	5.0000	65	8.0623	105	10.2470
26	5.0990	66	8.1240	106	10.2956
27	5.1962	67	8.1854	107	10.3441
28	5.2915	68	8.2462	108	10.3923
29	5.3852	69	8.3066	109	10.4403
30	5.4772	70	8.3666	110	10.4881
31	5.5678	71	8.4262	111	10.5357
32	5.6569	72	8.4853	112	10.5830
33	5.7446	73	8.5440	113	10.6301
34	5.8310	74	8.6023	114	10.6771
35	5.9161	75	8.6603	115	10.7238
36	6.0000	76	8.7178	116	10.7703
37	6.0828	77	8.7750	117	10.8167
38	6.1644	78	8.8318	118	10.8628
39	6.2450	79	8.8882	119	10.9087
40	6.3246	80	8.9443	120	10.9545

(b) Make a table to generate Pythagorean triples.

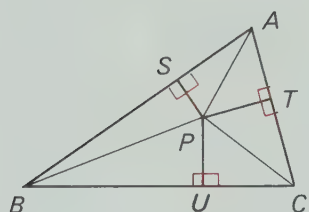
m	n	$m^2 - n^2$	$2mn$	$m^2 + n^2$
2	1	3	4	
3	1	8		
3	2			



7. Given: Rectangle $ABCD$.

Required: To prove

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$$

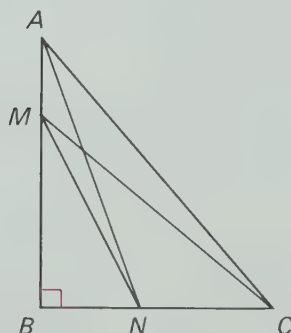


8. Given: $\triangle ABC$ and any point P in the interior.

$PS \perp AB$, $PU \perp BC$, $PT \perp AC$.

Required: To prove

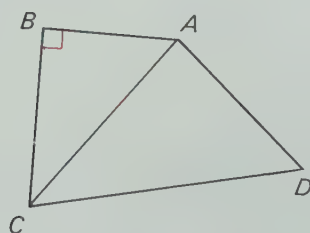
$$AS^2 + BU^2 + CT^2 = SB^2 + UC^2 + TA^2$$



9. Given: $\triangle ABC$, $AB \perp BC$ M and N are points on AB and BC respectively.

Required: To prove

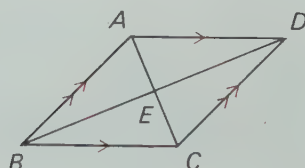
$$AC^2 + MN^2 = AN^2 + CM^2$$



10. Given: Quadrilateral $ABCD$, $\angle ABC = 90^\circ$,

$AB = 12$, $BC = 16$, $AD = 15$, $CD = 25$.

Required: To prove $\angle CAD = 90^\circ$.



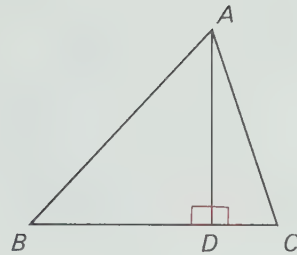
11. Given: $\square ABCD$, $AC = 10$, $BE = 12$, $AB = 13$.

Required: To prove (i) $AE \perp BE$ (ii) $ABCD$ is a rhombus.

12. Given: $\triangle ABC$, D is on BC , $AD \perp BC$.

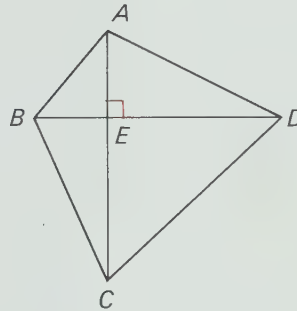
Required: To prove

$$AC^2 + BD^2 = AB^2 + CD^2$$

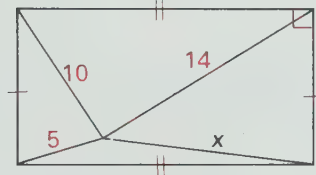


13. Given: Quadrilateral $ABCD$. $AC \perp BD$, AC and BD intersect at E .

Required: To prove $AB^2 + CD^2 = BC^2 + AD^2$.



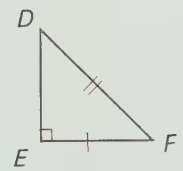
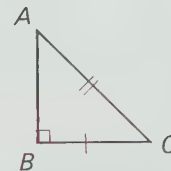
14. Find the value of x .



15. Prove.

(HS) Hypotenuse-Side Congruence.

If the hypotenuse and one side of one right triangle are respectively equal to the hypotenuse and one side of another right triangle, then the triangles are congruent.

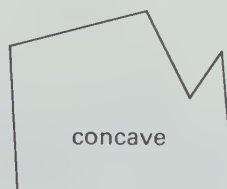


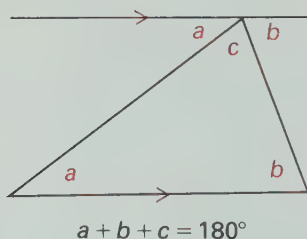
7.7 ANGLES OF A POLYGON

A polygonal path is formed when someone completes a “follow-the-dot” puzzle. If a polygonal path of three or more line segments terminates at the initial point, then the figure formed is a polygon.

A simple polygon can be

- (i) convex or
- (ii) concave.





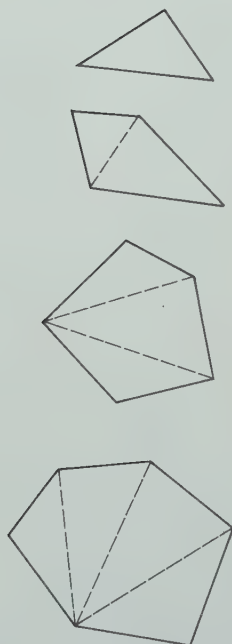
Polygons are named according to the number of sides as in Table 7-1.

We have seen that the sum of the interior angles of a triangle is 180° , and we can prove this using the parallel line theorem (TPT).

Polygons	
Number of sides	Name
3	Triangle
4	Quadrilateral
5	Pentagon
6	Hexagon
7	Heptagon
8	Octagon
9	Nonagon
10	Decagon
.	.
.	.
.	.
n	n -gon

Table 7-1

By dividing a polygon into triangles, we can find the sum of the angles of the polygon as in Table 7-2. We first divide the polygons using diagonals from the same vertex.



Number of sides	Diagonals drawn from one vertex	Triangles formed	Interior Angle Sum in Degrees
3	0	1	180°
4	1	2	360°
5	2	3	540°
6	3	4	720°
7	4	5	900°
8	5	6	1080°
.	.	.	.
100	97	.	.
.	.	.	.
n	$n - 3$	$n - 2$	$(n - 2)180^\circ$

Table 7-2

The results of Table 7-2 show that the sum of the angles of a polygon with n sides is $(n - 2)180^\circ$. If the polygon is *regular* (all sides equal) then each angle is equal to $\frac{(n - 2)180^\circ}{n}$.

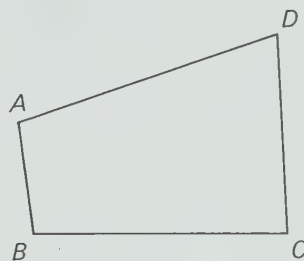
EXERCISE 7-7

- B** 1. Find the sum of the angles of a polygon with
 (a) 10 sides (b) 13 sides (c) 20 sides
2. A regular polygon has all angles equal. Find the degree measure of each angle of each of the following regular polygons.
 (a) equilateral triangle
 (b) square
 (c) regular pentagon
 (d) regular hexagon
 (e) regular 40-gon

3. Given: Quadrilateral $ABCD$

Required: To prove

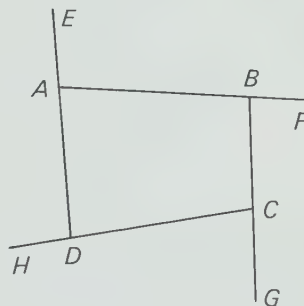
$$\angle ABC + \angle BCD + \angle CDA + \angle DAB = 360^\circ$$



4. Given: Quadrilateral $ABCD$, and exterior angles $\angle EAB$, $\angle FBC$, $\angle GCD$, and $\angle HDA$.

Required: To prove

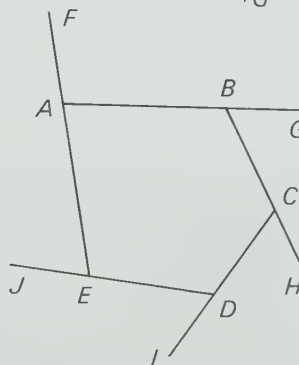
$$\angle EAB + \angle FBC + \angle DCG + \angle HDA = 360^\circ$$



5. Given: Pentagon $ABCDE$ and exterior angles $\angle FAB$, $\angle GBC$, $\angle HCD$, $\angle IDE$, and $\angle JEA$.

Required: To prove

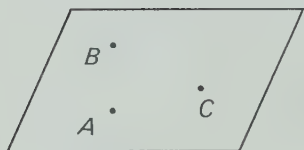
$$\angle FAB + \angle GBC + \angle HCD + \angle IDE + \angle JEA = 360^\circ$$



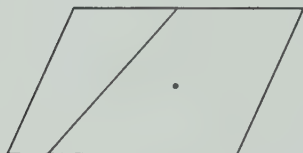
6. If the sum of the interior angles of a n -gon is 7380° , how many sides does it have?
7. Is it possible to have a polygon, the sum of whose interior angles are
 (a) 980° (b) 1800° (c) 1250°
8. Each interior angle of a regular polygon is 108° . How many sides does it have?
9. Is it possible to have a regular polygon having each interior angle equal to 100° ? Explain.

7.8 APPLICATIONS IN THREE DIMENSIONS

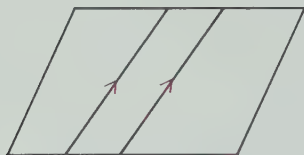
In the preceding sections, we studied geometry in the plane. Plane geometry has many applications in three dimensions. We extend our study to the third dimension by first stating some definitions and postulates.



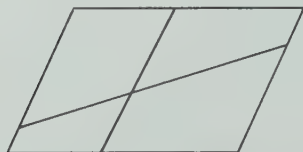
- I. A plane is determined by:
(i) three non-collinear points



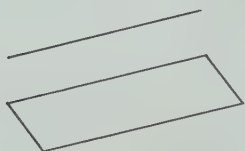
- (ii) a line and a point not on the line



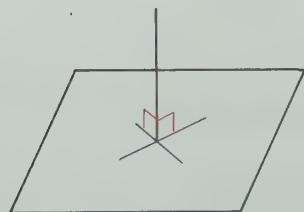
- (iii) two parallel lines



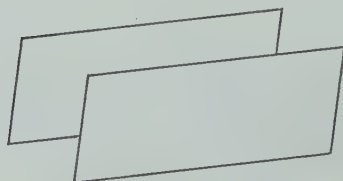
- (iv) two intersecting lines



- II. A line and a plane
(i) are parallel if they do not intersect.

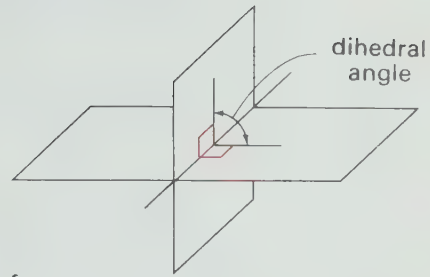


- (ii) are perpendicular if they intersect and the line is perpendicular to every line in the plane passing through the point of intersection.



- III. Two planes:
(i) are parallel if they do not intersect

- (ii) are perpendicular if the dihedral angle is a right angle.



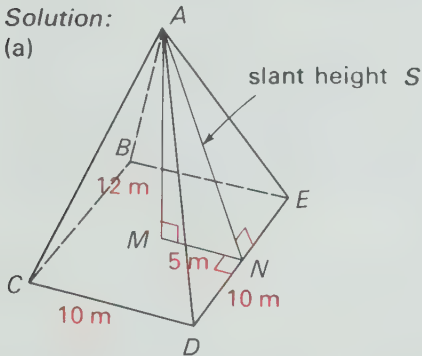
EXAMPLE 1. Find (i) the total area (ii) the volume of

(a) a regular pyramid with height 12 m and the base a square with sides 10 m

(b) a right circular cone with height 10 m and the base a circle with diameter 8 m.

Solution:

(a)



(i) We use the theorem of Pythagoras in $\triangle AMN$ to find the slant height s .

$$s^2 = 12^2 + 5^2 = 169$$

$$s = 13$$

Area of a triangle $= \frac{1}{2}bh$

$$\triangle ADE = \frac{1}{2} \times 10 \text{ m} \times 13 \text{ m}$$

$$= 65 \text{ m}^2$$

$$\begin{aligned} \text{Lateral Area} &= \triangle ABC + \triangle ACD + \triangle ADE + \triangle AEB \\ &= 4\triangle ADE \\ &= 4(65 \text{ m}^2) \\ &= 260 \text{ m}^2 \end{aligned}$$

[It is left to the student to show that $\triangle ABC$, $\triangle ACD$, $\triangle ADE$, and $\triangle AEB$ are congruent.]

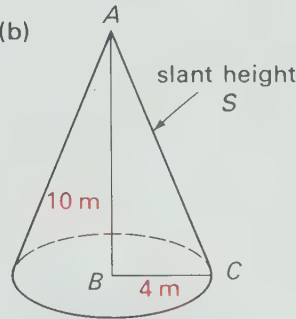
$$\begin{aligned} \text{Base area} &= BCDE \\ &= 10 \text{ m} \times 10 \text{ m} \\ &= 100 \text{ m}^2 \end{aligned}$$

$$\begin{aligned} \text{Total area} &= 100 \text{ m}^2 + 260 \text{ m}^2 \\ &= 360 \text{ m}^2 \end{aligned}$$

(ii) Volume $= \frac{1}{3}Ah$, where A is the area of the base.

$$\begin{aligned} V &= \frac{1}{3}(100 \text{ m}^2)12 \text{ m} \\ &= 400 \text{ m}^3 \end{aligned}$$

(b)



(i) We use the theorem of Pythagoras in $\triangle ABC$ to find the slant height s .

$$s^2 = 10^2 + 4^2 = 116$$

$$s \doteq 10.8$$

$$\begin{aligned} \text{Lateral Area} &= \frac{1}{2} \left(\begin{array}{c} \text{perimeter} \\ \text{of base} \end{array} \right) \times \left(\begin{array}{c} \text{slant} \\ \text{height} \end{array} \right) \\ &= \frac{1}{2}(2\pi 4 \text{ m}) \times (10.8 \text{ m}) \\ &\doteq 135.6 \text{ m}^2 \end{aligned}$$

$$\begin{aligned} \text{Base Area} &= \pi(4 \text{ m})^2 \\ &\doteq 50.2 \text{ m}^2 \end{aligned}$$

$$\begin{aligned} \text{Total area} &= 135.6 \text{ m}^2 + 50.2 \text{ m}^2 \\ &= 185.8 \text{ m}^2 \end{aligned}$$

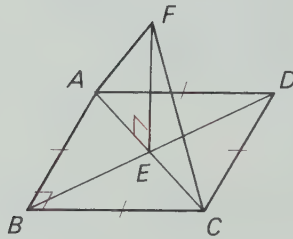
(ii) Volume $= \frac{1}{3}Ah$, where $A = \pi r^2$, the area of the base.

$$\begin{aligned} V &= \frac{1}{3} \times 3.14(4 \text{ m})^2 \times 10 \text{ m} \\ &\doteq 167.5 \text{ m}^3 \end{aligned}$$

The lateral area of the pyramid and the cone can also be found directly, using the formula

$$S = \frac{1}{2}ps$$

where S is the lateral area; p is the perimeter of the base and s is the slant height.



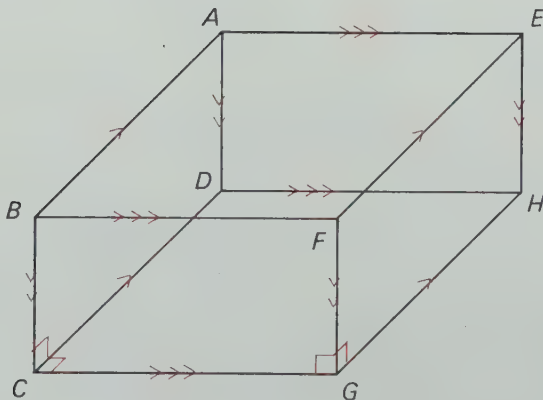
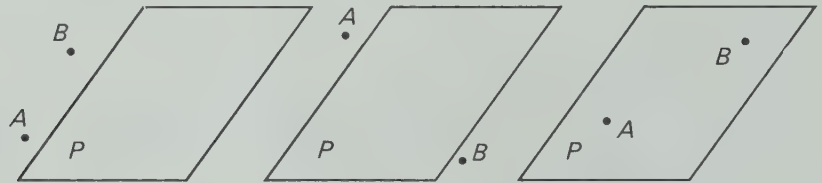
EXAMPLE 2. Given: $ABCD$ is a square, AC and BD intersect at E , $FE \perp AC$
Required: To prove $AF = CF$

Solution
Proof

Statements	Reasons
$ABCD$ is a parallelogram	definition
AC and BD intersect at E	given
$\therefore AE = CE$	PT
In $\triangle AEF$ and $\triangle CEF$,	
$AE = CE$	proved above
$\angle AEF = \angle CEF = 90^\circ$	given
$EF = EF$	common
$\therefore \triangle AEF \cong \triangle CEF$	SAS
$\therefore AF = CF$	

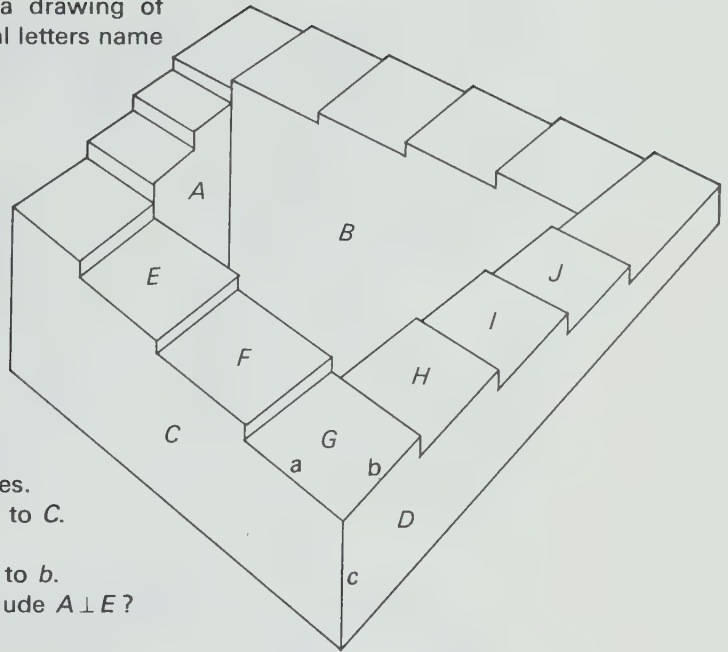
EXERCISE 7-8

- A** 1. A and B are two points and P is a plane. How would you determine
- whether A and B are on the same side of P ?
 - whether A and B are on opposite sides of P ?
 - whether A and B lie in the plane P ?



2. In the accompanying diagram, name
- three sets of parallel faces.
 - a line parallel to a plane.
 - a line perpendicular to a plane.
 - two perpendicular planes.
 - three planes intersecting at a point.
 - two equal dihedral angles.

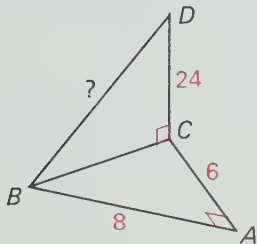
3. The accompanying diagram is a drawing of Escher's impossible staircase. Capital letters name planes and small letters name lines.



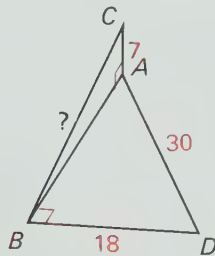
- Name four pairs of parallel planes.
- Name four planes perpendicular to C .
- Name a plane parallel to C .
- Name two planes perpendicular to b .
- If $A \perp C$ and $E \perp C$ can you conclude $A \perp E$?

4. Find the indicated length in each of the following:

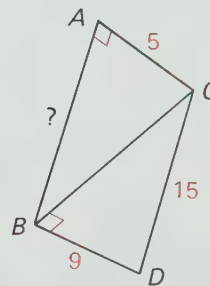
(a) Find BD



(b) Find BD

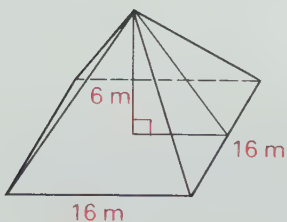


(c) Find AB

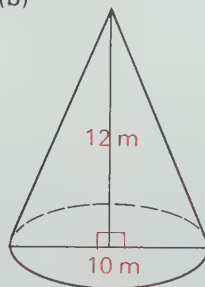


5. Find the lateral area of the following:

(a)



(b)

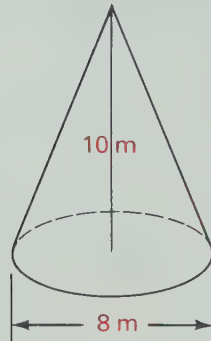


(c)

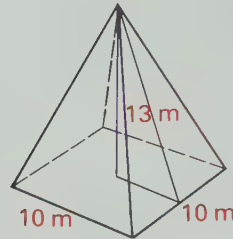


6. Find the volume of the following:

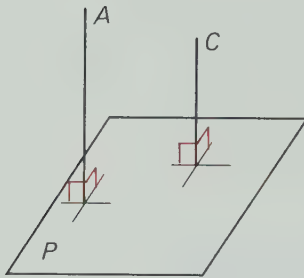
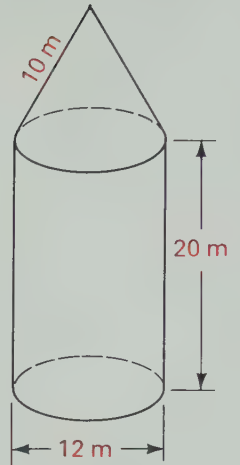
(a)



(b)

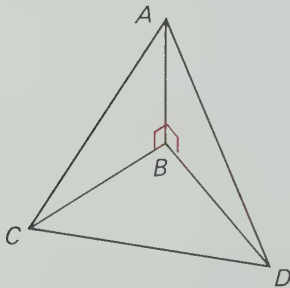


(c)



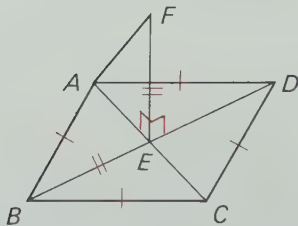
7. Given: Lines AB and CD are perpendicular to plane P .

Required: To prove $AB \parallel CD$



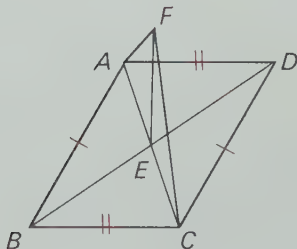
8. Given: AB perpendicular to the plane containing CB and BD . $CB = BD$.

Required: To prove $\triangle ACD$ is isosceles.



9. Given: Square $ABCD$, AC intersects BD at E . FE is perpendicular to plane $ABCD$, $FE = BE$.

Required: To prove $AB = AF$

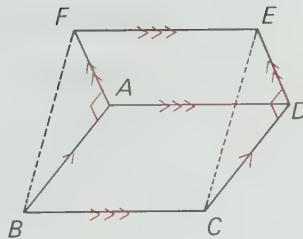


10. Given: FE is perpendicular to the plane containing rectangle $ABCD$. AC and BD intersect at E .

Required: To prove $AF = CF$

11. Given: $\parallel^{\text{gm}} ABCD$ and $\parallel^{\text{gm}} FADE$ with the dihedral angle $\angle FAB = 90^\circ$

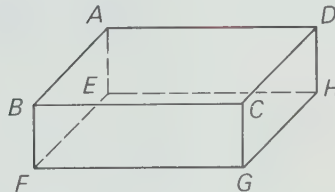
Required: To prove $FBCE$ is a parallelogram.



C 12. Given: A rectangular parallelepiped $ABCDEFGH$.

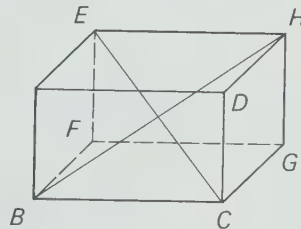
A rectangular parallelepiped is a solid having congruent rectangles for opposite faces.

Required: To prove $BD \perp DH$.



13. Given: A rectangular parallelepiped $ABCDEFGH$

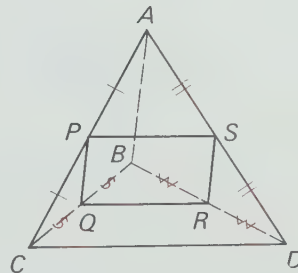
Required: To prove $BH = EC$.



14. Given: $P, Q, R,$ and S are the midpoints of line segments AC, CB, BD and AD in tetrahedron $ABCD$.

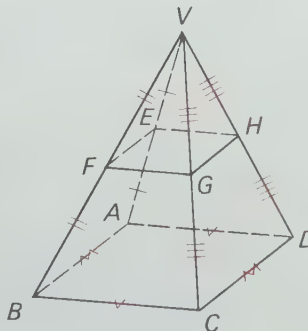
A tetrahedron is a pyramid with a triangular base.

Required: To prove $PQRS$ is a parallelogram.



15. Given: $E, F, G,$ and H are the midpoints of AV, BV, CV and DV , the lateral edges of the rectangular pyramid $VABCD$

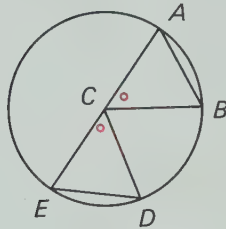
Required: To prove $EFGH$ is a parallelogram.



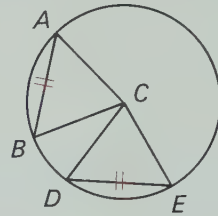
REVIEW EXERCISE

1. Name pairs of congruent triangles and give reasons.

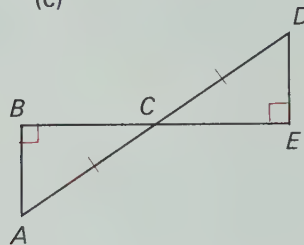
(a)



(b)



(c)



Put the numbers from 1 to 9 in the spaces to make the statements true.

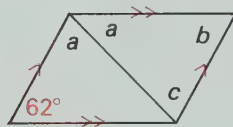
$$\square - \square + \square = 8$$

$$\square \div \square + \square = 8$$

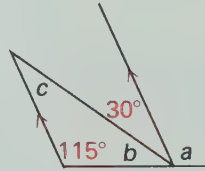
$$\square \times \square + \square = 8$$

2. Calculate the values of a , b , and c in the following.

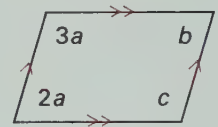
(a)



(b)



(c)



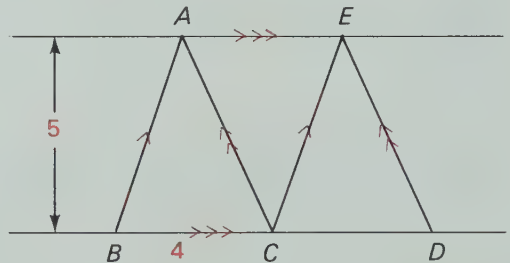
3. Find the area of

(a) $\parallel^{\text{gm}} ABCE$

(b) $\triangle ACE$

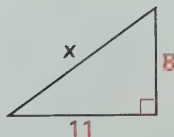
(c) $\triangle ECD$

(d) trapezoid $ABDE$

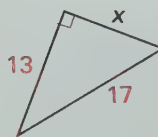


4. Find the value of x :

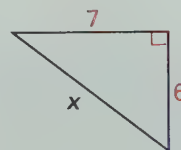
(a)



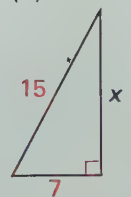
(b)



(c)



(d)



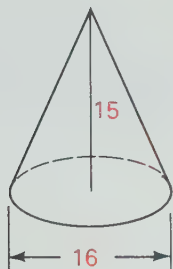
5. A regular polygon has 24 sides. Find

(a) the sum of the interior angles.

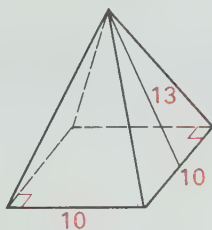
(b) the measure of the interior angles.

6. Find (i) the lateral area (ii) the volume of the following:

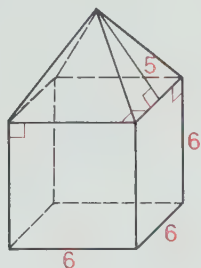
(a)



(b)



(c)

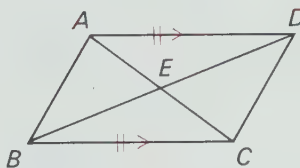


7. Given: $\triangle ABC$, D is on BC , $AD = BD = CD$.

Required: To prove $\angle BAC = 90^\circ$.

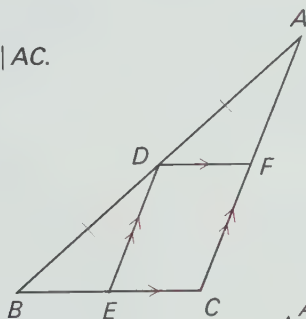
8. Given: $AD \parallel BC$, $AD = BC$, AC and BD intersect at E .

Required: To prove $\triangle AED \cong \triangle CEB$.

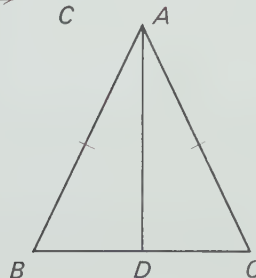


9. Given: $\triangle ABC$, $AD = DB$, $DF \parallel BC$, $DE \parallel AC$.

Required: To prove $EF \parallel BA$.



10. In $\triangle ABC$, $AB = AC$. D is a point on BC and $\angle BAD \neq \angle CAD$. Prove that $BD \neq DC$.

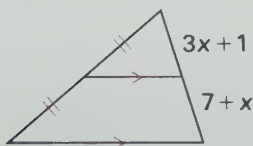


11. Given: $\triangle ABC$, $CA = CB$, M is the midpoint of AB . X and Y are on CA and CB so that $MX \perp CA$ and $MY \perp CB$.

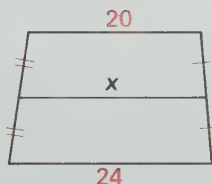
Required: To prove $MX = MY$.

12. Find the value of x in the following.

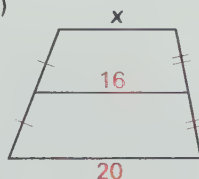
(a)

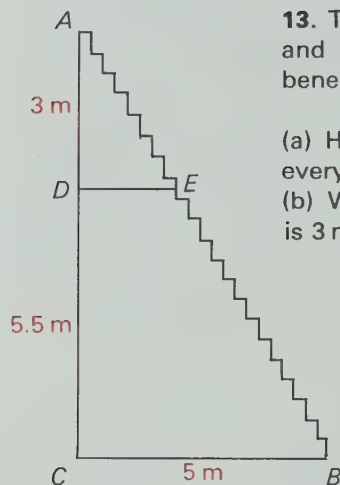


(b)



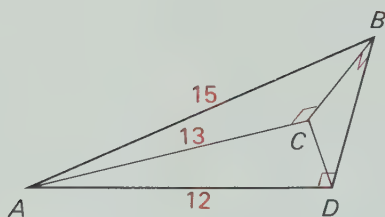
(c)



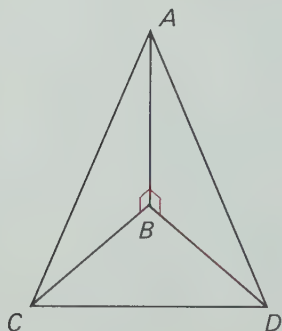


13. The vertical height of a fire escape is 8.5 m and it reaches the ground 5 m from the point beneath the top step.

- (a) How many steps are there if there is one step every 30 cm measured along AB ?
 (b) What is the length of the support DE if D is 3 m below A ?



14. Find the dimension CD to three significant figures if $AB = 15$, $AC = 13$, $AD = 12$ and $\angle ADB = \angle ACB = \angle DBC = 90^\circ$.



15. Figure $ABCD$ is a tetrahedron, with AB perpendicular to the plane containing $\triangle BCD$, and $AC = AD$. Prove $\triangle BCD$ is isosceles.

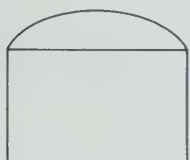
REVIEW AND PREVIEW TO CHAPTER 8

EXERCISE 1

NETWORKS

Try to draw the following figures without lifting your pencil and without retracing any lines.

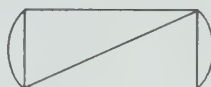
1.



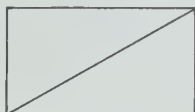
2.



3.



4.



5.



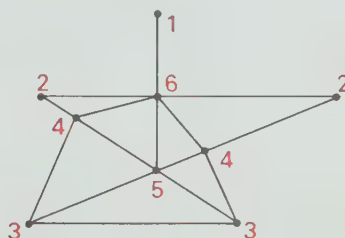
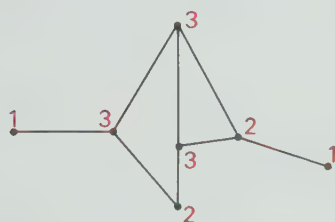
6.



EXERCISE 2

THE ORDER OF A VERTEX

The **order** of a vertex in a network is the number of lines (or arcs) which end at the vertex. In the following figures each vertex has been labelled with its order.

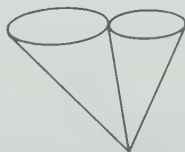


Copy each of the following figures and label each vertex with its order.

1.



2.



3.



4.



5.



6.



EXERCISE 3

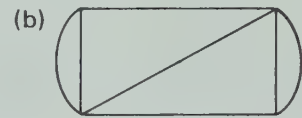
TRAVERSIBLE NETWORKS

A network is called *traversable* if it can be drawn in one stroke (i.e., without lifting the pencil) and without retracing any lines.

If a network is traversable, then it has at most 2 vertices whose orders are odd numbers.

This is true because if a network is traversable there is a path which passes through all vertices (except the first and last). Every "pass through" accounts for 2 line ends (one entering and one leaving). Thus each vertex, except for the first and last, must have an order which is an even number.

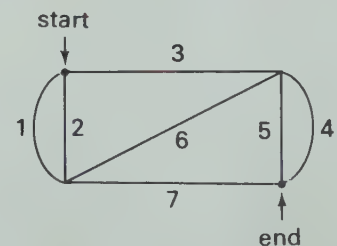
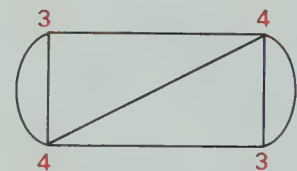
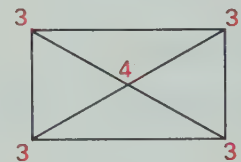
EXAMPLE 1. Determine whether the following networks are traversable. If a network is traversable indicate a path by numbering its lines in order.



Solution

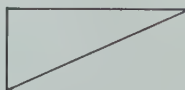
(a) Label each vertex with its order. Since there are more than 2 vertices with odd order, this network is not traversable.

(b) Label each vertex with its order. This network is traversable since there are only 2 vertices with odd order. To find a path, we start and end at the odd vertices. One such path is indicated by numbering the lines in order.



Determine whether the following networks are traversable. If a network is traversable, indicate a path by numbering its lines in order.

1.



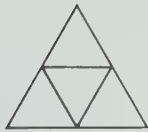
2.



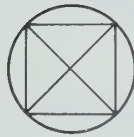
3.



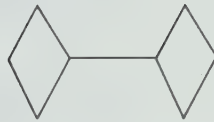
4.



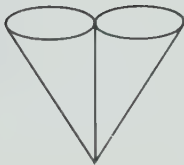
5.



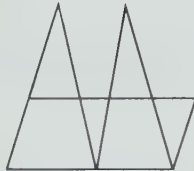
6.



7.



8.



9.



EXERCISE 4

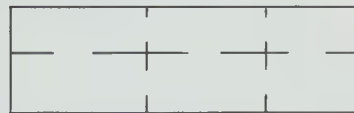
HOUSE PROBLEMS

Is it possible to find a path which goes through each door of the house once and only once? If the answer is yes, draw a path that will do it. *Hint:* Represent the rooms as points and the doorways as lines connecting the points.

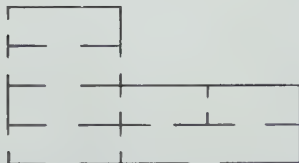
1.



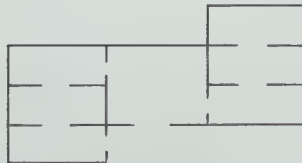
2.



3.



4.



Similarity

In studying the procedure of geometrical thought we may hope to reach what is most essential in man's mind.

Henri Poincaré

When it comes to applying geometry to the real world, some of the most useful facts are the properties of similar triangles. In this chapter you will first learn the properties of ratio and proportion, and then use them to study similar triangles and their applications.

8.1 RATIO AND PROPORTION

The ratio of one number to another is a way of comparing two numbers by division. We sometimes use the notation $a : b$ for the ratio of a to b . For instance the ratio of 4 to 7 is $4 : 7 = \frac{4}{7}$.

If $a, b \in R$ and $b \neq 0$, the ratio of a to b is

$$a : b = \frac{a}{b}.$$

If there are 18 boys and 15 girls in a mathematics class, then the ratio of boys to girls is $18 : 15$, or equivalently $6 : 5$ since

$$18 : 15 = \frac{18}{15} = \frac{6}{5}$$

A rope hangs over the side of the ship and is submerged 2 m. If the tide rises 0.6 m/h, how much of the rope is submerged after 4 h?

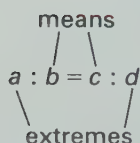
A proportion is an equality of two ratios

$$a : b = c : d$$

or, equivalently,

$$\frac{a}{b} = \frac{c}{d}$$

where $b \neq 0$ and $d \neq 0$.



The numbers a, b, c, d are called the first, second, third, and fourth terms in the proportion. The first and fourth terms, a and d , are called the **extremes**, and the second and third terms, b and c , are called the **means** of the proportion. If the means are equal, i.e., if $b = c$, then either b or c is called the **mean proportional** between a and d . For instance, in the proportion $4 : 6 = 6 : 9$, 6 is the mean proportional between 4 and 9.

EXAMPLE 1. Notice the following things about the proportion

$$\frac{8}{12} = \frac{26}{39}$$

(a) $8 \cdot 39 = 312$

$12 \cdot 26 = 312$

The product of the extremes is equal to the product of the means.

(b) $\frac{12}{8} = \frac{39}{26}$

If we turn both ratios upside down, the new proportion is still correct.

(c) $\frac{8}{26} = \frac{12}{39}$

If we interchange the means, the new proportion is still correct.

(d) $\frac{39}{12} = \frac{26}{8}$

If we interchange the extremes, the new proportion is still correct.

(e) $\frac{8+12}{12} = \frac{26+39}{39}$

If we add the denominators to the numerators, the new proportion is still correct.

Properties of Proportions

Suppose that none of the real numbers a, b, c, d is zero.

(a) If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.

(b) If $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$.

(c) If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$.

(d) If $\frac{a}{b} = \frac{c}{d}$, then $\frac{d}{b} = \frac{c}{a}$.

(e) If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{b} = \frac{c+d}{d}$.

(f) If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a-b}{b} = \frac{c-d}{d}$.

product of extremes
= product of means

$$\frac{a}{b} = \frac{c}{d}$$

$$\frac{a}{b} = \frac{c}{d}$$

$$\frac{a}{b} = \frac{c}{d}$$

$$\frac{a+b}{b} = \frac{c+d}{d}$$

$$\frac{a-b}{b} = \frac{c-d}{d}$$

Proof of (a): $\frac{a}{b} = \frac{c}{d}$ (given)

$$\therefore bd \times \frac{a}{b} = bd \times \frac{c}{d} \quad (\text{multiply both sides by } bd)$$

$$\therefore ad = bc$$

Proof of (e): Adding 1 to both sides of $\frac{a}{b} = \frac{c}{d}$, we get

$$\frac{a}{b} + 1 = \frac{c}{d} + 1 \quad \text{or} \quad \frac{a+b}{b} = \frac{c+d}{d}$$

The proofs of the other properties are left to Exercise 8-1. You will see later in this chapter that these properties are useful in geometry.

EXAMPLE 2. Solve the proportion $4 : 7 = x : 21$ for x .

Solution

$$\begin{aligned}\frac{4}{7} &= \frac{x}{21} \\ 7x &= 4 \cdot 21 \quad (\text{property (a)}) \\ x &= \frac{4 \cdot 21}{7} \\ &= 4 \cdot 3 = 12\end{aligned}$$

EXAMPLE 3. Find the mean proportional between 4 and 36.

Solution

Let x be the mean proportional.

$$\begin{aligned}\text{Then } \frac{4}{x} &= \frac{x}{36} \\ \therefore x^2 &= 4 \cdot 36 = 144 \quad (\text{property (a)}) \\ \therefore x &= 12 \quad \text{or} \quad -12.\end{aligned}$$

EXERCISE 8-1

A 1. Complete the following.

- | | |
|--|--|
| (a) If $\frac{a}{b} = \frac{7}{4}$, then $\frac{b}{a} = \boxed{}.$ | (b) If $\frac{x}{6} = \frac{2}{5}$, then $5x = \boxed{}.$ |
| (c) If $\frac{2}{x} = \frac{3}{8}$, then $3x = \boxed{}.$ | (d) If $\frac{x}{y} = \frac{3}{7}$, then $\frac{x}{3} = \boxed{}.$ |
| (e) If $\frac{m}{n} = \frac{2}{9}$, then $\frac{m+n}{n} = \boxed{}.$ | (f) If $\frac{h}{3} = \frac{k}{10}$, then $\frac{h-3}{3} = \boxed{}.$ |
| (g) If $\frac{c}{d} = \frac{1}{9}$, then $\frac{9}{d} = \boxed{}.$ | (h) If $\frac{27}{x+1} = \frac{4}{y}$, then $27y = \boxed{}.$ |
| (i) If $\frac{r}{6} = \frac{7}{s}$, then $\frac{s}{7} = \boxed{}.$ | (j) If $7c = 2d$, then $\frac{c}{d} = \boxed{}.$ |
| (k) If $\frac{1}{t} = \frac{t}{9}$, then $t = \boxed{}.$ | (l) If $3x = 4y$, then $\frac{x}{\boxed{}} = \frac{y}{\boxed{}}.$ |

B 2. Solve the following proportions for x .

- | | | |
|-------------------------------------|------------------------------------|-------------------------------------|
| (a) $3 : 5 = 9 : x$ | (b) $x : 3 = 9 : 10$ | (c) $6 : 17 = 2x : 51$ |
| (d) $x - 4 : 12 = 6 : 8$ | (e) $\frac{x}{12} = \frac{9}{16}$ | (f) $\frac{4}{3x} = \frac{15}{21}$ |
| (g) $\frac{12}{38} = \frac{2}{x}$ | (h) $\frac{x}{x+2} = \frac{2}{3}$ | (i) $\frac{1}{x} = \frac{5}{3x+8}$ |
| (j) $\frac{6+x}{7} = \frac{1}{4}$ | (k) $\frac{a}{x} = \frac{r}{s}$ | (l) $\frac{1+x}{1-x} = \frac{5}{3}$ |
| (m) $\frac{x+1}{6} = \frac{4}{x-1}$ | (n) $\frac{1+x}{2} = \frac{3x}{7}$ | (o) $\frac{a-x}{x} = \frac{b}{c}$ |

3. Find x and y in each of the following cases.

(a) $\frac{x}{15} = \frac{14}{y} = \frac{2}{9}$

(b) $\frac{x}{15} = \frac{y}{21} = 6$

(c) $\frac{2}{x} = \frac{4}{3y} = 10$

(d) $\frac{x}{8} = \frac{32}{x} = y$

4. If $\frac{a}{3} = \frac{b}{5}$, evaluate

(a) $\frac{2a}{3b}$

(b) $\frac{a^2}{b^2}$

(c) $\frac{a+b}{2a+b}$

5. Find the value of $\frac{x}{y}$ in the following:

(a) $3x = 5y$

(b) $5x + 3y = 5y - 2x$

(c) $16x^2 = 25y^2$

(d) $\frac{x+y}{x-y} = \frac{4}{3}$

6. Find the mean proportionals between the following pairs of numbers.

(a) 9 and 16

(b) 4 and 25

(c) $\sqrt{2}$ and $2\sqrt{2}$

(d) 3 and 21

(e) x and x^3

(f) $3x$ and $12x$

7. Find the fourth proportionals to the following sets of numbers.

(a) 3, 4, 7

(b) 2, 5, 8

(c) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}$

(d) a, b, c

8. Find two numbers whose sum is 50 and whose ratio is 3 : 7.

9. The ratio of girls to boys in a class was 4 : 5. Then four more girls are added to the class, making equal numbers of boys and girls. What is the present size of the class?

10. If $a : b = 5 : 6$, $b : c = 3 : 7$, and $c : d = 5 : 12$, find $a : d$.

11. Prove properties (b), (c), (d), and (f) of proportions.

12. If b is a mean proportional between a and c , prove that $b = \pm\sqrt{ac}$.

C 13. If $\frac{a}{b} = \frac{c}{d}$, use the properties of proportions to prove $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.

8.2 MULTIPLE RATIOS

Suppose $a : b = 2 : 3$ and $b : c = 3 : 5$. We write this for short as $a : b : c = 2 : 3 : 5$. These are called three-term ratios.

Since $\frac{a}{b} = \frac{2}{3}$, we have $\frac{a}{2} = \frac{b}{3}$. (property (c))

Since $\frac{b}{c} = \frac{3}{5}$, we have $\frac{b}{3} = \frac{c}{5}$. (property (c))

Therefore $\frac{a}{2} = \frac{b}{3} = \frac{c}{5}$.

Let k be the common value of these three ratios:

$$\frac{a}{2} = \frac{b}{3} = \frac{c}{5} = k.$$

Then $a = 2k$, $b = 3k$, and $c = 5k$.

In general, a three-term ratio $a : b : c$ makes sense if $b \neq 0$ and $c \neq 0$.

Multiple Proportions

$$a : b : c = d : e : f$$

means there is a constant $k \neq 0$ such that $a = kd$, $b = ke$, and $c = kf$.
If the numbers are all nonzero, then this can be written as

$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f}.$$

EXAMPLE 1. What are the missing numbers in the proportion $6 : 9 : 15 = 2 : \blacksquare : \blacksquare$?

Solution

Since $6 = 3 \cdot 2$, $9 = 3 \cdot 3$, and $15 = 3 \cdot 5$,
 $\therefore 6 : 9 : 15 = 2 : 3 : 5$.

EXAMPLE 2. Find the values of x and y if $x : 7 : 2 = 3 : y : 8$.

Solution

$$\begin{aligned} \frac{x}{3} &= \frac{7}{y} = \frac{2}{8} \\ \frac{x}{3} &= \frac{1}{4} & \frac{7}{y} &= \frac{1}{4} \\ \therefore x &= \frac{3}{4} & \therefore y &= 7 \cdot 4 = 28 \end{aligned}$$

EXAMPLE 3. A bag of fertilizer contains nitrogen, phosphoric acid, and potash in the ratio $18 : 6 : 9$. If it contains 4 kg of nitrogen, how much phosphoric acid and potash does it contain?

Solution

Let the mass of phosphoric acid be x kg and the mass of potash y kg.
Then $4 : x : y = 18 : 6 : 9$

Commercial fertilizer ratios are not reduced to lowest terms because they represent contents by percentage.

$$\begin{aligned} \therefore \frac{4}{18} &= \frac{x}{6} = \frac{y}{9} \\ \frac{2}{9} &= \frac{x}{6} & \frac{2}{9} &= \frac{y}{9} \\ \therefore 9x &= 12 & \therefore 9y &= 18 \\ \therefore x &= \frac{12}{9} = \frac{4}{3} & \therefore y &= 2 \end{aligned}$$

The mass of phosphoric acid is $1\frac{1}{3}$ kg and the mass of potash is 2 kg.

EXAMPLE 4. The sum of three numbers in the ratio $3 : 4 : 9$ is 320. Find the numbers.

Solution

Let the numbers be x , y , and z .

Then

$$x + y + z = 320$$

and

$$x : y : z = 3 : 4 : 9.$$

∴ there is a constant k such that

$$x = 3k, \quad y = 4k, \quad z = 9k.$$

$$\therefore 3k + 4k + 9k = 320$$

$$16k = 320$$

$$k = 20$$

The numbers are 60, 80, 180.

Multiple ratios with more than three terms can also be defined. For instance

$$1 : 3 : 5 : 7 = 4 : 12 : 20 : 28$$

since

$$\frac{1}{4} = \frac{3}{12} = \frac{5}{20} = \frac{7}{28}.$$

Place a pack of 52 playing cards on a table so that the bottom card is flush with the edge of the table. By offsetting successive cards from the bottom to the top, get the topmost card to overhang the table's edge by more than one card length.

EXERCISE 8-2

A 1. State the missing numbers in the following proportions.

(a) $1 : 2 : 3 = 3 : \blacksquare : \blacksquare$

(b) $2 : 6 : 10 = 1 : \blacksquare : \blacksquare$

(c) $7 : 3 : 1 = \blacksquare : 6 : \blacksquare$

(d) $1 : 3 : 2 = \blacksquare : \blacksquare : 8$

(e) $6 : \blacksquare : \blacksquare = 2 : 5 : 2$

(f) $\blacksquare : \frac{1}{4} : \blacksquare = 4 : 1 : 3$

(g) $2 : \blacksquare : 3 = 10 : 5 : \blacksquare$

(h) $2 : 3 : \blacksquare = \blacksquare : 3 : 4$

B 2. Solve the following proportions for x , y , and z .

(a) $x : 2 : 6 = 6 : 18 : y$

(b) $7 : x : 2 = y : 27 : 9$

(c) $2x : 3y : 7 = 12 : 1 : 3$

(d) $1 : 2 : 3 = x + 1 : y - 2 : 2$

(e) $1 : 2 : 4 : x = y : z : 18 : 3$

(f) $4 : 7 : 10 : 13 = 10 : x : y : z$

(g) $2 : 5 : x = 6 : x + y : 12$

(h) $x + y : x - y : 9 = 4 : 12 : 18$

3. If $a : b : c = 6 : 4 : 5$, find the value of $\frac{3a - b}{4b + c}$.

4. A certain brand of fertilizer contains nitrogen, phosphoric acid, and potash in the ratio $10 : 6 : 8$. If a bag of this fertilizer contains 2 kg of potash, how much nitrogen and phosphoric acid does it contain?

5. The M and N Nut Company advertises that its tins of mixed nuts contain peanuts, cashews, and almonds in the ratio $5 : 3 : 2$ by mass.

(a) If a tin contains 250 g of cashews, find the mass of peanuts and the mass of almonds in the tin.

(b) A smaller tin contains a total mass of 500 g of nuts. Find the mass of each kind of nut in this tin.

6. Find three numbers in the ratio $7 : 3 : 2$ whose sum is 72.

7. Find four numbers in the ratio $3 : 4 : 6 : 9$ whose sum is 330.

8. The measures of the three angles of a triangle are in the ratio $1 : 3 : 5$. Find the size of each angle.

C 9. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, prove that

$$\frac{a + c + e}{b + d + f} = \frac{a}{b}.$$

8.3 THE DIVISION OF SIDES THEOREM

In Figure 8-1, suppose that a triangular lot ABC was surveyed and the following measurements taken:

$$AD = 20 \text{ m}, \quad DB = 35 \text{ m}, \quad AE = 25 \text{ m}.$$

If $DE \parallel BC$, is it possible to calculate EC without measuring?

Yes. The following theorem allows us to make the calculation.

(DST) Division of Sides Theorem

A line parallel to one side of a triangle divides the other two sides in the same ratio.

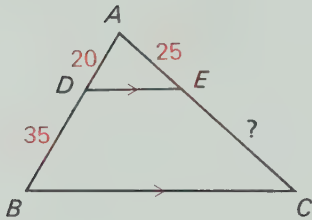


Figure 8-1

Given $\triangle ABC$ with $DE \parallel BC$

Required To prove $\frac{AD}{DB} = \frac{AE}{EC}$.

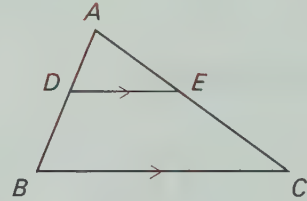


Figure 8-2

Proof

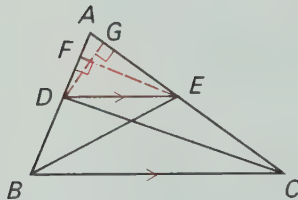
Draw BE and CD .

Triangles ADE and DBE have the same altitude EF . Therefore the ratio of their areas is equal to the ratio of their bases, i.e.,

$$\frac{\triangle ADE}{\triangle DBE} = \frac{AD}{DB}. \quad (1)$$

Likewise triangles ADE and CDE have the same altitude DG and so

$$\frac{\triangle ADE}{\triangle CDE} = \frac{AE}{EC}. \quad (2)$$



But triangles DBE and CDE have the same base DE and equal altitudes (since $DE \parallel BC$).

$$\therefore \triangle DBE = \triangle CDE$$

Thus

$$\frac{\triangle ADE}{\triangle DBE} = \frac{\triangle ADE}{\triangle CDE}.$$

Substituting into this equation from equations (1) and (2), we get

$$\frac{AD}{DB} = \frac{AE}{EC}$$

Corollary: A line parallel to one side of a triangle cuts off segments proportional to the other two sides of the triangle.

In terms of Figure 8-2, the corollary says that if $DE \parallel BC$, then

$$\frac{AD}{AB} = \frac{AE}{AC} \quad \text{and} \quad \frac{DB}{AB} = \frac{EC}{AC}.$$

The proof is left to Exercise 8-3.

EXAMPLE 1. At the beginning of this section $\triangle ABC$ was given with $DE \parallel BC$, $AD = 20$ m, $DB = 35$ m, and $AE = 25$ m. (See Figure 8-1.) Calculate EC .

Solution

Let x be the length of EC in metres.

Since $DE \parallel BC$,

$$\frac{AD}{DB} = \frac{AE}{EC} \quad (\text{DST})$$

$$\frac{20}{35} = \frac{25}{x}$$

$$20x = 25 \cdot 35$$

$$x = \frac{25 \cdot 35}{20}$$

$$= \frac{5 \cdot 35}{4} = 43.75$$

Thus the length of EC is 43.75 m.

Converse of DST

If a line divides two sides of a triangle in the same ratio, then it is parallel to the third side of the triangle.

Given : $\triangle ABC$ with D on AB and E on AC so that

$$\frac{AD}{DB} = \frac{AE}{EC}$$

Required : To prove $DE \parallel BC$.

Proof

Statements

Reasons

Let F be the point on AC such that

$$BF \parallel DE.$$

In $\triangle ABF$, $DE \parallel BF$.

$$\therefore \frac{AD}{DB} = \frac{AE}{EF}$$

DST

$$\text{But } \frac{AD}{DB} = \frac{AE}{EC}$$

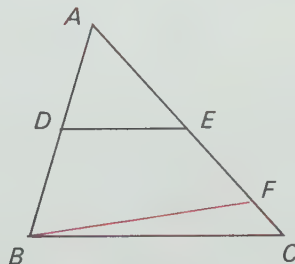
given

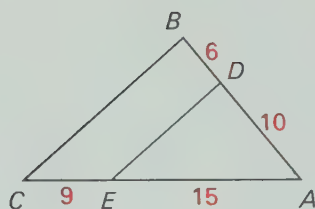
$$\therefore EF = EC$$

$\therefore F$ and C are the same point

F and C are on the same side of A

$$\therefore DE \parallel BC$$





EXAMPLE 2. Prove that $DE \parallel BC$.

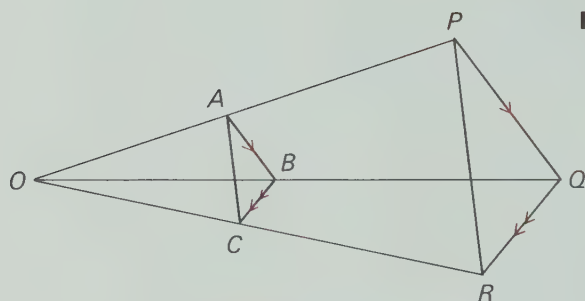
Solution

$$\frac{AD}{DB} = \frac{10}{6} = \frac{5}{3}$$

$$\frac{AE}{EC} = \frac{15}{9} = \frac{5}{3}$$

$$\therefore \frac{AD}{DB} = \frac{AE}{EC}$$

$$\therefore DE \parallel BC \quad (\text{Converse of DST})$$



Analysis

We can prove lines parallel using Converse of DST. In $\triangle OPR$, let us try to prove that

$$\frac{OA}{AP} = \frac{OC}{CR}$$

EXAMPLE 3. Given: $AB \parallel PQ$

$BC \parallel QR$

Required: To prove $AC \parallel PR$.

Proof

Statements	Reasons
In $\triangle OPQ$, $AB \parallel PQ$	given
$\therefore \frac{OA}{AP} = \frac{OB}{BQ}$	DST
In $\triangle OQR$, $BC \parallel QR$	given
$\therefore \frac{OB}{BQ} = \frac{OC}{CR}$	DST
$\therefore \frac{OA}{AP} = \frac{OC}{CR}$	
$\therefore AC \parallel PR$	Converse of DST

EXERCISE 8-3

A 1. Complete the following proportions.

(a) $\frac{PR}{RT} = \frac{\square}{\square}$

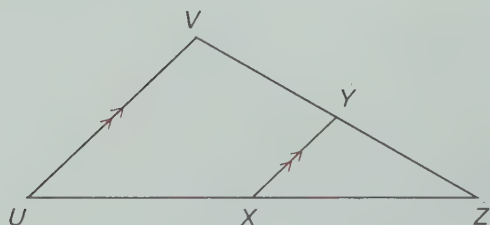
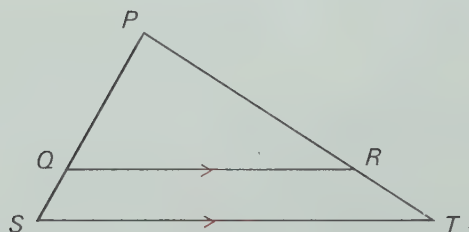
(b) $\frac{QS}{PQ} = \frac{\square}{\square}$

(c) $\frac{PQ}{PS} = \frac{\square}{\square}$

(d) $\frac{UX}{XZ} = \frac{\square}{\square}$

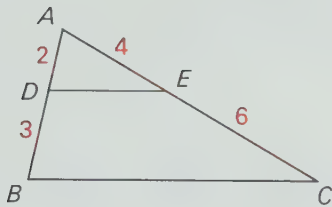
(e) $\frac{VY}{YZ} = \frac{\square}{\square}$

(f) $\frac{UX}{UZ} = \frac{\square}{\square}$

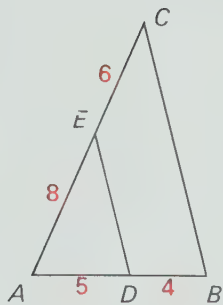


2. In which of the following figures is $DE \parallel BC$?

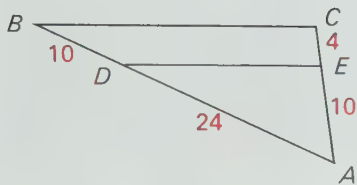
(a)



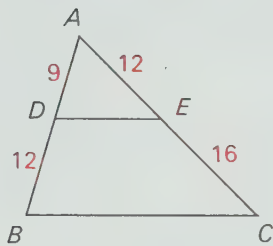
(b)



(c)

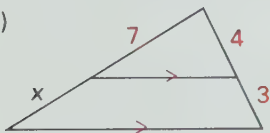


(d)

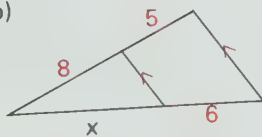


B 3. Find x in each of the following figures.

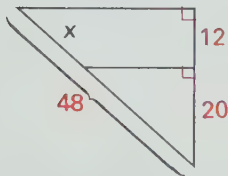
(a)



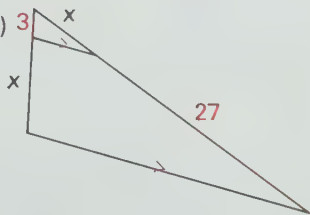
(b)



(c)

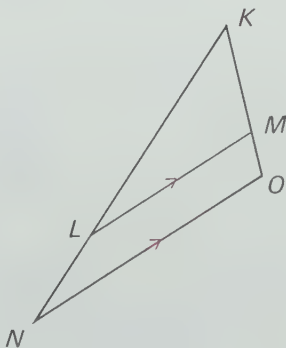


(d)



4. Complete the following table in your notebook.

KL	LN	KN	KM	MO	KO
5	3		4		
6	4				7
	2		2.5		4
12		20			15
a	b			c	
a			b	c	

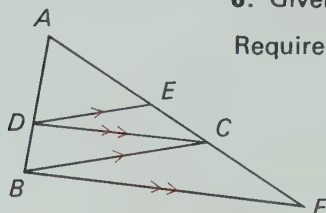


5. Prove the corollary to DST.

Hint: Use DST and properties (b) and (e) of proportions.

6. Given: $\triangle ABF$ with $DE \parallel BC$ and $DC \parallel BF$

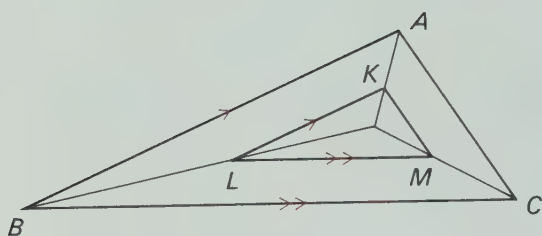
Required: To Prove $\frac{AE}{EC} = \frac{AC}{CF}$



7. $ABCD$ is a parallelogram, E is the midpoint of AD , F is the midpoint of BC .

(a) Prove that $AF \parallel EC$.

(b) Prove that the diagonal BD is divided into three segments of equal length by AF and EC .



8. Given: $AB \parallel KL$

$BC \parallel LM$

Required: To prove $AC \parallel KM$

9. O is any point inside a quadrilateral $ABCD$. K is any point on AO . KL meets OB at L and $KL \parallel AB$. LM meets OC at M and $LM \parallel BC$. MN meets OD at N and $MN \parallel CD$. Prove that $KN \parallel AD$.

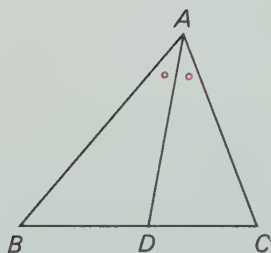
10. $ABCD$ is a quadrilateral and P, Q, R, S are the midpoints of the sides AB, BC, CD, DA respectively. Prove that $PQRS$ is a parallelogram.

C 11. Prove the *Angle Bisector Theorem*: The angle bisector in a triangle divides the opposite side into segments that have the same ratio as the other two sides.

Given: $\angle BAD = \angle DAC$

Required: To prove $\frac{BD}{DC} = \frac{AB}{AC}$

Hint: Produce CA and draw $BE \parallel DA$ so that BE meets CA produced at E .



8.4 SIMILAR TRIANGLES

In Figure 8-3, triangles ABC and DEF are called similar triangles because they have the same shape.

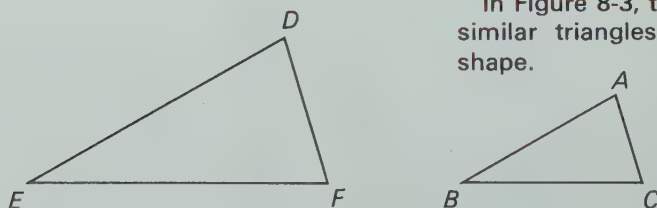


Figure 8-3

A more precise definition is the following.

Two triangles are called *similar* if there is a correspondence between their vertices such that corresponding angles are equal and corresponding sides are proportional.

The notation for this is

$$\triangle ABC \sim \triangle DEF$$

which is read: "Triangle ABC is similar to triangle DEF ." Corresponding vertices are written in the same order.

$\triangle ABC \sim \triangle DEF$ means that (i) $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$ and

(ii) $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$.

To define similar polygons all we have to do is replace "triangles" by "polygons" in the definition of similar triangles.

EXAMPLE 1. If (a) $\triangle FBI \sim \triangle SLT$ and (b) quadrilateral $ABCD \sim$ quadrilateral $WXYZ$, state (i) the equal angles (ii) the proportions relating the corresponding sides.

Solution

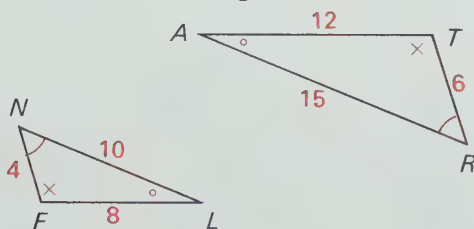
(a) (i) $\angle F = \angle S$, $\angle B = \angle L$, $\angle I = \angle T$

(ii) $\frac{FB}{SL} = \frac{BI}{LT} = \frac{IF}{TS}$

(b) (i) $\angle A = \angle W$, $\angle B = \angle X$, $\angle C = \angle Y$, $\angle D = \angle Z$

(ii) $\frac{AB}{WX} = \frac{BC}{XY} = \frac{CD}{YZ} = \frac{DA}{ZW}$

EXAMPLE 2. Show that the triangles are similar.



Solution

In $\triangle ART$ and $\triangle LNF$, we have

(i) $\angle A = \angle L$, $\angle R = \angle N$, $\angle F = \angle T$

(ii) $\frac{AR}{LN} = \frac{15}{10} = \frac{3}{2}$, $\frac{RT}{NF} = \frac{6}{4} = \frac{3}{2}$, $\frac{TA}{FL} = \frac{12}{8} = \frac{3}{2}$

$\therefore \frac{AR}{LN} = \frac{RT}{NF} = \frac{TA}{FL}$

$\therefore \triangle ART \sim \triangle LNF$

1. Write your age.
2. Multiply by 4.
3. Add 10.
4. Multiply by 25.
5. Add your "change" less than one dollar.
6. Subtract the number of days in a regular year.
7. Add 115

Explain your answer.

Corresponding sides are opposite equal angles.

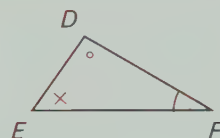
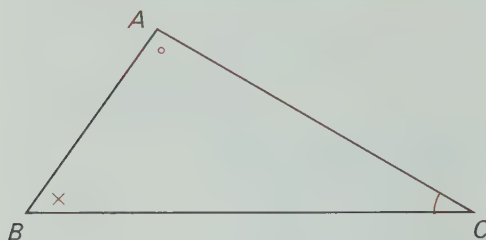
In Example 2 we had to prove two things to show that $\triangle ART \sim \triangle LNF$:

- (i) corresponding angles are equal,
- (ii) corresponding sides are proportional.

However, as it happens, in order to prove that triangles are similar it is sufficient to prove that (i) holds. This fact is our next theorem.

(AAA ~) Angle, Angle, Angle Similarity Theorem

If the three angles of one triangle are equal to the three angles of another triangle, then the triangles are similar.



Analysis

We copy the smaller triangle, $\triangle DEF$, in one corner of the larger triangle, $\triangle ABC$. Then we can use DST.

Given: $\triangle ABC$ and $\triangle DEF$ $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$.

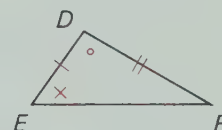
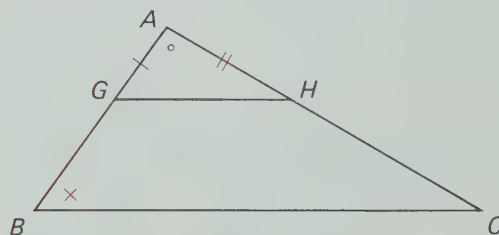
Required: To prove $\triangle ABC \sim \triangle DEF$.

Proof

Choose G on AB and H on AC so that

$$AG = DE \quad \text{and} \quad AH = DF$$

Join GH .



Statements

Reasons

Since $\angle A = \angle D$ is given,

$$\therefore \triangle AGH \cong \triangle DEF$$

SAS

$$\therefore \angle AGH = \angle DEF$$

But $\angle DEF = \angle B$

given

$$\therefore \angle AGH = \angle B$$

$$\therefore GH \parallel BC$$

TPT

$$\therefore \frac{AG}{AB} = \frac{AH}{AC}$$

DST Corollary

But $AG = DE$ and $AH = DF$

$$\therefore \frac{DE}{AB} = \frac{DF}{AC}$$

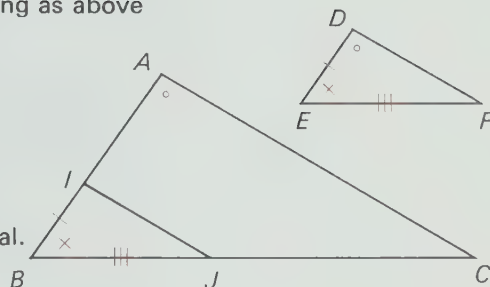
If we copy $\triangle DEF$ again as shown, then the same reasoning as above shows that

$$\frac{DE}{AB} = \frac{EF}{BC}$$

$$\therefore \frac{DE}{AB} = \frac{EF}{BC} = \frac{FD}{CA}$$

Also, the angles of $\triangle DEF$ and $\triangle ABC$ are given as equal.

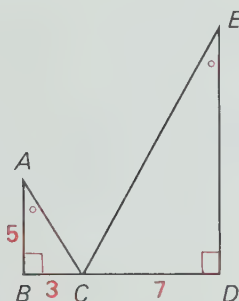
$$\therefore \triangle DEF \sim \triangle ABC.$$



Corollary: (AA~) Angle, Angle Similarity Theorem

If two angles of one triangle are equal to two angles of another triangle, then the triangles are similar.

This corollary follows from AAA~ because if two angles of one triangle are equal to two angles of another, then the third pair of angles are equal by ASTT.



EXAMPLE 3. Find DE.

Solution

Since $\angle A = \angle E$ and $\angle B = \angle D$,

$$\therefore \triangle ABC \sim \triangle EDC \quad (\text{AA} \sim)$$

$$\therefore \frac{DE}{BA} = \frac{DC}{BC}$$

$$\therefore DE = BA \cdot \frac{DC}{BC}$$

$$= 5 \cdot \frac{7}{3} = 11\frac{2}{3}$$

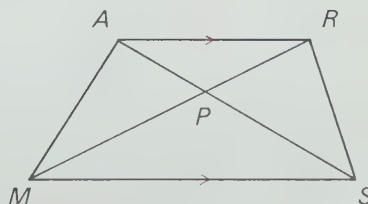
EXAMPLE 4

Given: trapezoid MARS with $MS \parallel AR$ and diagonals intersecting at P.

Required: To prove $MP \cdot PA = SP \cdot PR$

Proof

Statements	Reasons
$MS \parallel AR$	given
$\angle MSP = \angle PAR$	TPT
and $\angle SMP = \angle PRA$	TPT
$\therefore \triangle MPS \sim \triangle RPA$	AA~
$\therefore \frac{MP}{RP} = \frac{PS}{PA}$	
$\therefore MP \cdot PA = SP \cdot PR$	



Analysis

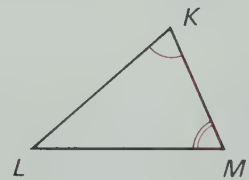
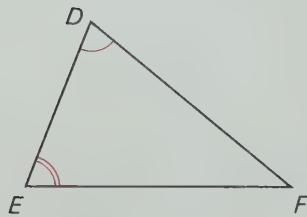
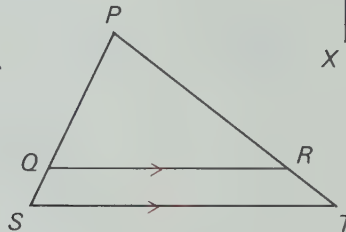
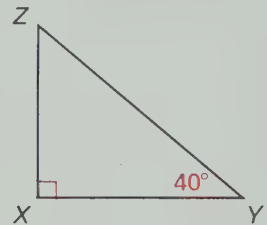
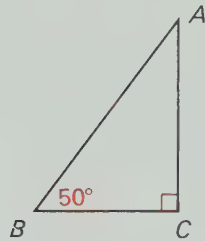
Rewrite the conclusion as a proportion:

$$\frac{MP}{PR} = \frac{SP}{PA}$$

DST does not apply here, and so we look for similar triangles.

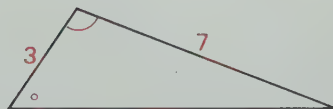
EXERCISE 8-4

- A** 1. State (i) the equal angles
(ii) the proportions relating the corresponding sides for each of the following similarities.
- (a) $\triangle XYZ \sim \triangle SIT$ (b) pentagon $ABCDE \sim$ pentagon $KLMNO$
(c) $\triangle LAM \sim \triangle PQR$ (d) quadrilateral $SOFT \sim$ quadrilateral $HARD$
2. If $\triangle STU \sim \triangle MOP$ and $\triangle RAN \sim \triangle BED$, complete the following statements.
- (a) $\frac{US}{PM} = \frac{TU}{\square}$ (b) $\angle E = \angle \square$ (c) $\frac{NA}{DE} = \frac{RA}{\square}$
(d) $\frac{MO}{ST} = \frac{MP}{\square}$ (e) $\angle U = \angle \square$ (f) $\frac{\square}{RN} = \frac{BE}{\square}$
- B** 3. State pairs of similar triangles and the corresponding proportions.

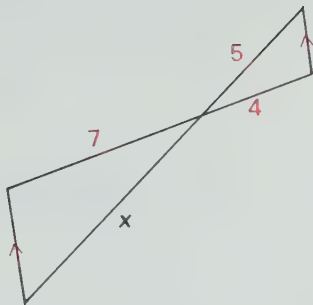


4. Find the values of x in the following figures.

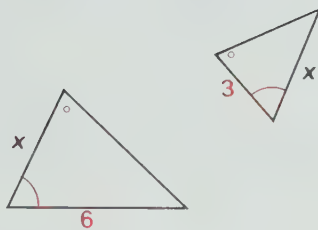
(a)



(b)

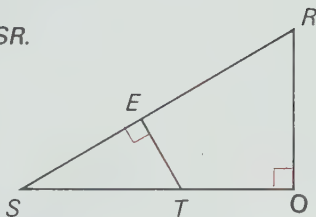


(c)



5. Given: $\triangle SOR$ with $RO \perp SO$ and $TE \perp SR$.

Required: To prove $\triangle SET \sim \triangle SOR$.



6. Prove that any two isosceles triangles with equal vertical angles are similar.

7. In an acute $\triangle ABC$, AD meets BC at D , BE meets AC at E , $AD \perp BC$, and $BE \perp AC$. AD and BE intersect at F .

Prove the following.

(a) $\triangle AEF \sim \triangle BDF$

(b) $\triangle ACD \sim \triangle BCE$

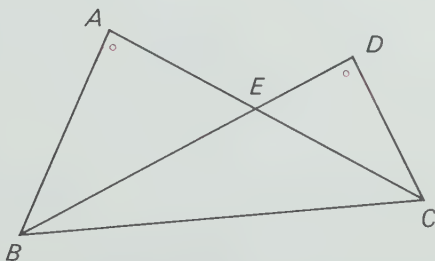
(c) $\triangle AFE \sim \triangle ACD$.

8. Given: $\angle A = \angle D$.

Required: To prove

(a) $\frac{CD}{AB} = \frac{DE}{AE}$

(b) $AE \cdot EC = BE \cdot ED$



9. Prove that corresponding altitudes of similar triangles have the same ratio as corresponding sides.

10. If D, E, F are the midpoints of BC, CA, AB respectively in $\triangle ABC$, prove that $\triangle ABC \sim \triangle DEF$.

11. If $\triangle ABC \sim \triangle DEF$, $AB = 3$ cm, $BC = 5$ cm, $CA = 7$ cm, and the perimeter of $\triangle DEF$ is 24 cm, find the lengths of the sides of $\triangle DEF$.

12. $ABCD$ is a trapezoid with $AB \parallel CD$. The diagonals intersect at E , and a line drawn through E parallel to AB intersects BC at F .

Prove that $\frac{AB}{CD} = \frac{BF}{FC}$.

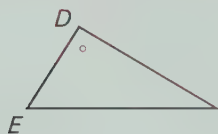
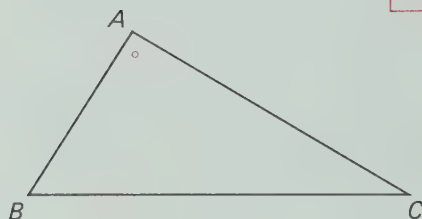
13. If $\triangle ABC \sim \triangle DEF$ and $\triangle DEF \sim \triangle KLM$, prove that $\triangle ABC \sim \triangle KLM$.

8.5 MORE SIMILARITY THEOREMS

The theorems in this section will give you two more ways of proving triangles similar.

(SAS ~) Side, Angle, Side Similarity Theorem

If two triangles have two pairs of corresponding sides proportional, and the included angles are equal, then the triangles are similar.



Given: $\triangle ABC$ and $\triangle DEF$ with

$$\frac{AB}{DE} = \frac{AC}{DF}$$

and $\angle A = \angle D$.

Required: To prove $\triangle ABC \sim \triangle DEF$.

Proof

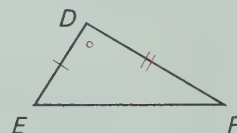
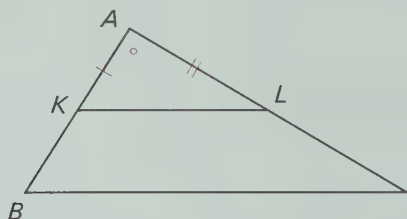
Chose K on AB and L on AC so that

$$AK = DE \quad \text{and} \quad AL = DF$$

Join KL .

Analysis

We copy the smaller triangle, $\triangle DEF$, in the top corner of the larger triangle, $\triangle ABC$. Then we can use the Converse of DST.



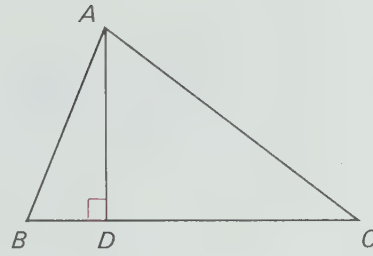
Statements	Reasons
Since $\angle A = \angle D$ is given,	
$\therefore \triangle AKL \cong \triangle DEF$	SAS
$\therefore \angle AKL = \angle DEF$	
Since $\frac{AB}{DE} = \frac{AC}{DF}$	given
and $DE = AK$ and $DF = AL$,	
$\therefore \frac{AB}{AK} = \frac{AC}{AL}$	
$\therefore \frac{KB}{AK} = \frac{LC}{AL}$	property (f) of proportions
$\therefore \frac{AK}{KB} = \frac{AL}{LC}$	property (b) of proportions
$\therefore KL \parallel BC$	converse of DST
$\therefore \angle B = \angle AKL$	
$\therefore \angle B = \angle DEF$	
But $\angle A = \angle D$	given
$\therefore \triangle ABC \sim \triangle DEF$	AA ~

EXAMPLE 1

Given $\triangle ABC$ with $AD \perp BC$ and $AD^2 = BD \cdot DC$

Required: To prove $\angle BAC = 90^\circ$

Proof



Statements

Reasons

$$AD^2 = BD \cdot DC$$

given

$$AD \cdot AD = BD \cdot DC$$

$$\therefore \frac{BD}{AD} = \frac{AD}{DC}$$

$$\text{also } \angle BDA = \angle ADC$$

right angles

$$\therefore \triangle BDA \sim \triangle ADC$$

SAS ~

$$\therefore \angle ABD = \angle DAC$$

$$\text{But } \angle ABD + \angle DAB = 90^\circ$$

right $\triangle$

$$\therefore \angle DAC + \angle DAB = 90^\circ$$

$$\therefore \angle BAC = 90^\circ$$

(SSS ~) *Side, Side, Side Similarity Theorem*

If corresponding sides of two triangles are proportional, then the triangles are similar.

Given: $\triangle ABC$ and $\triangle DEF$ with $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$

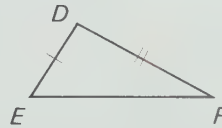
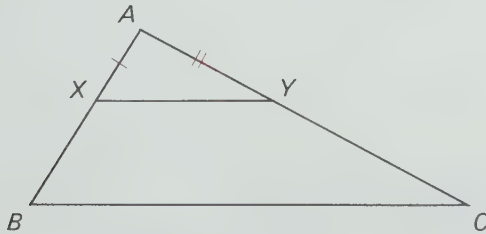
Required: To prove $\triangle ABC \sim \triangle DEF$.

Proof:

Choose X on AB and Y on AC so that

$$AX = DE \quad \text{and} \quad AY = DF.$$

Join XY .



Statements

Reasons

$$\frac{AB}{DE} = \frac{AC}{DF}$$

given

$$\therefore \frac{AB}{AX} = \frac{AC}{AY}$$

$$\text{Also } \angle BAC = \angle XAY$$

$$\therefore \triangle ABC \sim \triangle AXY$$

SAS ~

$$\therefore \frac{AB}{AX} = \frac{BC}{XY}$$

But $AX = DE$

$$\therefore \frac{AB}{DE} = \frac{BC}{XY}$$

Now $\frac{AB}{DE} = \frac{BC}{EF}$

given

$$\therefore \frac{BC}{XY} = \frac{BC}{EF}$$

$$\therefore XY = EF$$

$$\therefore \triangle AXY \cong \triangle DEF$$

SSS

$\therefore \triangle AXY$ and $\triangle DEF$ have equal angles

But $\triangle AXY$ and $\triangle ABC$ have equal angles (similar $\triangle$'s)

$\therefore \triangle ABC$ and $\triangle DEF$ have equal angles.

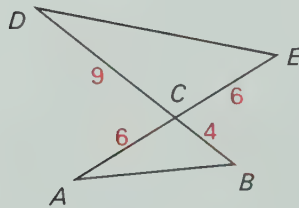
Also, we are given that the sides are proportional

$\therefore \triangle ABC \sim \triangle DEF$.

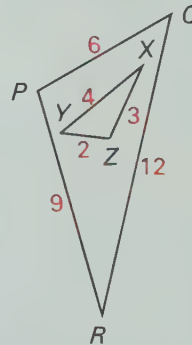
EXERCISE 8-5

- 1.** State whether or not the following pairs of triangles are similar. If the triangles are similar, state which theorem proves it.

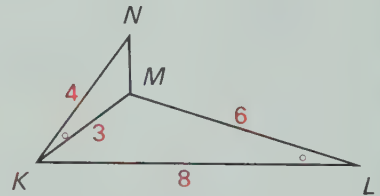
(a)



(b)



(c)



- 2.** The lengths of corresponding sides of pairs of triangles are given. In which cases are the triangles similar?

(a) 4, 7, 9; 12, 21, 27

(b) 9, 5, 7.5; 4.5, 2.5, 3.75

(c) 2, 5, 8; 5, 12, 20

(d) 48, 36, 24; 32, 24, 16

(e) 12, 9, 8; 4, 3, 2.5

- 3.** If X and Y are the midpoints of AB and AC in $\triangle ABC$, prove that $XY = \frac{1}{2}BC$.

- 4.** In quadrilateral $ABCD$, the diagonals AC and BD intersect at O and $AO : OC = BO : OD$. Prove that $AB \parallel CD$.

- 5.** If D , E , F are the midpoints of BC , CA , and AB respectively, prove that $\triangle ABC \sim \triangle DEF$ using the theorems of this section.

- 6.** Prove that corresponding medians of similar triangles have the same ratio as corresponding sides.

7. $\triangle ABC$ is isosceles with $AB = AC$. CB is produced to D and BC is produced to E so that $AB^2 = BD \cdot CE$. Prove that $\triangle ABD \sim \triangle ECA$.
8. In $\triangle ABC$, AB is produced to D and AC is produced to E so that $AB \cdot AD = AC \cdot AE$. Prove that $\angle BCE + \angle BDE = 180^\circ$.

8.6 MEAN PROPORTIONALS IN A RIGHT TRIANGLE

Figure 8-4 shows a right triangle ABC with the altitude to the hypotenuse drawn. This forms two new triangles. It turns out that all three triangles are similar.

The altitude to the hypotenuse of a right triangle forms two triangles that are similar to each other and to the original triangle.

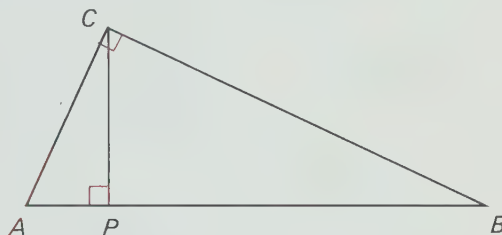


Figure 8-4

Given: $\triangle ABC$ with $\angle C = 90^\circ$ and $CP \perp AB$.

Required: To prove $\triangle ABC \sim \triangle ACP \sim \triangle CBP$.

Proof

Statements	Reasons
In $\triangle ABC$ and $\triangle ACP$,	
$\angle A$ is common,	
and $\angle APC = 90^\circ = \angle ACB$.	
$\therefore \triangle ABC \sim \triangle ACP$	AA ~
In $\triangle ABC$ and $\triangle CBP$,	
$\angle B$ is common,	
and $\angle CPB = 90^\circ = \angle ACB$.	
$\therefore \triangle ABC \sim \triangle CBP$	AA ~
Thus $\triangle ACP \sim \triangle ABC \sim \triangle CBP$.	

Corollary 1: Mean Proportional Theorem

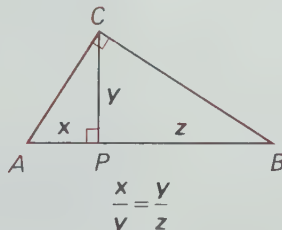
The altitude to the hypotenuse of a right triangle is the mean proportional between the segments of the hypotenuse.

Given: $\triangle ABC$ with $\angle C = 90^\circ$ and $CP \perp AB$.

Required: To prove $\frac{AP}{CP} = \frac{CP}{PB}$

Proof: Since $\triangle ACP \sim \triangle CBP$,

$$\therefore \frac{AP}{CP} = \frac{CP}{BP}.$$



Corollary 2: Each leg of a right triangle is the mean proportional between the hypotenuse and the segment of the hypotenuse adjacent to that leg.

Given: $\triangle ABC$ with $\angle C = 90^\circ$ and $CP \perp AB$

Required: To prove

$$\frac{AB}{AC} = \frac{AC}{AP} \quad \text{and} \quad \frac{AB}{BC} = \frac{BC}{BP}$$

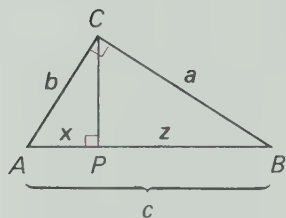
Proof

Since $\triangle ABC \sim \triangle ACP$,

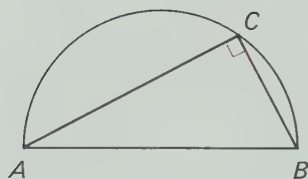
$$\therefore \frac{AB}{AC} = \frac{AC}{AP}$$

Since $\triangle ABC \sim \triangle CBP$

$$\therefore \frac{AB}{CB} = \frac{BC}{BP}$$



$$\frac{c}{b} = \frac{b}{x}, \quad \frac{c}{a} = \frac{a}{z}$$



In a semicircle
 $\angle C = 90^\circ$.

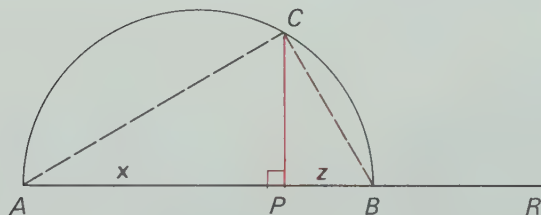
Solution

Given: Segments with lengths x and z .

Required: To construct a segment whose length is the mean proportional between x and z .

Construction:

1. Draw a ray AR , and mark off $AP = x$ and $PB = z$.
2. Construct a semicircle on AB as diameter.
3. At P , construct $PC \perp AB$ meeting the semicircle at C . Then $\angle BCA = 90^\circ$.
4. PC is the required mean proportional (by the Mean Proportional Theorem).

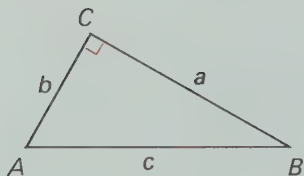


EXAMPLE 2. Use Corollary 2 to give another proof of the Pythagorean Theorem: In a right triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.

Solution

Given: Right $\triangle ABC$ with legs of lengths a and b and hypotenuse of length c .

Required: To prove $c^2 = a^2 + b^2$.



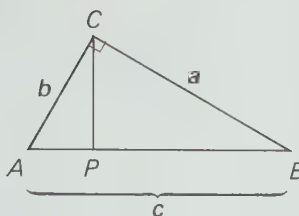
Proof:

Draw $CP \perp AB$ and let $x = AP$, $z = PB$. Then, by Corollary 2,

$$\frac{c}{b} = \frac{b}{x} \quad \text{and} \quad \frac{c}{a} = \frac{a}{z}$$

$$\therefore b^2 = cx \quad \text{and} \quad a^2 = cz$$

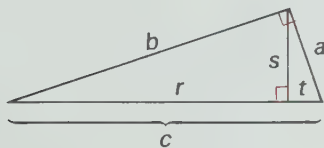
$$\begin{aligned} \therefore a^2 + b^2 &= cz + cx = c(z + x) \\ &= c \cdot c = c^2 \end{aligned}$$



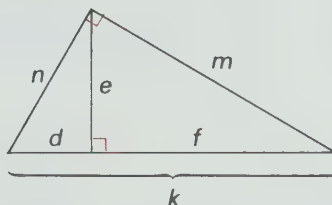
EXERCISE 8-6

- A** 1. For each triangle below, state the proportion given by the Mean Proportional Theorem.

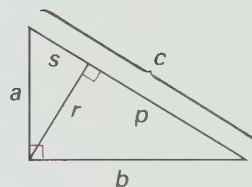
(a)



(b)



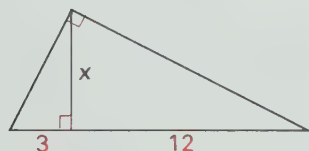
(c)



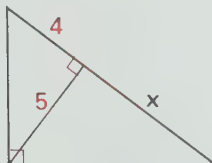
2. For each triangle above, state the proportions given by Corollary 2.

- B** 3. Find x in each of the following triangles.

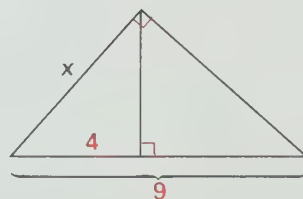
(a)



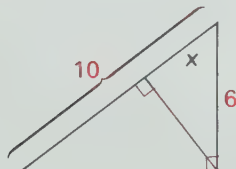
(b)



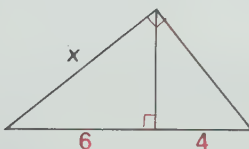
(c)



(d)



(e)



4. Draw two line segments and construct a segment whose length is the mean proportional between them.

5. Draw two line segments. Let their lengths be a and b .

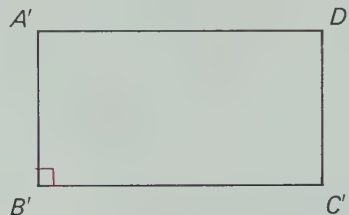
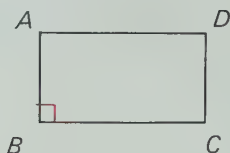
- (a) Construct the mean proportional between $2a$ and $3b$.

- (b) How long is the mean proportional in terms of a and b ?

- C** 6. If the lengths of the two legs of a right triangle are in the ratio $1 : 2$, prove that the altitude to the hypotenuse divides the hypotenuse in the ratio $1 : 4$.

8.7 AREAS AND VOLUMES OF SIMILAR FIGURES

How do the areas of similar figures compare? Let us start with an easy figure, a rectangle.



A' is read "A prime".

If rectangles $ABCD$ and $A'B'C'D'$ are similar, then there is a number k such that each side of $A'B'C'D'$ is k times the corresponding side of $ABCD$:

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = k.$$

Thus

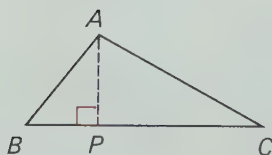
$$\begin{aligned} \frac{\text{area of } A'B'C'D'}{\text{area of } ABCD} &= \frac{A'B' \cdot B'C'}{AB \cdot BC} \\ &= \frac{A'B'}{AB} \cdot \frac{B'C'}{BC} \\ &= k^2. \end{aligned}$$

For instance if the sides of a rectangle are tripled, then its area is not tripled. The area is multiplied by 9. (How is the perimeter affected?)

The same is true for other similar figures, but the proof is not as easy.

(AST) Areas of Similar Triangles Theorem

The ratio of the areas of two similar triangles is equal to the square of the ratio of the corresponding sides.



Given : $\triangle ABC \sim \triangle DEF$.

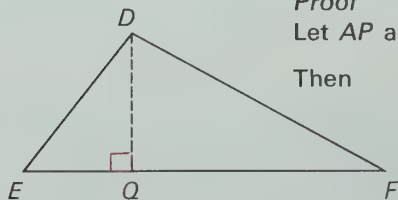
Required: To prove $\frac{\triangle DEF}{\triangle ABC} = \left(\frac{DE}{AB}\right)^2 = \left(\frac{EF}{BC}\right)^2 = \left(\frac{FD}{CA}\right)^2$

Proof

Let AP and DQ be the altitudes of the two triangles.

Then

$$\begin{aligned} \frac{\triangle DEF}{\triangle ABC} &= \frac{\frac{1}{2}EF \cdot DQ}{\frac{1}{2}BC \cdot AP} \\ &= \frac{EF}{BC} \cdot \frac{DQ}{AP} \end{aligned}$$



We know that corresponding altitudes of similar triangles have the same ratio as corresponding sides. (Question 9, Exercise 8-5)

$$\therefore \frac{DQ}{AP} = \frac{EF}{BC}$$

$$\therefore \frac{\triangle DEF}{\triangle ABC} = \left(\frac{EF}{BC}\right)^2$$

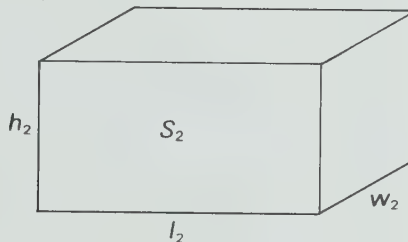
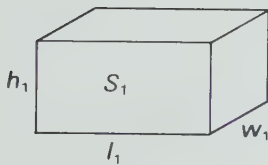
But

$$\frac{EF}{BC} = \frac{DE}{AB} = \frac{FD}{CA}$$

$$\therefore \frac{\triangle DEF}{\triangle ABC} = \left(\frac{DE}{AB}\right)^2 = \left(\frac{EF}{BC}\right)^2 = \left(\frac{FD}{CA}\right)^2$$

So far we have only considered the similarity of plane figures, but it is also possible to consider similar solids. Two solids are similar if they have the same shape. For such simple solids as cones, cylinders, and rectangular boxes we can make this definition precise as follows: two solids are similar if their corresponding dimensions are proportional. For instance, two rectangular solids S_1 and S_2 with dimensions l_1, w_1, h_1 and l_2, w_2, h_2 are similar if there is a number k such that

$$\frac{l_2}{l_1} = \frac{w_2}{w_1} = \frac{h_2}{h_1} = k.$$



How do the volumes V_1 and V_2 of S_1 and S_2 compare?

$$\begin{aligned} \frac{V_2}{V_1} &= \frac{l_2 w_2 h_2}{l_1 w_1 h_1} \\ &= \left(\frac{l_2}{l_1}\right) \left(\frac{w_2}{w_1}\right) \left(\frac{h_2}{h_1}\right) \\ &= k^3 \end{aligned}$$

Find two different numbers such that the sum of their squares is a cube, and the sum of their cubes is a square.

For instance, if the dimensions of a box are tripled, then the volume is multiplied by $3^3 = 27$.

We have proved a special case of the following general theorem. Other special cases are investigated in Exercise 8-8.

The ratio of the volumes of two similar solids is equal to the cube of the ratio of any pair of corresponding segments.

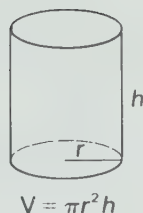
An application of this theorem occurred in *Gulliver's Travels* by Jonathan Swift. Gulliver's first voyage was to Lilliput where the inhabitants were very tiny, and he tells of the Emperor of Lilliput's decree concerning his food as follows:

"The Emperor stipulates to allow me a Quantity of Meat and Drink sufficient for the Support of 1728 Lilliputians. Some time after, asking a Friend at Court how they came to fix on that determinate Number, he told me that his Majesty's Mathematicians, having taken the Height of my Body... and finding it to exceed theirs in the proportion of

EXERCISE 8-7

- 250 fmt : intermediate

13. (a) Show that the ratio of the volume of two similar circular cylinders is equal to the cube of the ratio of their heights.



(b) Do the same for two similar cones.

14. An Egyptian pharaoh decided to build a new pyramid by doubling all the dimensions of an older pyramid. How much heavier was the new pyramid compared to the old one?

8.8 DIVISION OF A LINE SEGMENT



Suppose that P lies on AB and $AP = 15$ cm, $PB = 10$ cm. Then

$$\frac{AP}{PB} = \frac{15}{10} = \frac{3}{2}.$$

We say that P divides the segment AB internally in the ratio $3 : 2$.

Now suppose that P lies on AB produced and $AB = 25$ cm, $BP = 10$ cm.



Then
$$\frac{AP}{PB} = \frac{35}{10} = \frac{7}{2}$$

Here we say that P divides the segment AB externally in the ratio $7 : 2$.

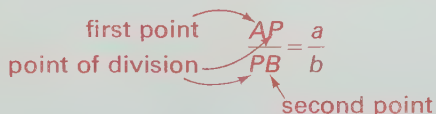
On the other hand if P lies on BA produced and $PA = 10$ cm, $AB = 25$ cm. Then



$$\frac{AP}{PB} = \frac{10}{35} = \frac{2}{7}.$$

Here P divides the segment AB externally in the ratio $2 : 7$

P divides AB in the ratio $a : b$ if



If P lies on AB , the division is internal.

If P lies on AB produced, the division is external and $a > b$.

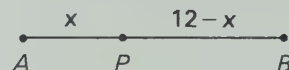
If P lies on BA produced, the division is external and $a < b$.

EXAMPLE 1. If P divides AB internally in the ratio $5 : 3$ and $AB = 12$ cm, find the length of AP and PB .

Solution

Let $AP = x$.

Then $PB = 12 - x$.



$$\frac{AP}{PB} = \frac{5}{3}$$

$$\frac{x}{12 - x} = \frac{5}{3}$$

$$3x = 5(12 - x) = 60 - 5x$$

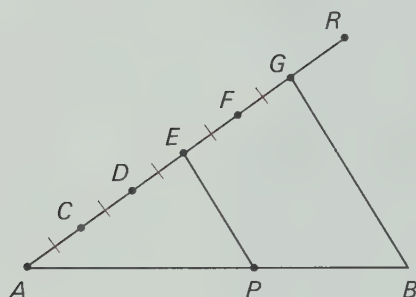
$$8x = 60$$

$$x = 7.5$$

The length of AP is 7.5 cm.

The length of PB is 4.5 cm.

EXAMPLE 2. Given a line segment AB , locate the point P which divides AB internally in the ratio $3 : 2$.



Solution

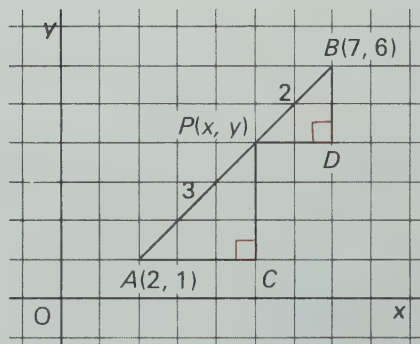
1. From A draw any ray AR at an angle to AB .
2. With any suitable radius r and centre A , draw an arc cutting AR at C .
3. With the same radius r , draw an arc with centre C cutting AR at D .
4. Continue drawing arcs so that

$$AC = CD = DE = EF = FG = r.$$

5. Join GB .
6. Construct $EP \parallel GB$.
7. Then $\frac{AP}{PB} = \frac{AE}{EG} = \frac{3}{2}$ (DST)

$\therefore P$ divides AB internally in the ratio $3 : 2$.

EXAMPLE 3. Find the point $P(x, y)$ that divides the segment joining $A(2, 1)$ to $B(7, 6)$ internally in the ratio of $3 : 2$.



Solution

Since $P(x, y)$ is the point that divides AB internally in the ratio of $3 : 2$ then

$$\frac{AP}{PB} = \frac{3}{2}$$

Draw the rise and run of AP and BP .

Since $AC \parallel PD$, $\angle PAC = \angle BPD$

$$\angle C = \angle D = 90^\circ$$

$\therefore \triangle PAC \sim \triangle BPD$ (AA $\sim$)

$$\therefore \frac{AP}{PB} = \frac{AC}{PD} = \frac{PC}{BD} = \frac{3}{2}$$

$$\frac{AC}{PD} = \frac{3}{2}$$

$$\frac{PC}{BD} = \frac{3}{2}$$

$$\therefore \frac{x-2}{7-x} = \frac{3}{2}$$

$$\frac{y-1}{6-y} = \frac{3}{2}$$

$$2x-4=21-3x$$

$$2y-2=18-3y$$

$$5x=25$$

$$5y=20$$

$$x=5$$

$$y=4$$

$\therefore P(5, 4)$ divides AB internally in the ratio of 3 : 2.

EXAMPLE 4. Find the point $P(x, y)$ that divides the segment joining $A(2, 2)$ to $B(5, 6)$ externally in the ratio of 4 : 3.

Solution

Since $P(x, y)$ is the point that divides AB externally in the ratio of 4 : 3 then

$$\frac{AP}{PB} = \frac{4}{3}$$

Draw the rise and run of AB and BP .

Since $AC \parallel BD$, $\angle BAC = \angle PBD$

$$\angle C = \angle D = 90^\circ$$

$$\therefore \triangle BAC \sim \triangle PBD \quad (AA \sim)$$

$$\therefore \frac{BA}{PB} = \frac{AC}{BD} = \frac{BC}{PD} = \frac{1}{3}$$

$$\frac{AC}{BD} = \frac{1}{3}$$

$$\frac{BC}{PD} = \frac{1}{3}$$

$$\frac{5-2}{x-5} = \frac{1}{3}$$

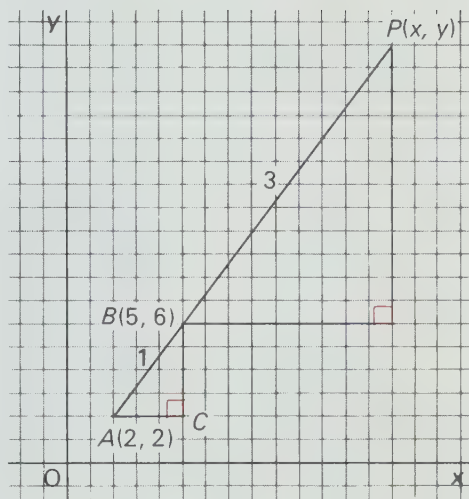
$$\frac{6-2}{y-6} = \frac{1}{3}$$

$$15-6=x-5$$

$$18-6=y-6$$

$$x=14$$

$$y=18$$

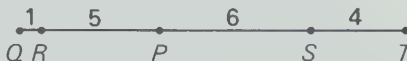
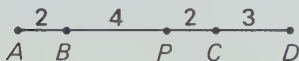


$\therefore P(14, 18)$ divides AB externally in the ratio of 4 : 3.

EXERCISE 8-8

- A** 1. State the ratios in which P divides the following line segments. State whether the division is internal or external.

- (a) BC (b) AB (c) AC (d) CD (e) BD
(f) AD (g) RS (h) RT (i) ST (j) QR



- B** 2. In what ratio does P divide AB internally in the following cases?

- (a) $AB = 16$ cm, $PB = 12$ cm (b) $AP = 6$ cm, $PB = 27$ cm
(c) $AB = 30$ cm, $AP = 22.5$ cm

3. If Q divides the segment CD internally in the ratio 2 : 7 and $CD = 18$ cm, find the lengths of CQ and QD .

4. If P divides AB externally in the ratio $6 : 5$ and $AB = 8$ cm, find the lengths of BP and AP .
5. If S divides PQ externally in the ratio $4 : 7$ and $PQ = 18$ cm find the lengths of SP and SQ .
6. Draw any line segment and use the method of Example 2 to divide the segment into three equal parts.
7. Construct the points which divide a given line segment internally in the following ratios.
 - (a) $1 : 2$
 - (b) $4 : 3$
 - (c) $2 : 7$

8. Construct the points which divide a given line segment externally in the following ratios.

- (a) $7 : 5$
- (b) $3 : 5$

9. For each of the following determine the point $P(x, y)$ that divides the given segment in the given ratio.

- (a) $A(2, 2)$ to $B(5, 3)$ internally in the ratio of $1 : 2$.
- (b) $C(-3, -5)$ to $D(4, 0)$ internally in the ratio of $1 : 4$.
- (c) $E(5, -2)$ to $F(-4, 5)$ internally in the ratio of $1 : 3$.
- (d) $G(-1, -3)$ to $H(7, 5)$ internally in the ratio of $4 : 5$.
- (e) $A(-7, 3)$ to $B(7, -1)$ internally in the ratio of $3 : 2$.
- (f) $C(6, -5)$ to $D(-3, -1)$ internally in the ratio of $4 : 3$.

10. For each of the following determine the point $P(x, y)$ that divides the given segment in the given ratio.

- (a) $A(-4, 2)$ to $B(4, 4)$ externally in the ratio of $5 : 3$.
- (b) $C(2, 1)$ to $D(5, 3)$ externally in the ratio of $3 : 1$.
- (c) $E(-6, 2)$ to $F(4, -2)$ externally in the ratio of $1 : 3$.
- (d) $G(4, -2)$ to $H(-6, 2)$ externally in the ratio of $2 : 3$.
- (e) $C(-5, -2)$ to $D(0, -4)$ externally in the ratio of $2 : 5$.
- (f) $A(3, 0)$ to $B(-2, 6)$ externally in the ratio of $7 : 2$.

C 11. The line segment joining $(2, 2)$ and $(-7, -4)$ is trisected. Find the points of division.

12. The line segment joining $A(-4, -5)$ to $B(2, -2)$ is extended to C . If $BC = 3AB$, find the location of C .

13. Find the point $P(x, y)$ that divides the segment joining $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$ internally in the ratio of $a : b$ where $a, b > 0$.

8.9 APPLICATIONS

The theorems of this chapter are very useful in computing various lengths, distances, areas, and volumes in practical situations. For example, astronomers use similar triangles to calculate the distances of the stars, and the depths of craters on the moon can be computed from the lengths of shadows in photographs. In this section, however, we shall deal with more down-to-earth applications.

EXAMPLE 1. A man whose eye-level E is 180 cm above the ground wants to know the height TR of a Douglas fir tree in British Columbia. He places a mirror face-up on the ground 30 m from the tree so that it is level. Then he moves further away until he can see the top of the tree reflected in the mirror. He measures his distance from the mirror and it is $MG = 90$ cm. How tall is the Douglas fir?

WORD LADDER

Start with the word "hare" and change one letter at a time to form a new word until you reach "soup". The best solution has the fewest steps.

h a r e

s o u p

Solution

$\angle EMG$ and $\angle TMR$ are the angles of incidence and reflection of a beam of light striking a mirror. A law of physics states that they are equal.

In $\triangle TRM$ and $\triangle EGM$,

$$\angle TMR = \angle EMG$$

and $\angle TRM = \angle EGM$ (right angles)

$$\therefore \triangle TRM \sim \triangle EGM \quad (AA \sim)$$

$$\therefore \frac{TR}{EG} = \frac{RM}{GM}$$

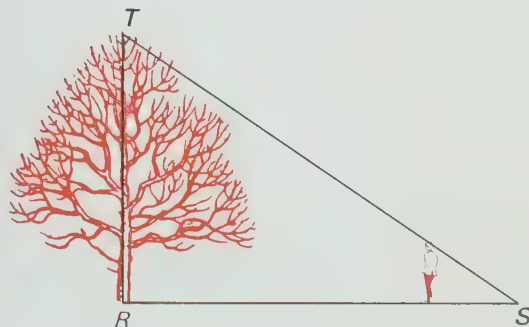
$$TR = \frac{EG \cdot RM}{GM}$$

But the lengths in meters are

$$RM = 30, \quad EG = 1.8, \quad GM = 0.9$$

$$TR = \frac{(1.8) \times 30}{0.9} = 60$$

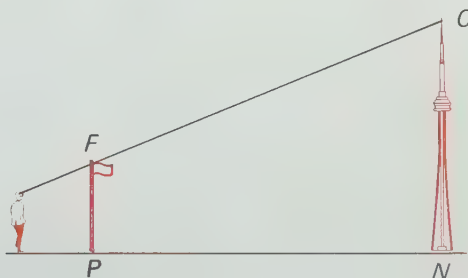
The tree is 60 m high.



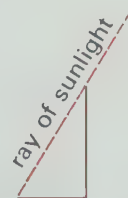
EXERCISE 8-9

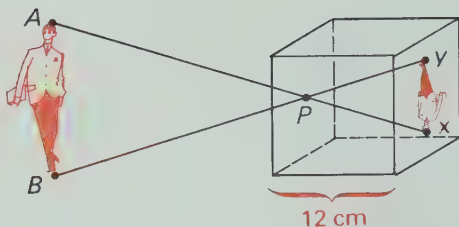
- B 1.** A man whose height is 190 cm decides to measure the height of a tree by walking away from the tree along its shadow until his head is in line with the top T of the tree and the tip S of the shadow. If his distance from the tree is 50 m and the length of the tree's shadow is 55 m, find the height of the tree.

- 2.** A man whose eye-level is 2 m above the ground decides to find the height of the CN Tower. He notices that if he stands 4 m from a flagpole, the top of the pole appears to be in line with the top of the CN Tower. He measures the distance from the flagpole to the base of the tower and finds that $PN = 133$ m. He happens to know that the flagpole is 18 m high. How high is the CN Tower?

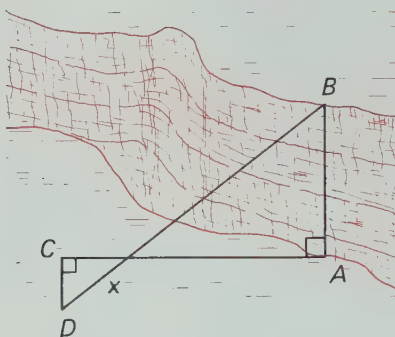


- 3.** A vertical post, 1.2 m high, casts a shadow 80 cm long at 14 30. At the same time a nearby tree casts a shadow 17 m long. Assuming that the sun's rays are parallel, find the height of the tree.



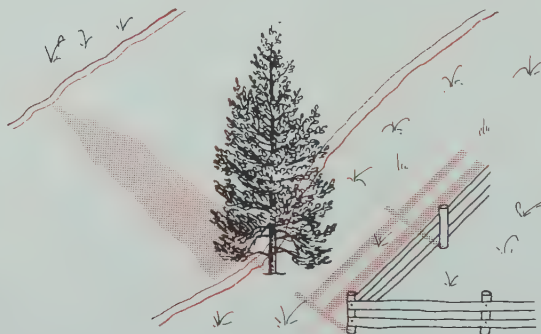


4. A pinhole camera can be made by poking a pinhole P in the centre of one side of a box. Suppose the distance from the object to the pinhole is 2 m and the camera is 12 cm long. If the height of the object is $AB = 1.2$ m, find the height of the image XY .



5. A cowboy named Bill and his faithful horse Bullet suddenly come to a canyon and stop at point A . Bill wonders if they can jump across the canyon. Luckily Bill remembers the AA Similarity Theorem and so he places a stake in the ground at X and observes that while standing at D , the stake X is in line with B and D . He paces out the distances $AX = 15$ paces, $XC = 5$ paces, $CD = 4$ paces, where $CD \perp CA$ and $BA \perp CA$. Bill's pace length is 1 m. He calculates that the distance from A to B is equal to Bullet's previous best jump. They must be in Dodge City by sundown and so Bill and Bullet attempt the jump. Bullet jumps 13 m. Did they make it?

6. A farmer wants to measure the distance across a river on his farm. At 09 : 15 one morning he notices that the shadow of a tree on one bank falls directly across the river so that the tip of the shadow just touches the opposite bank. At the same instant he measures the length of the shadow of a nearby fence post and finds it to be 4.5 m. He knows that the tree is 12 m tall and the post is 2 m tall. How wide is the river?

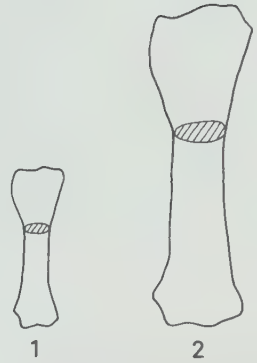


7. A photographer wants to have a picture enlarged to double its area. By what number would the length and width of the picture have to be multiplied in order to double the area?

8. A legend says that the citizens of Athens were told by the Delphic oracle in 430 B.C. that in order to rid Athens of the plague they must double the size of the altar of Apollo. By what number would the dimensions of the altar have to be multiplied in order to double the volume?

9. The strengths of two similar bones are proportional to the areas of their cross-sections. Suppose that the two bones illustrated are from animals with similar shapes, and the second bone is twice as long as the first.

- How much stronger is the second bone than the first?
- How many times heavier than the first animal is the second animal?
- Which animal is better able to support its own mass?



REVIEW EXERCISE

A 1. Complete the following.

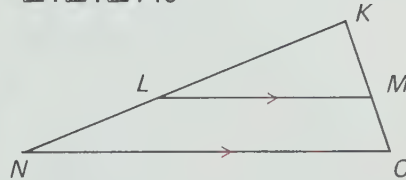
- If $\frac{x}{3} = \frac{6}{7}$, then $7x = \blacksquare$.
- If $\frac{x}{8} = \frac{y}{9}$, then $\frac{9}{y} = \blacksquare$.
- If $\frac{x}{8} = \frac{y}{9}$, then $\frac{x}{y} = \blacksquare$.
- If $\frac{p}{5} = \frac{q}{4}$, then $5q = \blacksquare$.
- If $\frac{a}{7} = \frac{b}{8}$, then $\frac{a+7}{7} = \blacksquare$.
- If $\frac{c}{d} = \frac{8}{3}$, then $\frac{c-d}{d} = \blacksquare$.
- If $3x = 10y$, then $\frac{y}{x} = \blacksquare$.
- If $r = \frac{8}{s}$, then $s = \blacksquare$.

2. State the missing numbers in the following proportions.

- $1 : 3 = 2 : \blacksquare$
- $3 : 6 : 12 = 1 : \blacksquare : \blacksquare$
- $\blacksquare : 2 = 4 : 8$
- $6 : \blacksquare : \blacksquare = 3 : 2 : 1$
- $\blacksquare : 10 : \blacksquare = 1 : 2 : 5$
- $1 : 2 : 3 : 4 = \blacksquare : \blacksquare : \blacksquare : 16$

3. Complete the following proportions.

- $\frac{KL}{LN} = \blacksquare$
- $\frac{KO}{KM} = \blacksquare$
- $\frac{MO}{KM} = \blacksquare$



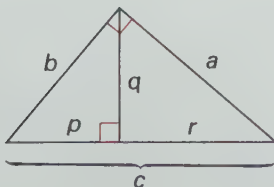
4. If $\triangle ABC \sim \triangle DEF$ and $\triangle PQR \sim \triangle LUV$, complete the following statements.

- $\angle B = \angle \blacksquare$
- $\frac{QR}{UV} = \frac{PR}{\blacksquare}$
- $\frac{DE}{AB} = \frac{FD}{\blacksquare}$
- $\angle R = \angle \blacksquare$
- $\frac{BC}{EF} = \frac{\blacksquare}{DE}$
- $\frac{PQ}{\blacksquare} = \frac{\blacksquare}{UV}$

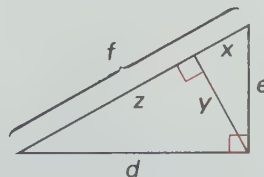
5. For each triangle below, state

(a) the proportion given by the Mean Proportional Theorem, (b) the proportions given by Corollary 2 of §8.7.

(i)



(ii)



B 6. Solve the following proportions for x and y .

(a) $6 : x = 4 : 7$

(b) $x : y : 3 = 2 : 3 : 5$

(c) $9 : 20 = 3x : 48$

(d) $5 : 2 : x = 2y : 6 : 8$

(e) $\frac{x-1}{18} = \frac{8}{6}$

(f) $\frac{x}{15} = \frac{8}{y} = \frac{3}{10}$

(g) $\frac{x}{2} = \frac{x+3}{7}$

(h) $\frac{2}{x+2} = \frac{x-2}{6}$

(i) $\frac{a}{b} = \frac{c}{x}$

(j) $\frac{x+3}{x-2} = \frac{1}{5}$

7. Find the ratio $\frac{x}{y}$ in the following cases.

(a) $4x = 6y + x$

(b) $\frac{x+y}{x+2y} = \frac{5}{7}$

8. Find the mean proportionals between the following pairs of numbers.

(a) 5 and 125

(b) 4 and 20

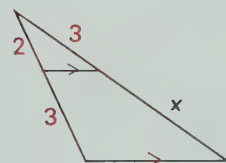
(c) $4x$ and $36x$

9. Find two numbers whose sum is 161 and whose ratio is $2 : 5$.

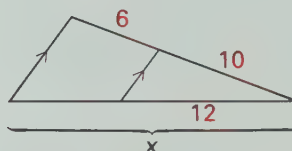
10. The measures of the three angles of a triangle are in the ratio $2 : 3 : 7$. Find the size of each angle.

11. Find x and y in the following figures.

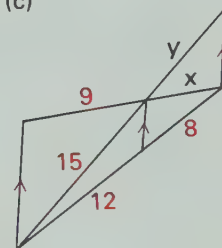
(a)



(b)



(c)

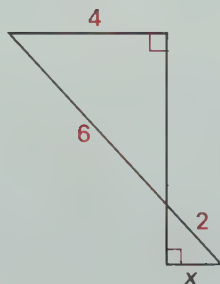


12. If P divides the segment XY internally in the ratio $8 : 5$ and $XY = 21$ cm, find the lengths of XP and PY .

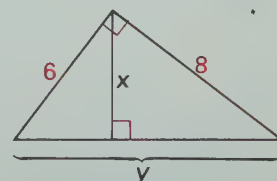
13. Draw any line segment and construct the point which divides it internally in the ratio $5 : 2$.

14. Find the values of x , y , z in the following figures.

(a)

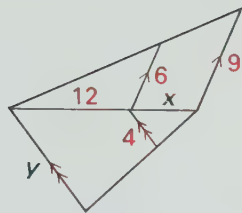


(b)

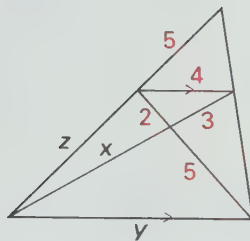


Find the sum of all the numbers that can be formed using the digits 1 to 9 once and only once.

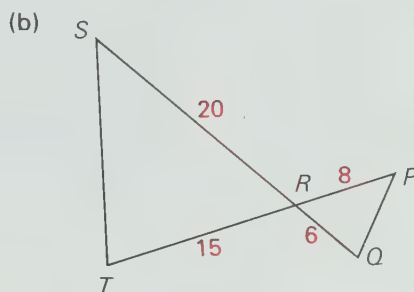
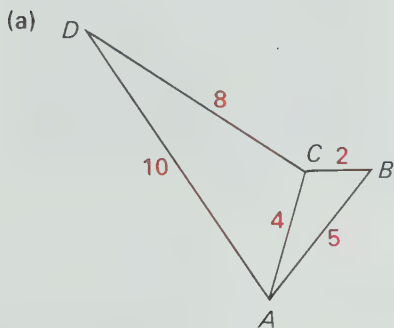
(c)



(d)



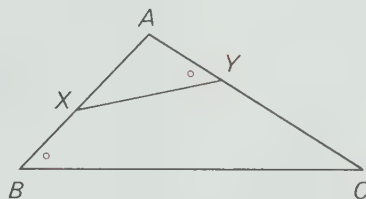
15. State whether or not the following pairs of triangles are similar. If the triangles are similar, state which theorem proves it.



16. Given: $\angle ABC = \angle AYX$

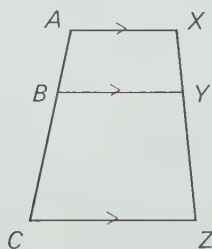
Required: To Prove (a) $\triangle AYX \sim \triangle ABC$

(b) $AX \cdot BC = AC \cdot XY$



17. Given: $AX \parallel BY \parallel CZ$.

Required: To prove $\frac{AB}{BC} = \frac{XY}{YZ}$



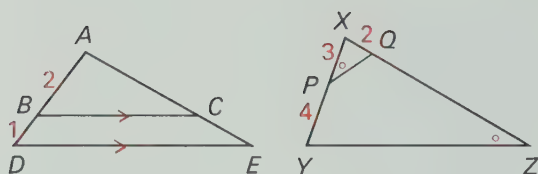
18. In $\triangle KLM$, LX and MY are altitudes intersecting at Z . Prove that (a)

$$\frac{KL}{LX} = \frac{KM}{MY}, \text{ and (b) } \frac{LZ}{MZ} = \frac{LY}{MX}$$

19. P is any point on the diagonal BD of a quadrilateral $ABCD$. A line through P parallel to AD meets AB at Q . A line through P parallel to CD meets BC at R . Prove that $QR \parallel AC$.

20. $ABCD$ is a parallelogram. A line through D intersects BC at X and AB produced at Y . Prove that $\frac{BX}{BY} = \frac{CX}{AB}$.

21. $ABCD$ is a rhombus. A line through C intersects AD produced at P and AB produced at Q . If $DP = \frac{1}{2}AB$, prove that $BQ = 2AB$.

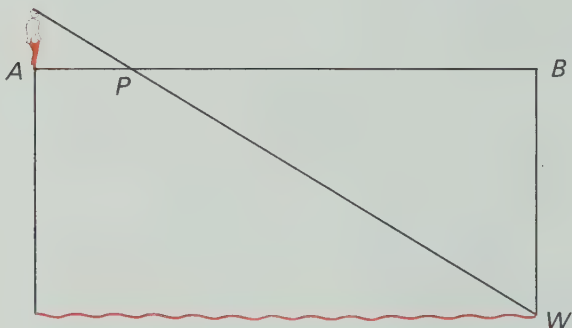


22. Find the ratios

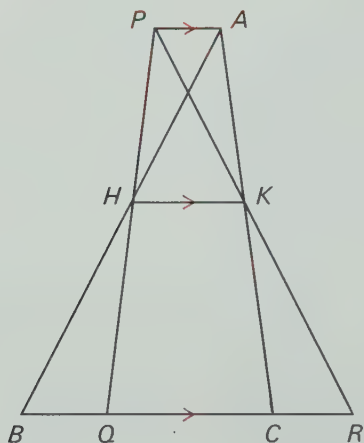
(a) $\frac{\triangle ABC}{\triangle ADE}$,

(b) $\frac{\triangle XPQ}{\triangle XYZ}$.

23. A potter has been making $\frac{1}{2}$ L jugs, and wants to make a 2 L jug in the same shape. By what factor will he have to multiply all the dimensions of the smaller jug?



24. A boy, whose eye-level is 1.5 m above the ground, stands at one end of a bridge and notices that point P on the bridge is in line with the water-level W at the other end of the bridge. He measures and finds that $AP = 4$ m and $PB = 30$ m. How high is the bridge above the water?



25. Given: $AP \parallel HK \parallel BC$

Required: To prove $BQ = CR$.

26. Find the points of division that would divide the line segment with end points $(-4, -6)$ and $(8, 10)$ into 4 equal parts.

27. If P divides AB internally in the ratio of $4 : 5$ and $AB = 54$ cm, find the length of AP and PB .

28. If Q divides CD externally in the ratio of $7 : 3$ and $CD = 20$ cm, find the lengths of CQ and DQ .

29. Determine the point $P(x, y)$ that divides the line segment joining $A(-2, 2)$ to $B(6, 4)$ internally in the ratio of $3 : 2$.

30. Determine the point $P(x, y)$ that divides the line segment joining $A(-4, 4)$ to $B(4, -2)$ externally in the ratio of $2 : 3$.

REVIEW AND PREVIEW TO CHAPTER 9

EXERCISE 1

MATRICES

Perform the indicated operations.

1. $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 6 \end{pmatrix}$
2. $\begin{pmatrix} 3 & 6 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 7 & -7 \\ 2 & -2 \end{pmatrix}$
3. $\begin{pmatrix} 8 & -6 \\ -2 & 4 \end{pmatrix} - \begin{pmatrix} 4 & 2 \\ -4 & 4 \end{pmatrix}$
4. $\begin{pmatrix} -2.5 & 0 \\ 1 & 6.3 \end{pmatrix} + \begin{pmatrix} 8.6 & 2.8 \\ -1.2 & 1.9 \end{pmatrix}$
5. $\begin{pmatrix} 1 & 7 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix}$
6. $\begin{pmatrix} -3 & 4 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \end{pmatrix} + \begin{pmatrix} 5 \\ -3 \end{pmatrix}$
7. $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 6 & 0 \end{pmatrix}$
8. $\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$
9. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$
10. $\begin{pmatrix} 6 & -1 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & -4 \end{pmatrix}$
11. $\begin{pmatrix} 1 & 0 \\ 8 & 2 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 3 & 5 \end{pmatrix}$
12. $\begin{pmatrix} 4 & -3 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 1 & -6 \\ 3 & -4 \end{pmatrix}$

EXERCISE 2

MATRIX EQUATIONS

Solve for x , y , z , t .

1. $\begin{pmatrix} x & y \\ 2 & 5 \end{pmatrix} + \begin{pmatrix} 3 & -6 \\ 1 & z \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ t & 3 \end{pmatrix}$
2. $\begin{pmatrix} x & 2y \\ 2z & t \end{pmatrix} + \begin{pmatrix} y & 3x \\ t & z \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 6 & 5 \end{pmatrix}$
3. $\begin{pmatrix} 3x & x \\ z & -t \end{pmatrix} + \begin{pmatrix} 2y & 5 \\ 3 & z \end{pmatrix} = \begin{pmatrix} 0 & y \\ 8t & 4 \end{pmatrix}$
4. $\begin{pmatrix} x & y \\ z & x \end{pmatrix} + \begin{pmatrix} y & z \\ t & z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 6 & 3 \end{pmatrix}$
5. $\begin{pmatrix} x & y \\ t & z \end{pmatrix} + \begin{pmatrix} y & z \\ y & t \end{pmatrix} = \begin{pmatrix} z & 10 \\ x & y \end{pmatrix}$
6. $\begin{pmatrix} x & 2 \\ 3 & y \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} z & 5 \\ 5 & t \end{pmatrix}$
7. $\begin{pmatrix} 5 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ 3 & -6 \end{pmatrix} = \begin{pmatrix} 1 & 8 \\ z & t \end{pmatrix}$
8. $\begin{pmatrix} x & y \\ y & x \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$
9. $\begin{pmatrix} 4 & y \\ 3 & y \end{pmatrix} \begin{pmatrix} x \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ 7 \end{pmatrix}$

You have 12 identical coins, one of which is counterfeit. The counterfeit coin is either lighter or heavier than the rest. Using a simple balance 3 times, find the counterfeit coin.



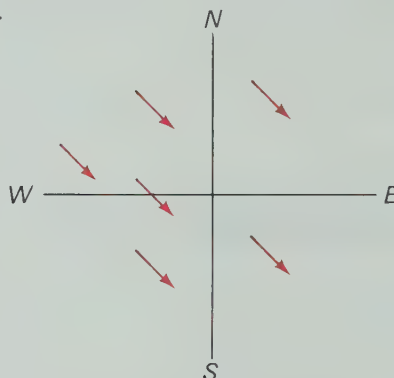
Vectors and Transformations

A mathematician, like a painter or a poet, is a maker of patterns.

G. H. Hardy

9.1 VECTORS

Physical quantities such as velocity or force are described by giving both their magnitude and their direction. For instance if a wind is blowing across a large field with a constant velocity of 50 km/h from the north-west, we might represent it by a number of parallel arrows of the same length. Each arrow is a directed line segment pointing in the same direction.



A **vector** is a mathematical object which has both magnitude and direction. It can be represented geometrically by a **directed line segment**, i.e., a line segment pointing in a specified direction, or by a set of directed line segments all of which have the same length and point in the same direction.

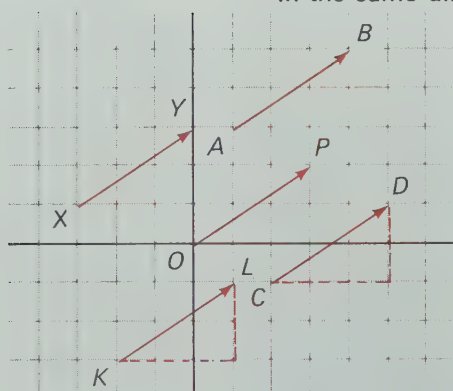


Figure 9-1

We denote a vector by an arrow on a letter, such as $\vec{v}$, or by an arrow on a line segment, such as $\overrightarrow{AB}$. The equation $\vec{v} = \overrightarrow{AB}$ indicates that the vector $\vec{v}$ is represented by the directed line segment from A to B. A is the **initial point** and B is the **terminal point**. Figure 9-1 shows several parallel line segments which are equivalent to $\overrightarrow{AB}$ in the sense that they have the same length and direction as $\overrightarrow{AB}$. All of them represent the same vector $\vec{v}$ and so we can also write $\vec{v} = \overrightarrow{KL}$ or $\vec{v} = \overrightarrow{CD}$.

All of the directed line segments in Figure 9-1 have the common property that the terminal point is reached from the initial point by a displacement of 3 units to the right and 2 units upward. We indicate this by writing $\vec{v} = [3, 2]$.

In general, if a and b are any real numbers, then $\vec{v} = [a, b]$ defines a vector which is represented geometrically by a displacement of a horizontal units and b vertical units.

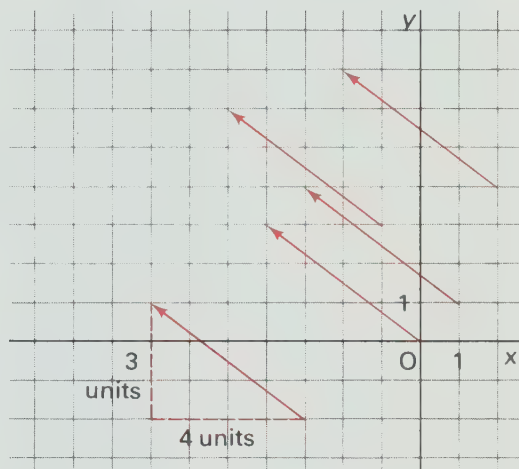
EXAMPLE 1. Represent the vector $\vec{v} = [-4, 3]$ geometrically by line segments with the following initial points.

- (a) $(1, 1)$ (b) $(-3, -2)$ (c) $(0, 0)$ (d) $(-1, 3)$ (e) $(2, 4)$

Solution

$\vec{v} = [-4, 3]$ represents a displacement of 4 units to the left and 3 units upward. If the initial point is (x, y) then the terminal point must be $(x - 4, y + 3)$.
 $\therefore$ the terminal points are

- (a) $(-3, 4)$
 (b) $(-7, 1)$
 (c) $(-4, 3)$
 (d) $(-5, 6)$
 (e) $(-2, 7)$



The **magnitude** $|\vec{v}|$ of a vector $\vec{v}$ is the length of any line segment which represents it. If $\vec{v} = \overline{AB}$, then $|\vec{v}|$ is the length of AB . In Example 1 we could use the Pythagorean Theorem to compute the magnitude of $\vec{v} = [-4, 3]$.

$$\begin{aligned} |\vec{v}|^2 &= (-4)^2 + 3^2 \\ &= 16 + 9 = 25 \\ |\vec{v}| &= 5 \end{aligned}$$

The magnitude of the vector $\vec{v} = [a, b]$ is

$$|\vec{v}| = \sqrt{a^2 + b^2}.$$

There are two ways of determining whether vectors are equal.
geometrically

If $\vec{u} = \overline{AB}$ and $\vec{v} = \overline{CD}$,

then $\vec{u} = \vec{v}$ if and only if

(i) $|\vec{u}| = |\vec{v}|$, i.e., $AB = CD$

and (ii) $AB \parallel CD$ and have the same direction.

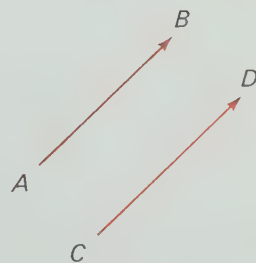
algebraically

If $\vec{u} = [a, b]$ and $\vec{v} = [c, d]$,

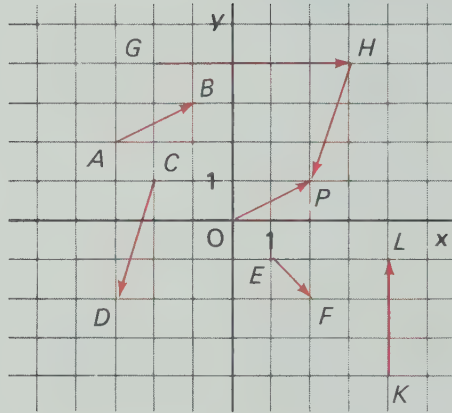
then $\vec{u} = \vec{v}$ if and only if

(i) $a = b$ and

(ii) $c = d$.



EXERCISE 9-1



- A** 1. (a) Express each of the following vectors in algebraic form $[a, b]$.

$\overrightarrow{AB}$, $\overrightarrow{CD}$, $\overrightarrow{EF}$, $\overrightarrow{GH}$, $\overrightarrow{OP}$, $\overrightarrow{HP}$, $\overrightarrow{KL}$.

- (b) Which of the vectors in (a) are equal?

2. Find the terminal points of the directed line segments which represent the vector $[5, 2]$ if the initial points are the following.

(a) $(0, 0)$ (b) $(2, 4)$ (c) $(-1, 6)$ (d) $(7, -2)$ (e) (x, y)

3. In each of the following cases name in algebraic form $[a, b]$ the vector $\vec{v} = \overrightarrow{AB}$.

(a) $A(1, 1)$ $B(6, 7)$

(b) $A(1, 1)$ $B(1, -3)$

(c) $A(-2, 5)$ $B(3, 6)$

(d) $A(1, -6)$ $B(0, 0)$

(e) $A(0, 0)$ $B(9, -8)$

(f) $A(-1, -1)$ $B(-8, 3)$

4. State the values of a , b , c , d , e , f if

(i) $[3, 8] = [a, b]$

(ii) $[c, -1] = [2, d]$

(iii) $[4, 3e] = [2f, 6]$.

- B** 5. Draw three geometric vectors (i.e., directed line segments) to represent each of the following vectors.

(a) $[5, 7]$

(b) $[-1, 1]$

(c) $[2, -3]$

(d) $[-6, -4]$

(e) $[3, 4.5]$

6. (a) Plot the points $A(3, 3)$, $B(6, -1)$, $C(-4, 4)$, $D(-2, -3)$.

(b) Draw the following vectors and also express each of them in algebraic form.

(i) $\overrightarrow{AB}$

(ii) $\overrightarrow{BC}$

(iii) $\overrightarrow{CD}$

(iv) $\overrightarrow{DA}$

(v) $\overrightarrow{AC}$

(vi) $\overrightarrow{BA}$

(vii) $\overrightarrow{AD}$

(viii) $\overrightarrow{DB}$

(c) Find the magnitudes of each of the vectors in (b).

7. Given the points $A(-2, -1)$, $B(5, 1)$, $C(0, 3)$.

(a) Find the coordinates of the point D such that $\overrightarrow{AB} = \overrightarrow{CD}$.

(b) Find the coordinates of the point E such that $\overrightarrow{AC} = \overrightarrow{BE}$.

(c) Find the coordinates of the point F such that $\overrightarrow{AF} = \overrightarrow{CB}$.

8. Find the magnitudes of the following vectors, given the points $A(1, 6)$, $B(7, 14)$, $C(-1, 10)$.

(a) $\vec{u} = [5, -12]$

(b) $\vec{v} = [-2, 8]$

(c) $\overrightarrow{AB}$

(d) $\overrightarrow{CA}$

(e) $\vec{w} = [-3, -6]$

(f) $\vec{t} = [6, 2.5]$

(g) $\overrightarrow{BC}$

(h) $\overrightarrow{AC}$

9. Express the following as vectors in algebraic form $[a, b]$.

(a) 260 km west

(b) 80 km south

(c) 100 km northeast

(d) 60 km southeast

(e) 400 km in a direction 30° west of north.

Solve

$$6751x + 3249y = 26\,751$$

$$3249x + 6751y = 23\,249$$

9.2 ADDITION AND SUBTRACTION OF VECTORS

Suppose a man walks from A to B along the vector $\overrightarrow{AB}$ and then walks from B to C along $\overrightarrow{BC}$. He could also have walked directly from A to C along the vector $\overrightarrow{AC}$. Therefore it seems reasonable to *define* the *sum* (or *resultant*) of these vectors as $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$. This definition of vector addition is called the Triangle Law. Notice that the initial point of the second vector coincides with the terminal point of the first vector.

The Triangle Law

If $\vec{u} = \overrightarrow{AB}$ and $\vec{v} = \overrightarrow{BC}$, then $\vec{u} + \vec{v}$ is represented by the third side of $\triangle ABC$: $\vec{u} + \vec{v} = \overrightarrow{AC}$.

In short, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

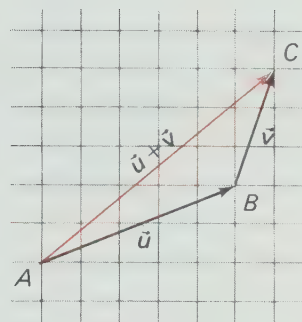
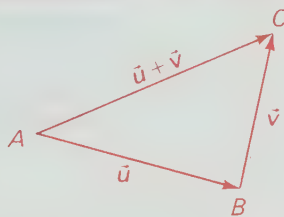


Figure 9-2

The Triangle Law permits us to add vectors geometrically. But vectors can also be represented in algebraic form. In Figure 9-2, $\vec{u} = [5, 2]$ and $\vec{v} = [1, 3]$. We can also see from the figure that $\vec{u} + \vec{v} = [6, 5]$. Therefore

$$\begin{aligned}[5, 2] + [1, 3] &= [6, 5] \\ &= [5 + 1, 2 + 3].\end{aligned}$$

It appears that vectors in algebraic form can be added simply by adding the corresponding components. This is confirmed by examining Figure 9-3.

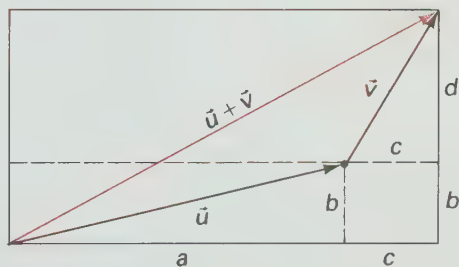


Figure 9-3

Algebraic Vector Addition

If $\vec{u} = [a, b]$ and $\vec{v} = [c, d]$,
then $\vec{u} + \vec{v} = [a + c, b + d]$.

In short, $[a, b] + [c, d] = [a + c, b + d]$.

In Figure 9-4 $ABCD$ is a parallelogram.

$$\therefore BC = AD$$

$$BC \parallel AD$$

Also BC and AD have the same direction.

$$\therefore \overrightarrow{BC} = \overrightarrow{AD}$$

By the Triangle Law, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

$$\therefore \overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$$

This proves the Parallelogram Law which gives a second geometric method for adding vectors.

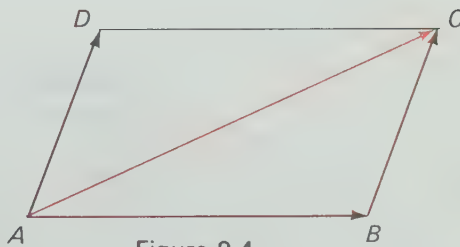
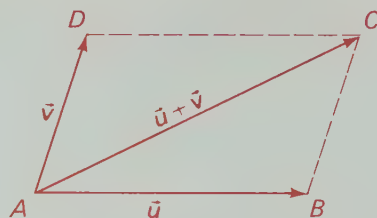


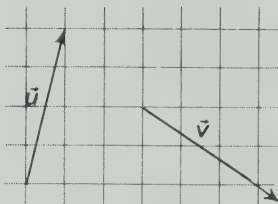
Figure 9-4

The Parallelogram Law

If $\vec{u} = \overrightarrow{AB}$ and $\vec{v} = \overrightarrow{AD}$, then $\vec{u} + \vec{v}$ is represented by the diagonal of the parallelogram determined by $\overrightarrow{AB}$ and $\overrightarrow{AD}$. In short, $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$.

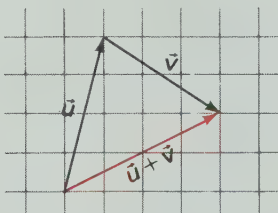


Notice that in applying the Parallelogram Law to add $\vec{u}$ and $\vec{v}$ we must arrange that $\vec{u}$ and $\vec{v}$ have the same initial point.



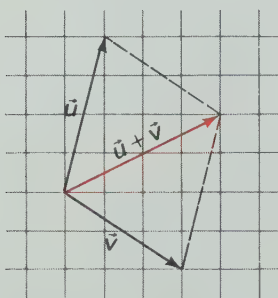
EXAMPLE 1. Find $\vec{u} + \vec{v}$ using

- the triangle law,
- the parallelogram law,
- algebraic vector addition.



Solution

(a) Move $\vec{v}$ parallel to itself so that its initial point coincides with the terminal point of $\vec{u}$. Then draw the resultant $\vec{u} + \vec{v}$ from the initial point of $\vec{u}$ to the terminal point of $\vec{v}$.



(b) Move $\vec{v}$ parallel to itself so that it has the same initial point as $\vec{u}$. Then complete the parallelogram and $\vec{u} + \vec{v}$ is the diagonal vector from the same initial point.

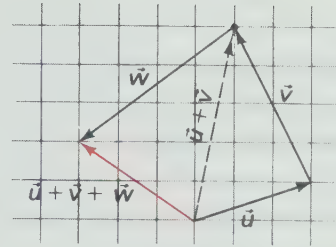
$$\begin{aligned} \text{(c)} \quad \vec{u} &= [1, 4], \vec{v} = [3, -2] \\ \vec{u} + \vec{v} &= [1, 4] + [3, -2] \\ &= [1+3, 4-2] \\ &= [4, 2] \end{aligned}$$

EXAMPLE 2. If $\vec{u} = [3, 1]$, $\vec{v} = [-2, 4]$, and $\vec{w} = [-4, -3]$, find $\vec{u} + \vec{v} + \vec{w}$ (a) algebraically and (b) geometrically. [It will be verified in Exercise 9-2 that $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ and so $\vec{u} + \vec{v} + \vec{w}$ is well-defined.]

Solution

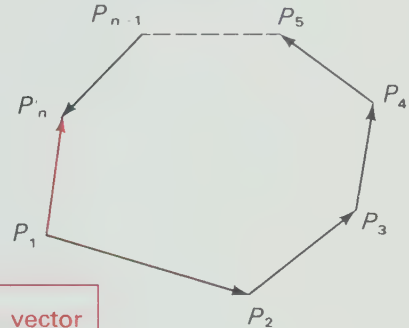
$$\begin{aligned} \text{(a)} \quad \vec{u} + \vec{v} + \vec{w} &= [3, 1] + [-2, 4] + [-4, -3] \\ &= [3-2-4, 1+4-3] \\ &= [-3, 2] \end{aligned}$$

(b) Arrange $\vec{u}$, $\vec{v}$, $\vec{w}$ so that the initial point of $\vec{v}$ coincides with the terminal point of $\vec{u}$ and the initial point of $\vec{w}$ coincides with the terminal point of $\vec{v}$. Apply the Triangle Law to construct $\vec{u} + \vec{v}$. Then apply it a second time to construct $(\vec{u} + \vec{v}) + \vec{w}$.



Generally we can add any number of vectors as in Example 2 by applying the triangle law repeatedly. Arrange the vectors joining the initial point of one vector to the terminal point of another. Then the resultant is represented by the remaining side of the polygon:

$$\overrightarrow{P_1P_2} + \overrightarrow{P_2P_3} + \overrightarrow{P_3P_4} + \cdots + \overrightarrow{P_{n-1}P_n} = \overrightarrow{P_1P_n}$$



The *opposite* (or *negative*) of the vector $\vec{v} = [a, b]$ is the vector $-\vec{v} = [-a, -b]$.

Notice that $-\vec{v}$ has the same magnitude as $\vec{v}$ but points in the opposite direction.

Also

$$\begin{aligned}\vec{v} + (-\vec{v}) &= [a, b] + [-a, -b] \\ &= [a - a, b - b] \\ &= [0, 0].\end{aligned}$$

This vector $[0, 0]$ is a special vector which we call the *zero vector* and denote by $\vec{0}$. It has magnitude $|\vec{0}| = 0$ but is not considered to have a direction.

$$\vec{0} = [0, 0]$$

Thus we can write $\vec{v} + (-\vec{v}) = \vec{0}$.

Definition of Subtraction of Vectors

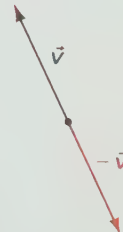
$$\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$$

If $\vec{u} = [a, b]$ and $\vec{v} = [c, d]$, then

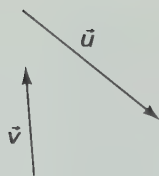
$$\begin{aligned}\vec{u} - \vec{v} &= \vec{u} + (-\vec{v}) \\ &= [a, b] + [-c, -d] \\ &= [a - c, b - d].\end{aligned}$$

Thus

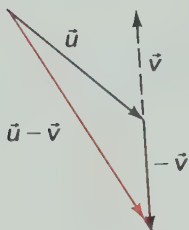
$$[a, b] - [c, d] = [a - c, b - d]$$



Compare with real numbers:
 $x - y = x + (-y)$
 if x, y are real numbers.



EXAMPLE 3. Find $\vec{u} - \vec{v}$ where $\vec{u}$ and $\vec{v}$ are as shown.



Solution:

Move $\vec{v}$ parallel to itself so that its initial point coincides with the terminal point of $\vec{u}$. Draw $(-\vec{v})$. Then the Triangle Law gives $\vec{u} + (-\vec{v})$ which is $\vec{u} - \vec{v}$.

EXERCISE 9-2

A 1. Find the following sums of vectors.

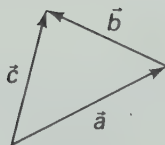
- | | |
|------------------------------|---------------------------|
| (a) $[5, 2] + [3, 4]$ | (b) $[-3, 6] + [4, 1]$ |
| (c) $[2, 4] + [-6, -1]$ | (d) $[16, 0] + [2, -9]$ |
| (e) $[1, -1] + [-7, 6]$ | (f) $[-2, -3] + [-1, -5]$ |
| (g) $[7.5, 8] + [1.5, -1.5]$ | (h) $[p, q] + [r, s]$ |

2. Find the following differences of vectors.

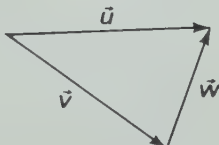
- | | |
|------------------------------|---------------------------|
| (a) $[6, 8] - [3, 8]$ | (b) $[1, 3] - [3, 7]$ |
| (c) $[-2, 9] - [6, -4]$ | (d) $[-5, 10] - [-5, 10]$ |
| (e) $[7.5, 8] - [1.5, -1.5]$ | (f) $[p, q] - [r, s]$ |

3. Express each vector as the sum of two other vectors.

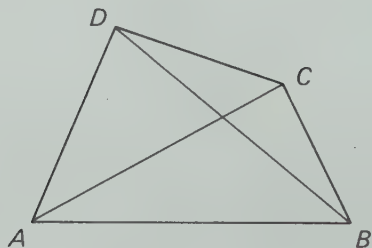
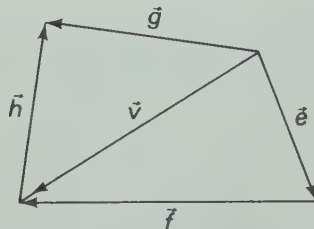
(a)



(b)



(c)

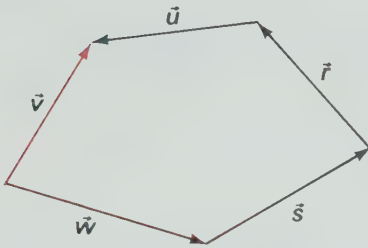


4. Name a single vector equal to

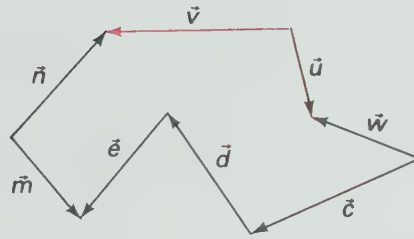
- | | |
|---|---|
| (a) $\overrightarrow{AB} + \overrightarrow{BC}$ | (b) $\overrightarrow{AB} + \overrightarrow{BD}$ |
| (c) $\overrightarrow{CD} + \overrightarrow{DA}$ | (d) $\overrightarrow{BC} + \overrightarrow{CD}$ |
| (e) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$ | |

5. Express $\vec{v}$ as the sum of other vectors.

(a)



(b)



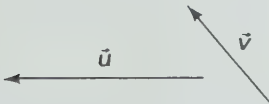
B 6. Using a suitable scale, find the sum of the following pairs of vectors geometrically.

- 7 km south followed by 10 km west
- 12 km east followed by 8 km north
- 6 km south followed by 15 km southeast
- 14 km north, 20 km west, 8 km southwest

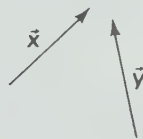
7. Redraw the following pairs of vectors and find their sum using

- the triangle law,
- the parallelogram law.

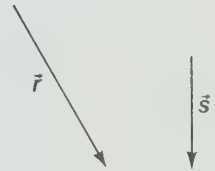
(a)



(b)

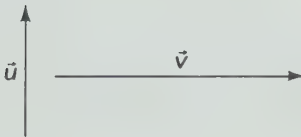


(c)



8. Redraw the following pairs of vectors and find their difference $\vec{u} - \vec{v}$ geometrically.

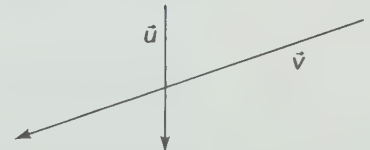
(a)



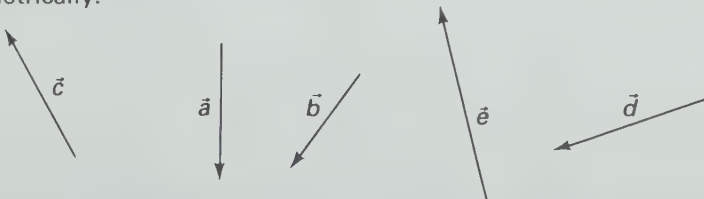
(b)



(c)



9. Redraw the following vectors and find the sum $\vec{a} + \vec{b} + \vec{c} + \vec{d} + \vec{e}$ geometrically.



10. Find the following sums and differences of vectors (i) algebraically, and (ii) geometrically.

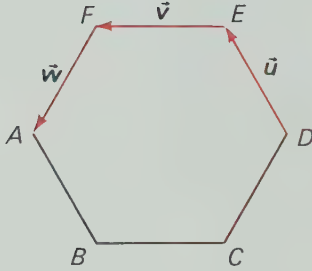
- | | |
|------------------------------|-------------------------|
| (a) $[6, 4] + [3, 0]$ | (b) $[8, 7] - [2, 5]$ |
| (c) $[-2, -3] + [5, 9]$ | (d) $[-3, 8] + [6, -1]$ |
| (e) $[2, -5.5] + [1.5, 2.5]$ | (f) $[12, 6] - [7, -2]$ |

- (g) $[2, 2] + [5, 3] + [1, -6]$
 (h) $[-3, 3] + [4, 2] + [5, -4] - [-6, 2]$

11. Given the vectors $\vec{u} = [1, 5]$, $\vec{v} = [-3, 4]$, and $\vec{w} = [2, -2]$.

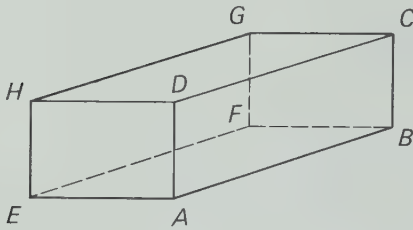
- (a) Find $\vec{u} + \vec{v}$, $\vec{v} + \vec{w}$, and $\vec{u} + \vec{w}$ algebraically and geometrically.
 (b) Find $|\vec{u}|$, $|\vec{v}|$, $|\vec{w}|$, $|\vec{u} + \vec{v}|$, $|\vec{v} + \vec{w}|$, $|\vec{u} + \vec{w}|$.

12. Give a geometric argument to show that $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$ for any two vectors $\vec{u}$, $\vec{v}$. When does the equal sign hold?



13. $ABCDEF$ is a regular hexagon. Express the following vectors in terms of $\vec{u}$, $\vec{v}$, $\vec{w}$.

- (a) $\vec{AB}$ (b) $\vec{DC}$ (c) $\vec{AC}$
 (d) $\vec{FB}$ (e) $\vec{EA}$ (f) $\vec{DA}$
 (g) $\vec{BE}$ (h) $\vec{DB}$



14. A rectangular box has parallel faces $ABCD$ and $EFGH$. Find single vectors equal to the following sums.

- (a) $\vec{AB} + \vec{BC}$ (b) $\vec{GF} + \vec{FB}$ (c) $\vec{AD} + \vec{AB}$
 (d) $\vec{HD} + \vec{AB}$ (e) $\vec{EC} + \vec{CB}$ (f) $\vec{AB} + \vec{AD} + \vec{AE}$
 (g) $\vec{HE} + \vec{AF}$ (h) $\vec{FD} + \vec{AE}$ (i) $\vec{HE} + \vec{AG} + \vec{FC}$

C 15. Prove both geometrically and algebraically that the commutative law holds for vector addition, i.e., $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ for any two vectors $\vec{u}$, $\vec{v}$.

16. Prove both algebraically and geometrically that the associative law holds for vector addition, i.e., $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ for any three vectors $\vec{u}$, $\vec{v}$, $\vec{w}$.

17. If $\vec{u} = [a, b]$ and $\vec{v} = [c, d]$, prove algebraically that $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$.

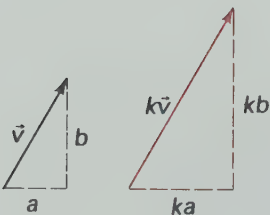
9.3 MULTIPLICATION OF A VECTOR BY A SCALAR

When we work with vectors we call real numbers *scalars* in order to distinguish them from vectors. If $\vec{v}$ is a vector and k is a scalar, how shall we define the product $k\vec{v}$? If x is a real number, then $2x = x + x$, and so if $\vec{v}$ is a vector we write $2\vec{v} = \vec{v} + \vec{v}$. If $\vec{v} = [a, b]$, then

$$\begin{aligned} 2\vec{v} &= \vec{v} + \vec{v} = [a, b] + [a, b] \\ &= [2a, 2b]. \\ \therefore 2[a, b] &= [2a, 2b] \end{aligned}$$

In general for any real number k we define $k\vec{v}$ as follows.

If $\vec{v} = [a, b]$, then $k\vec{v} = [ka, kb]$. In short, $k[a, b] = [ka, kb]$.



Geometrically how does $k\vec{v}$ compare with $\vec{v}$? Its magnitude is

$$\begin{aligned} |k\vec{v}| &= \sqrt{(ka)^2 + (kb)^2} \\ &= \sqrt{k^2a^2 + k^2b^2} \\ &= \sqrt{k^2(a^2 + b^2)} \\ &= \sqrt{k^2} \sqrt{a^2 + b^2} \\ &= |k| |\vec{v}|. \end{aligned}$$

So the magnitude of $k\vec{v}$ is $|k|$ times the magnitude of $\vec{v}$. The slope of

$k\vec{v}$ is $\frac{kb}{ka} = \frac{b}{a}$ = slope of $\vec{v}$ ($a \neq 0$). Thus $k\vec{v}$ is parallel to $\vec{v}$.

If $k > 0$, then a and ka have the same sign. Also b and kb have the same sign. So if $k > 0$, then $k\vec{v}$ and $\vec{v}$ have the same direction.

If $k < 0$, then a and ka have opposite signs and similarly for b and kb . So if $k < 0$, then $k\vec{v}$ and $\vec{v}$ are in opposite directions. This is consistent with our definition of $-\vec{v}$. Note that $(-1)\vec{v} = -\vec{v}$.

If $k = 0$, then $k\vec{v} = [0 \cdot a, 0 \cdot b] = [0, 0] = \vec{0}$ which has no direction.

Put the numbers from 1 to 9 in the spaces to make the statements true.

$$\square \times \square + \square = 7$$

$$\square \div \square + \square = 7$$

$$\square + \square - \square = 7$$

(i) The magnitude of the vector $k\vec{v}$ is $|k|$ times the magnitude of $\vec{v}$, i.e., $|k\vec{v}| = |k| |\vec{v}|$.

(ii) If $k > 0$, $k\vec{v}$ has the same direction as $\vec{v}$. If $k < 0$, $k\vec{v}$ has the opposite direction to $\vec{v}$. If $k = 0$, $k\vec{v} = \vec{0}$ has no direction.

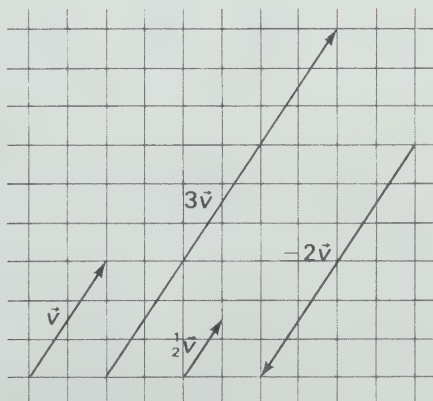
EXAMPLE 1. Given $\vec{v} = [2, 3]$, find (a) $3\vec{v}$, (b) $\frac{1}{2}\vec{v}$, (c) $-2\vec{v}$ and illustrate.

Solution

$$\begin{aligned} \text{(a) } 3\vec{v} &= 3[2, 3] \\ &= [3 \cdot 2, 3 \cdot 3] \\ &= [6, 9] \end{aligned}$$

$$\begin{aligned} \text{(b) } \frac{1}{2}\vec{v} &= \frac{1}{2}[2, 3] \\ &= [\frac{1}{2} \cdot 2, \frac{1}{2} \cdot 3] \\ &= [1, \frac{3}{2}] \end{aligned}$$

$$\begin{aligned} \text{(c) } -2\vec{v} &= (-2)[2, 3] \\ &= [-4, -6] \end{aligned}$$

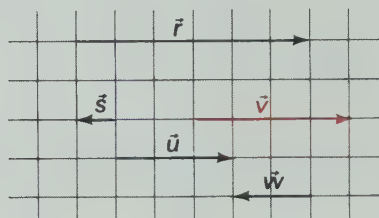


EXERCISE 9-3

A 1. Express each of the following vectors in the form $[a, b]$.

- | | | |
|---------------------------|---------------------------|-------------------------|
| (a) $4[6, -1]$ | (b) $5[2, 3]$ | (c) $6[-3, 8]$ |
| (d) $\frac{1}{2}[10, -6]$ | (e) $-3[5, -2]$ | (f) $\frac{2}{3}[3, 9]$ |
| (g) $0[-3, -8]$ | (h) $-\frac{1}{3}[-6, 0]$ | (i) $4[2.5, 4]$ |

2. Express the vectors $\vec{u}$, $\vec{w}$, $\vec{r}$, $\vec{s}$ in terms of $\vec{v}$.



- B** 3. Express each of the following vectors in the form $[a, b]$.

- (a) $6[2, 5] + 4[3, 7]$ (b) $9[-1, 8] - 5[2, 6]$
 (c) $\frac{3}{5}[5, -10] + \frac{1}{2}[9, 6]$ (d) $8[6.5, 8.5] - 1.6[5, 0]$
 (e) $2[-1, 7] + 3[8, -5] + \frac{1}{2}[16, -18]$
 (f) $\frac{2}{3}[18, 27] - \frac{1}{2}[16, 28] + \frac{3}{2}[6, -4]$

4. Prove the distributive laws for scalar multiplication of vectors,

- (a) $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$ (b) $(k + m)\vec{u} = k\vec{u} + m\vec{u}$,

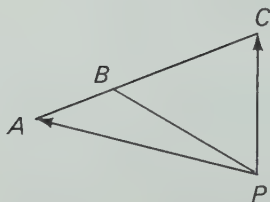
where k and m are scalars, by writing $\vec{u}$ and $\vec{v}$ in algebraic form, i.e., $\vec{u} = [a, b]$, $\vec{v} = [c, d]$.

5. Use the distributive laws proved in question 4 to express the following vectors in another form.

- (a) $6\vec{u} + 14\vec{u}$ (b) $7\vec{u} + 7\vec{v}$
 (c) $\frac{1}{2}(\vec{u} + \vec{v})$ (d) $3(2\vec{u} - \vec{v})$
 (e) $4(\vec{u} - 2\vec{v}) + 8\vec{v}$ (f) $\frac{1}{3}\vec{u} - \frac{2}{3}\vec{u}$

6. $ABCD$ is a parallelogram and P , Q are the midpoints of AB , DA . If $\vec{u} = \vec{BP}$ and $\vec{v} = \vec{AQ}$, express the following vectors in terms of $\vec{u}$ and $\vec{v}$.

- (a) $\vec{AD}$ (b) $\vec{PA}$ (c) $\vec{CD}$ (d) $\vec{PQ}$
 (e) $\vec{BD}$ (f) $\vec{PD}$ (g) $\vec{AC}$



7. Suppose that B divides the line segment AC internally in the ratio $1:2$, i.e., $\frac{AB}{BC} = \frac{1}{2}$. Express the following vectors in terms of $\vec{PA}$ and $\vec{PC}$.

- (a) $\vec{AC}$ (b) $\vec{AB}$ (c) $\vec{PB}$

- C** 8. Prove that $\vec{u} \parallel \vec{v}$ if and only if $\vec{u} = k\vec{v}$ for some scalar k .

9.4 GEOMETRY WITH VECTORS

Certain problems in Euclidean geometry can be conveniently solved using vector methods. This does not mean that vectors are useful in every geometrical problem, but if you try other methods to solve the examples and exercises in this section, you will see that in many cases vector methods give a shorter proof.

The two main facts about vectors which are needed in solving these problems are the following.

1. The triangle law for vector addition.
2. If $\vec{AB} = k\vec{CD}$, then $AB \parallel CD$ and $AB = kCD$.

EXAMPLE 1.

Given: AD is a median in $\triangle ABC$, i.e., $BD = DC$

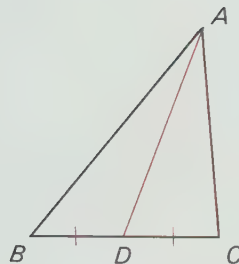
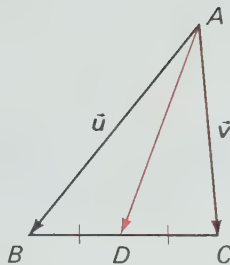
Required: To prove $\overrightarrow{AD} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$

Proof

Let $\vec{u} = \overrightarrow{AB}$, $\vec{v} = \overrightarrow{AC}$.

Then

$$\begin{aligned}\overrightarrow{AD} &= \overrightarrow{AB} + \overrightarrow{BD} \\ &= \vec{u} + \frac{1}{2}\overrightarrow{BC} \\ &= \vec{u} + \frac{1}{2}(-\vec{u} + \vec{v}) \\ &= \vec{u} - \frac{1}{2}\vec{u} + \frac{1}{2}\vec{v} \\ &= \frac{1}{2}\vec{u} + \frac{1}{2}\vec{v} \\ &= \frac{1}{2}(\vec{u} + \vec{v}) \\ &= \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})\end{aligned}$$



EXAMPLE 2. Prove that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and its length is one half the length of the third side.

Given: In $\triangle ABC$, $AD = DB$ and $AE = EC$.

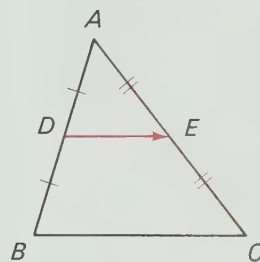
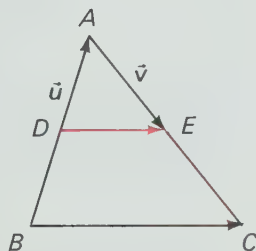
Required: To prove (a) $DE \parallel BC$

(b) $DE = \frac{1}{2}BC$

Proof

Let

$$\begin{aligned}\vec{u} &= \overrightarrow{DA}, \vec{v} = \overrightarrow{AE} \\ \overrightarrow{DE} &= \vec{u} + \vec{v} \\ &= \frac{1}{2}\overrightarrow{BA} + \frac{1}{2}\overrightarrow{AC} \\ &= \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{AC}) \\ \therefore \overrightarrow{DE} &= \frac{1}{2}\overrightarrow{BC} \\ \therefore DE &\parallel BC \text{ and } DE = \frac{1}{2}BC.\end{aligned}$$



EXERCISE 9-4

1. Prove that if two sides of a quadrilateral are equal and parallel, then the other two sides are also equal and parallel.

Given: Quadrilateral $ABCD$

$$AB = DC, AB \parallel DC$$

Required: To prove (a) $AD = BC$

(b) $AD \parallel BC$

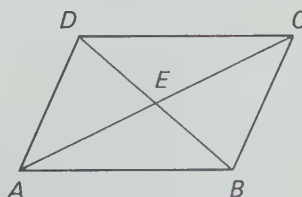
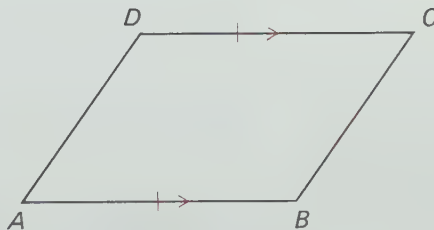
2. If the diagonals of a quadrilateral bisect each other, prove that the quadrilateral is a parallelogram.

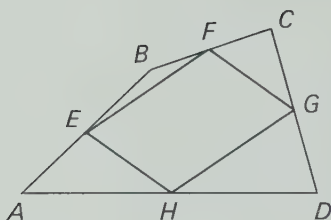
Given: Quadrilateral $ABCD$

AB and CD intersect at E .

$$AE = EC, BE = ED.$$

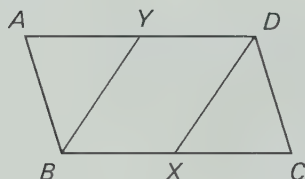
Required: To prove $ABCD$ is a parallelogram.





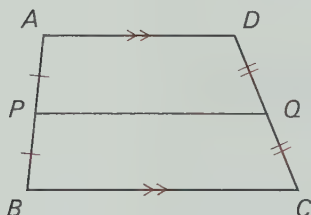
3. Given: Quadrilateral $ABCD$, and midpoints E, F, G, H of the four sides.

Required: To prove $EFGH$ is a parallelogram.
[Hint: Express $\overline{EH}$ and $\overline{FG}$ in terms of $\overline{BD}$.]



4. Given: In parallelogram $ABCD$, X and Y are midpoints of sides BC and AD .

Required: To prove $BY = XD$ and $BY \parallel XD$.



5. Given: Trapezoid $ABCD$ with $AD \parallel BC$. P, Q midpoints of AB, CD .

Required: To prove $PQ = \frac{1}{2}(AD + BC)$.

6. Prove that the diagonals of a parallelogram bisect each other. [Hint: Use Example 1.]

7. In $\triangle ABC$, D and E are points on AB and AC with $AD:DB = AE:EC = k:1$. Prove that $DE \parallel BC$ and $DE:BC = k:k+1$.

- C** 8. If $ABCD$ is a parallelogram and E is the midpoint of AD , prove that AC and BE trisect each other.

9.5 TRANSLATIONS

A **transformation** is a mapping whose domain and range are the same plane. If a transformation T maps a point P in the plane into another point P' , then P' is called the **image** of P and we write

$$T:P \rightarrow P' \quad \text{or} \quad T(P) = P'.$$

We shall be particularly interested in a type of transformation called a **rigid motion** or **isometry**. This is a transformation which preserves distances and angles.

Defining property of a rigid motion T

1. If $A \rightarrow A'$ and $B \rightarrow B'$, then $A'B' = AB$. (T preserves distances.)

What two whole numbers, neither containing any zeros, when multiplied together equal exactly 1 000 000 000?

The following properties of rigid motions are consequences of property 1.

2. If $A \rightarrow A'$, $B \rightarrow B'$, and $C \rightarrow C'$, then $\angle A'B'C' = \angle ABC$.

(T preserves angles.)

3. The image of a line is a line. (T preserves straightness.)

4. The images of parallel lines are parallel lines. (T preserves parallelism.)

5. The image of any figure has the same area as the original figure. (T preserves areas.)

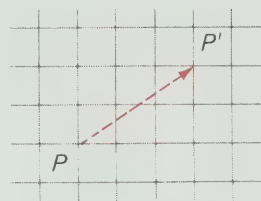
We now look at a specific kind of rigid motion called a **translation**. You can think of a translation as a transformation performed by sliding any object a certain distance in a certain direction. In other words a translation T is defined by a vector $\vec{v}$. To be specific, $T: P \rightarrow P'$ if $\overrightarrow{PP'} = \vec{v}$.

For example, if $\vec{v} = [3, 2]$, then the corresponding translation T displaces every point in the plane three units to the right and two units upward.

$$T: (x, y) \rightarrow (x+3, y+2).$$

Any vector $\vec{v}$ defines a translation $T: P \rightarrow P'$ where $\overrightarrow{PP'} = \vec{v}$. If $\vec{v} = [a, b]$, then

$$T: (x, y) \rightarrow (x+a, y+b).$$



EXAMPLE 1. Find the image of $\triangle ABC$ under the translation defined by the vector $\vec{v} = [6, -2]$ and show that the image triangle is congruent to $\triangle ABC$.

Solution

The vector $[6, -2]$ gives the translation

$$(x, y) \rightarrow (x+6, y-2).$$

$$\therefore A(-2, 4) \rightarrow A'(4, 2)$$

$$B(-4, -1) \rightarrow B'(2, -3)$$

$$C(1, 2) \rightarrow C'(7, 0)$$

$$AB = \sqrt{(-4+2)^2 + (-1-4)^2} \quad A'B' = \sqrt{(2-4)^2 + (-3-2)^2}$$

$$= \sqrt{4+25}$$

$$= \sqrt{4+25}$$

$$= \sqrt{29}$$

$$= \sqrt{29}$$

$$BC = \sqrt{(1+4)^2 + (2+1)^2}$$

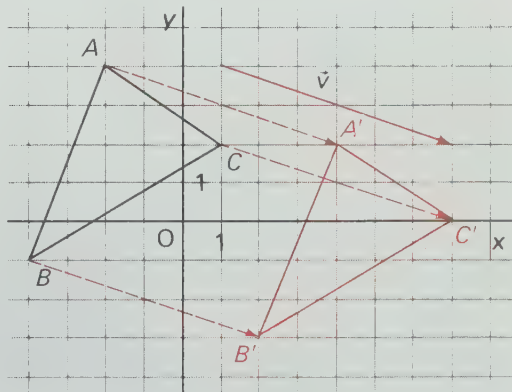
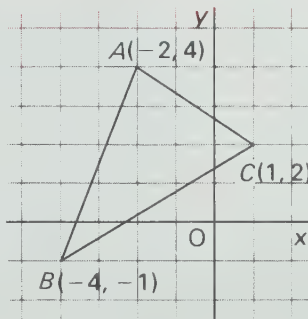
$$B'C' = \sqrt{(7-2)^2 + (0+3)^2}$$

$$= \sqrt{25+9}$$

$$= \sqrt{25+9}$$

$$= \sqrt{34}$$

$$= \sqrt{34}$$



$$AC = \sqrt{(1+2)^2 + (2-4)^2} \quad A'C' = \sqrt{(7-4)^2 + (0-2)^2}$$

$$= \sqrt{9+4} \quad = \sqrt{9+4}$$

$$= \sqrt{13} \quad = \sqrt{13}$$

$$\therefore AB = A'B', \quad BC = B'C', \quad AC = A'C'$$

$$\therefore \triangle ABC \cong \triangle A'B'C' \quad (\text{SSS})$$

WORD LADDER

Start with the word
"eye" and change one letter at a time to form a new word until you reach "lid".
The best solution has the fewest steps.

eye

lid

In Example 1 we saw that the translation maps line segments into equal line segments, and so it is a rigid motion. This is true of any translation, as will be seen in Exercise 9-5. Thus translations have the five properties of rigid motions listed above. They also enjoy two other properties not shared by all other rigid motions:

6. If $A \rightarrow A'$ and $B \rightarrow B'$, then $A'B' \parallel AB$. (A translation T maps any line into a parallel line.)

7. If $\triangle ABC \rightarrow \triangle A'B'C'$, then the sense A to B to C is the same as the sense A' to B' to C' . (T preserves sense.)

As an example of property 7 notice that in Example 1 the sense A to B to C is counterclockwise and the sense A' to B' to C' is also counterclockwise.

EXERCISE 9-5

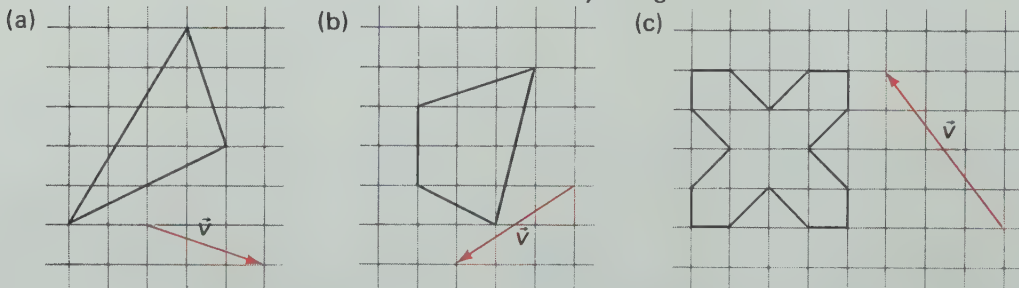
A 1. State the images of the following points under the translation $T: (x, y) \rightarrow (x+3, y-4)$.

- | | | | |
|------------|-------------|----------------------------------|--------------|
| (a) (1, 5) | (b) (2, -2) | (c) (-3, 4) | (d) (-2, -1) |
| (e) (0, 0) | (f) (2, 9) | (g) $(\frac{1}{2}, \frac{1}{2})$ | (h) (a, b) |

2. State the images of the point $(-2, 5)$ under the translations defined by the following vectors.

- | | | | |
|-------------|------------|-------------|--------------|
| (a) [3, 4] | (b) [6, 0] | (c) [9, -3] | (d) [-3, -6] |
| (e) [-6, 8] | (f) [9, 9] | (g) [5, -2] | (h) [a, b] |

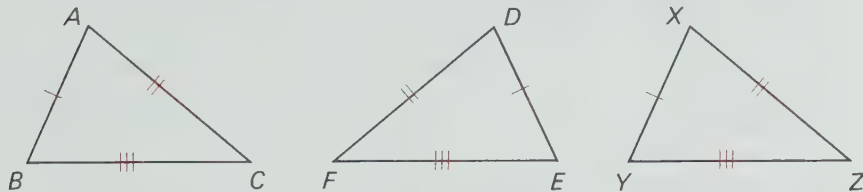
B 3. Redraw the following figures on graph paper and find their images under the translations defined by the given vectors $\vec{v}$.



4. Given the three points $A(-1, -1)$, $B(2, 3)$, $C(3, -2)$ and the translation $T: (x, y) \rightarrow (x-5, y+2)$.

- Find the images $A' = T(A)$, $B' = T(B)$, $C' = T(C)$.
- Graph $\triangle ABC$ and $\triangle A'B'C'$.
- Find the lengths AB , BC , CA , $A'B'$, $B'C'$, $C'A'$.
- Show that $\triangle ABC \cong \triangle A'B'C'$.
- Show that $AB \parallel A'B'$, $BC \parallel B'C'$, $CA \parallel C'A'$.

5. Repeat question 4 using the three points $A(0, 1)$, $B(2, 7)$, $C(5, 3)$ and the translation $T:(x, y) \rightarrow (x-3, y-2)$.
6. Is there a translation which maps $\triangle ABC \rightarrow \triangle DEF$?
Is there a translation which maps $\triangle ABC \rightarrow \triangle XYZ$?
If so, name a vector which defines the translation.



7. Prove that any translation is a rigid motion. [Hint: Let A' , B' be the images of $A(x_1, y_1)$, $B(x_2, y_2)$ under the translation $T:(x, y) \rightarrow (x+a, y+b)$ and show that $A'B' = AB$.]
8. If T is any translation and $T:A \rightarrow A'$, $B \rightarrow B'$, prove that $A'B' \parallel AB$.
9. Find the equation of the image of the line $3x+2y+1=0$ under the translation $(x, y) \rightarrow (x+4, y-1)$.

9.6 ROTATIONS AND COMPOSITION

In Figure 9-5 a point P has been rotated counterclockwise about the point O through an angle θ to a new position P' . This defines a transformation $R:P \rightarrow P'$ called a **rotation**. Notice that $OP = OP'$ since P and P' lie on the same circle with centre O . O is called the **centre of rotation** and θ is called the **angle of rotation**.

A rotation about a point O through an angle θ is a transformation $R:P \rightarrow P'$ such that

$$\angle POP' = \theta \quad \text{and} \quad OP = OP'.$$

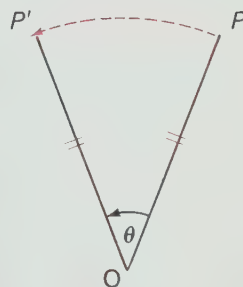


Figure 9-5

Figure 9-6 shows the effect of a rotation of a figure through an angle 70° counterclockwise. The same result could have been achieved by a clockwise rotation of 290° , in which case we say that the angle of rotation is -290° . The convention is that a **counterclockwise rotation is positive** and a **clockwise rotation is negative**.

If $\theta = 90^\circ$, then the rotation is called a **quarter-turn**, and if $\theta = 180^\circ$, the rotation is called a **half-turn**.

As will be seen in Exercise 9-6, every rotation is a rigid motion. In fact rotations enjoy all of the properties of translations that were listed in section 9-5 except for property 6. The image of a line segment under a rotation is a line segment which is not in general parallel to the original line segment.

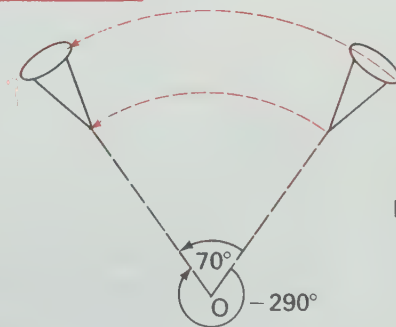


Figure 9-6

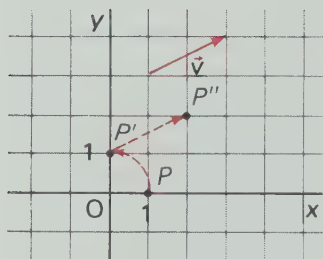


Figure 9-7

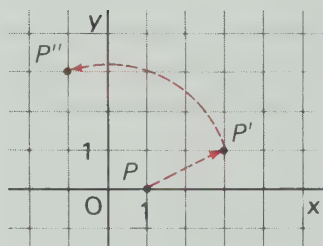


Figure 9-8

Now suppose that two transformations are applied successively. Figure 9-7 shows the result of first rotating a point P through 90° about the origin O and then translating the image point P' by means of the vector $\vec{v} = [2, 1]$ giving a new point P'' . The rotation is described by $R: P \rightarrow P'$ and the translation by $T: P' \rightarrow P''$. The new transformation obtained by first rotating and then translating is written as

$$T \circ R: P \rightarrow P''.$$

It is called the **composition** of the two transformations. Notice that in the transformation $T \circ R$, R is performed first and then T is performed.

Suppose that we now apply these transformations in the reverse order. Figure 9-8 shows the result of the transformation $R \circ T: P \rightarrow P''$. This means that we first translate and then rotate. Here P is translated to the point P' by the vector $\vec{v}$ and then P' is rotated 90° about O to give P'' . P'' has different locations in Figures 9-7 and 9-8, and so $T \circ R$ and $R \circ T$ are different transformations. Let us confirm this by finding the coordinates of P'' , the image of P , under $T \circ R$ and $R \circ T$.

$$T \circ R: P(1, 0) \xrightarrow{R} P'(0, 1) \xrightarrow{T} P''(2, 2)$$

$$R \circ T: P(1, 0) \xrightarrow{T} P'(3, 1) \xrightarrow{R} P''(-1, 3)$$

If T_1 and T_2 are any two transformations with $T_1: P \rightarrow P'$ and $T_2: P' \rightarrow P''$, then the transformation $T_2 \circ T_1$ obtained by applying first T_1 , then T_2 , is called the **composition** of T_1 and T_2 and maps $P \rightarrow P''$.

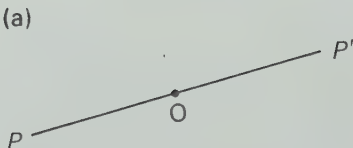
$$T_2 \circ T_1: P \rightarrow P''$$

$$\text{i.e., } (T_2 \circ T_1)(P) = T_2(T_1(P))$$

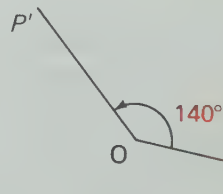
EXERCISE 9-6

- A** 1. In the following diagrams P' is the image of P under a rotation about O . In each case name two angles of rotation (counterclockwise and clockwise).

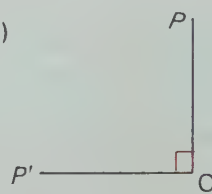
(a)



(b)

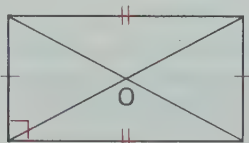


(c)

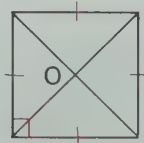


2. What angles of rotation will cause the following figures' to be mapped onto themselves if O is the centre of rotation?

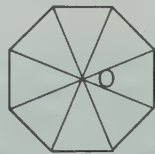
(a)

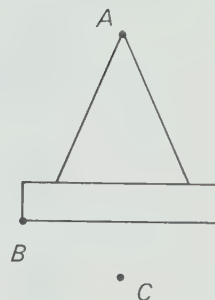
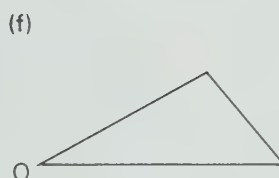
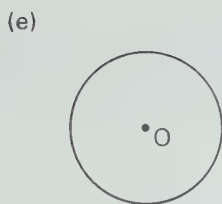
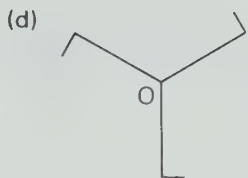


(b)



(c)





B 3. Copy the figure and draw its images under rotation through angles 90° , 180° , and 270°

(a) about A,

(b) about B,

(c) about C.

4. Construct $\triangle ABC$ so that $\angle A = 72^\circ$ and $AB = AC = 3$ cm. On the same diagram draw the images of $\triangle ABC$ under rotation about A through angles 72° , 144° , 216° , 288° .

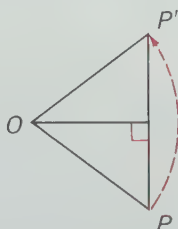
5. Complete the following table in your notebook by finding the image of each given point under rotation about the origin through the given angle.

	Angle of Rotation		
Point	90°	180°	270°
A(1, 0)			
B(3, 2)			
C(2, -5)			
D(-1, 4)			
P(x, y)			

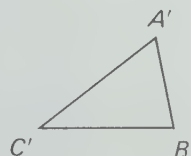
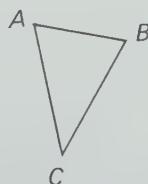
6. Find the images of the point $P(8, 10)$ under rotation about the point $A(5, 6)$ through angles 90° , 180° , and 270° .

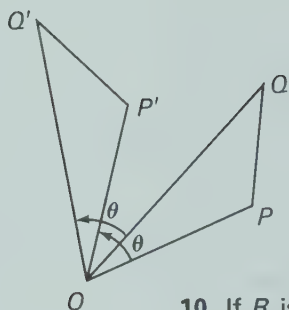
7. The square with vertices $O(0, 0)$, $P(1, 0)$, $Q(1, 1)$, $R(0, 1)$ is rotated through 45° about O . Find the coordinates of the images of the vertices of the square.

8. (a) Prove that if $R: P \rightarrow P'$ is a rotation about O , then O lies on the perpendicular bisector of PP' .



(b) If $\triangle ABC \rightarrow \triangle A'B'C'$ under a rotation about a point O , use part (a) to find O .





9. Prove that any rotation is a rigid motion.
 Given: $P \rightarrow P'$ and $Q \rightarrow Q'$ under a rotation through an angle θ about O .
 Required: To prove $PQ = P'Q'$.

10. If R is rotation through 60° about the origin and T is the translation defined by the vector $\vec{v} = [5, 4]$, draw the images of the triangle with vertices $O(0, 0)$, $P(4, 0)$, and $Q(3, 2)$ under the following transformations.

- (a) R (b) T (c) $T \circ R$ (d) $R \circ T$ (e) $R \circ R$
 (f) $T \circ T$

11. If R_θ is rotation about a point O through an angle whose degree measure is θ , what is

- (a) $R_{30} \circ R_{30}$ (b) $R_{45} \circ R_{80}$ (c) $R_\theta \circ R_\alpha$?

12. Given that R is rotation through 90° about the origin

T is the translation $(x, y) \rightarrow (x + 2, y - 5)$

S is the translation $(x, y) \rightarrow (x - 6, y)$,

complete the following table in your notebook by finding the coordinates of the images of the given points under the given transformations.

	$A(2, 2)$	$B(0, 5)$	$C(-1, 3)$	$P(x, y)$
R				
$T \circ R$				
$R \circ T$				
$S \circ T$				
$R \circ S$				

13. If T is the translation defined by a vector $\vec{u}$ and S is the translation defined by a vector $\vec{v}$, describe the transformation $S \circ T$.

9.7 REFLECTIONS AND SYMMETRY

Everyone is familiar with real-life examples of reflections. If you look in a mirror you see a reflection or image of yourself which appears to be located on the other side of the mirror at an equal distance from the mirror. It also appears as if an imaginary line joining an object to its image intersects the mirror at right angles. Thus it appears as if a mirror is the perpendicular bisector of the line joining an object to its image. We use this as the defining property of a reflection.

What English word contains each of the vowels a, e, i, o, u?

Reflection in a line m is the transformation that maps any point P into the point P' such that m is the perpendicular bisector of PP' .

The line m is called the **mirror line** because it acts like a mirror. Mirror lines will be indicated by double-ended arrows.

In Figure 9-9 $\triangle ABC$ has been reflected in the line m giving the image $\triangle A'B'C'$. It appears that $AB = A'B'$, $BC = B'C'$, and $AC = A'C'$, and this is readily verified. (See Exercise 9-7.) Thus $\triangle ABC \cong \triangle A'B'C'$ and the reflection is a rigid motion. However the reflection does not preserve sense. The sense A to B to C is clockwise but the sense A' to B' to C' is counterclockwise, and so the reflection reverses sense.

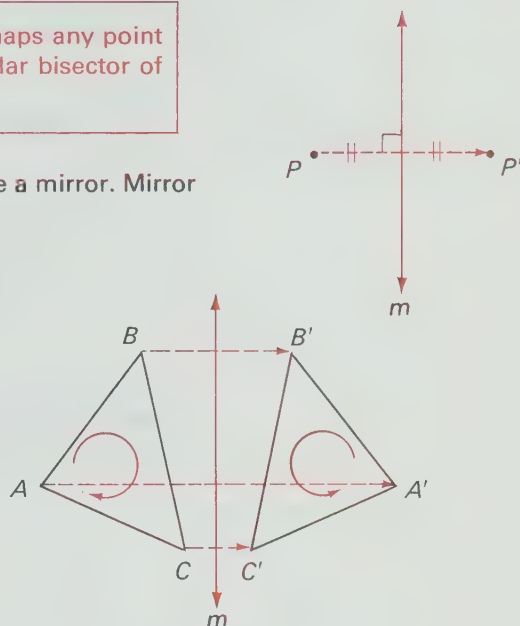


Figure 9-9

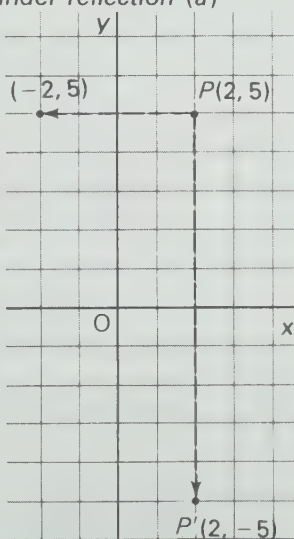
EXAMPLE 1. Find the image of the point $P(2, 5)$ under reflection (a) in the x -axis, (b) in the y -axis.

Solution

(a) The mirror line is the x -axis. If $P \rightarrow P'$ then the x -axis is the perpendicular bisector of PP' , and so P and P' are the same distance from the y -axis. Thus P' has the same x -coordinate as P , namely 2. Also P and P' must be the same distance from the x -axis but on opposite sides of it. Since the y -coordinate of P is 5, the y -coordinate of P' must be -5 .

$$\therefore P(2, 5) \rightarrow P'(2, -5)$$

(b) Similarly we can see that for reflection in the y -axis $P(2, 5) \rightarrow P'(-2, 5)$.



Reflection in the x -axis maps the point $P(x, y)$ into its mirror-image $P'(x, -y)$, i.e.,

$$(x, y) \rightarrow (x, -y).$$

Reflection in the y -axis maps $P(x, y)$ into $P'(-x, y)$, i.e.,

$$(x, y) \rightarrow (-x, y).$$

EXAMPLE 2. If F_1 is reflection in the x -axis and F_2 is reflection in the y -axis, show that $F_2 \circ F_1$ is rotation about the origin through 180° .

Solution

See Exercise 9-5, question 5.

If R is rotation through 180° about O , then

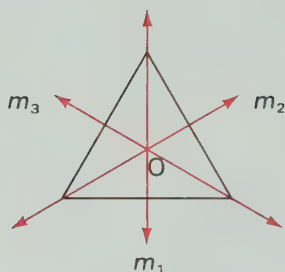
$$R:(x, y) \rightarrow (-x, -y).$$

But we also have

$$F_2 \circ F_1:(x, y) \xrightarrow{F_1} (x, -y) \xrightarrow{F_2} (-x, -y).$$

$$\therefore F_2 \circ F_1 = R$$

In general, as will be seen in Exercise 9-7, the composition of two reflections in two lines m_1 and m_2 is either a rotation (if m_1 and m_2 intersect) or a translation (if m_1 and m_2 are parallel).



Certain figures have the property that they are mapped onto themselves by a reflection or a rotation. These figures are called **symmetric**. For example an equilateral triangle is mapped onto itself by reflection in any median m_1 , m_2 , or m_3 , and we say the triangle is symmetric about any of its medians. The equilateral triangle is also mapped onto itself by a rotation of 120° or 240° or 360° about the point O where the medians intersect, and we say the triangle has rotational symmetry of order 3.

Reflectional Symmetry

A figure is called **symmetric about a line m** if the figure is mapped onto itself by reflection in m , and m is called a **line of symmetry** (or **axis of symmetry**).

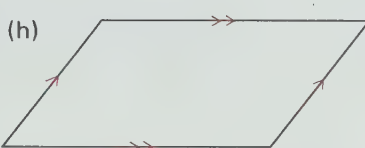
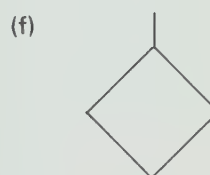
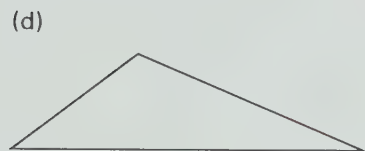
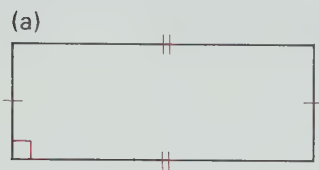
Rotational Symmetry

A figure is called **symmetric about a point O** if the figure is mapped onto itself by a rotation about O . This rotational symmetry is of **order n** if there are n rotations which map the figure onto itself.

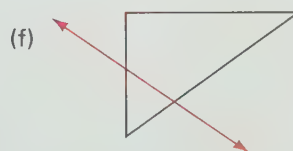
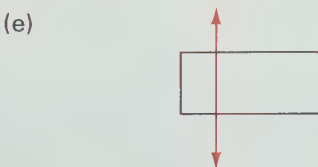
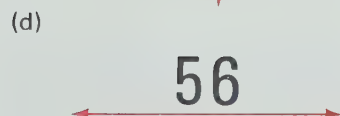
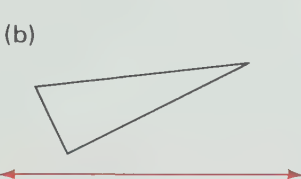
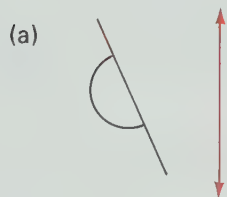
EXERCISE 9-7

- A**
- State the images of the following points after reflection in the x -axis.
 (a) $(3, 8)$ (b) $(6, -5)$ (c) $(4, 0)$ (d) $(-1, 1)$
 (e) $(0, 7)$ (f) $(-2, -3)$ (g) $(0, 0)$ (h) (a, b)
 - State the images of the above points after reflection in the y -axis.
 - (a) Which points are left unchanged by a reflection? (b) Which lines are left unchanged by a reflection?

4. For each of the following figures state the number of lines of symmetry and the order of rotational symmetry.



B 5. Copy the following figures and draw the reflection of each figure in the given mirror-line.



6. Given: Points $A(-2, 0)$, $B(4, 1)$, $C(2, 7)$

(a) Graph $\triangle ABC$.

(b) Find the images A' , B' , C' of A , B , C under reflection in the x -axis.

(c) Graph $\triangle A'B'C'$.

(d) Find the lengths AB , BC , CA , $A'B'$, $B'C'$, $C'A'$.

(e) Show that $\triangle ABC \cong \triangle A'B'C'$.

(f) Repeat parts (b), (c), (d) and (e) for reflection in the y -axis.

7. Prove that reflection in the x -axis is a rigid motion.

Given: Points $A(a, b)$ and $B(c, d)$ and their images A' , B' under reflection in the x -axis.

Required: To prove that $A'B' = AB$.

8. Find the image of the point $(7, 4)$ after reflection in the following lines.

(a) $x = 5$

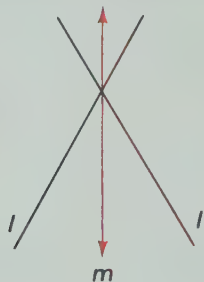
(b) $y = 5$

(c) $y = x$

(d) $y = -x$

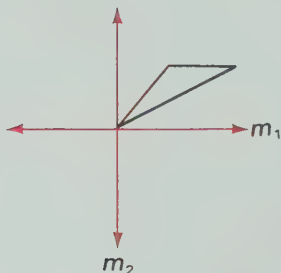
9. Show that the image of the point $(5, 2)$ after reflection in the line $2x + 3y - 3 = 0$ is $(1, -4)$.

10. Prove that if a line l is mapped into a line l' by reflection in a mirror-line m , then m bisects the angle between l and l' .

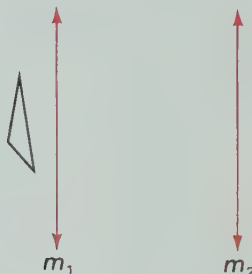


11. Copy the following figures. If F_1 is reflection in m_1 and F_2 is reflection in m_2 , draw the images of these figures under the transformations $F_2 \circ F_1$ and $F_1 \circ F_2$.

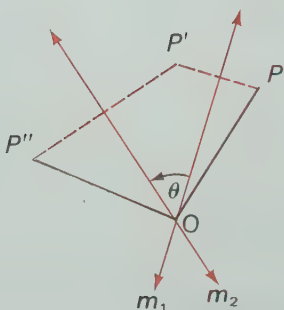
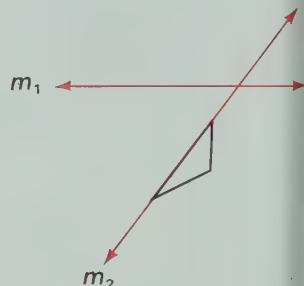
(a)



(b)



(c)



12. If m_1 and m_2 are lines which intersect in a point O , and F_1 is reflection in m_1 and F_2 is reflection in m_2 , show that $F_2 \circ F_1$ is a rotation about O through an angle which is twice the angle between m_1 and m_2 .

Given: $F_1: P \rightarrow P'$ and $F_2: P' \rightarrow P''$, θ is the angle between m_1 and m_2

Required: To prove (i) $OP = OP''$

(ii) $\angle POP'' = 2\theta$

13. If F is reflection in the x -axis,

G is reflection in the y -axis,

T is the translation $(x, y) \rightarrow (x+3, y-4)$,

R is rotation through 180° about the origin,

complete the following table in your notebook by finding the coordinates of the images of the given points under the given transformations.

	$A(1, 4)$	$B(-2, 3)$	$C(0, 7)$	$P(x, y)$
R				
$F \circ G$				
$F \circ T$				
$T \circ F$				
$R \circ T$				
$T \circ G$				

14. List the capital letters of the alphabet which are symmetric about
(a) a vertical line, (b) a horizontal line, (c) a point.

C 15. Prove that if F is reflection in the line $y = x$, then

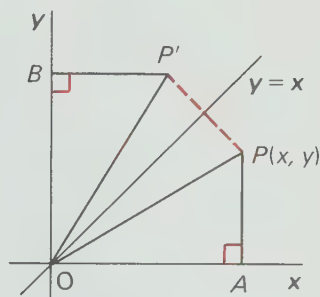
$$F: (x, y) \rightarrow (y, x).$$

Given: $F: P(x, y) \rightarrow P'$ i.e.,

$$OA = x, AP = y$$

Required: To prove $BP' = y, OB = x$

Hint: Prove $\triangle OAP \cong \triangle OBP'$



9.8 DILATATIONS

Glory is like a circle in the water,
Which never ceaseth to enlarge itself,
Till by broad spreading it disperse to nought.

William Shakespeare
(Henry VI, part I)

If you throw a stone into a body of water it creates a circle which grows larger. If a photograph is enlarged, everything in the picture is the same shape, but larger. If you wash clothes in hot water they may shrink to a slightly smaller size, but they have the same shape. These are all examples of dilatations. A dilatation is a transformation which changes the size of a figure but not its shape.

In Figure 9-10 $\triangle ABC$ has been enlarged to $\triangle A'B'C'$ by drawing $OA' = 2OA$, $OB' = 2OB$, and $OC' = 2OC$. In fact the image P' of any point P on $\triangle ABC$ is found by drawing $OP' = 2OP$. The transformation

$$D: P \rightarrow P'$$

defined by $OP' = 2OP$ is a dilatation with centre O and magnification factor 2.

In Figure 9-11, the transformation $D: P \rightarrow P'$ defined by $OP' = \frac{1}{2}OP$ is a dilatation which shrinks $\triangle ABC$ to $\triangle A'B'C'$ by a factor of $\frac{1}{2}$.

A dilatation $D: P \rightarrow P'$ is a transformation which maps any point P into a point P' such that $OP' = kOP$ and O , P , and P' are collinear. The fixed point O is called the *centre of the dilatation*, and the fixed positive number k is called the *factor* (or *magnification factor*, or *dilatation factor*).

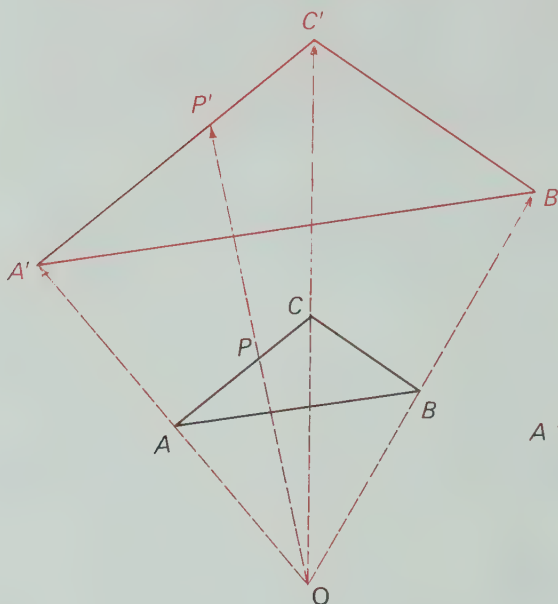


Figure 9-10

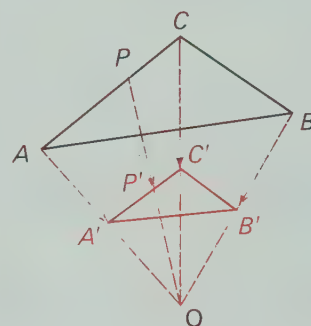


Figure 9-11

In Figures 9-10 and 9-11 it appears that the sides of $\triangle A'B'C'$ are parallel to the sides of $\triangle ABC$, and $\triangle A'B'C' \sim \triangle ABC$. We now prove this for any dilatation.

Properties of a Dilatation

For a dilatation with magnification factor $k > 0$,

- (a) the image of any line is a parallel line,
- (b) the image of any triangle is a similar triangle,
- (c) the length of any line segment is multiplied by k .

Given: Dilatation D with factor k which maps $\triangle ABC \rightarrow \triangle A'B'C'$

Required: To prove (a) $A'B' \parallel AB$, $B'C' \parallel BC$, $A'C' \parallel AC$

(b) $\triangle A'B'C' \sim \triangle ABC$

(c) $A'B' = kAB$

Proof: (a) Since D maps $A \rightarrow A'$ and $B \rightarrow B'$,

$$\therefore OA' = kOA \quad \text{and} \quad OB' = kOB$$

$$\therefore \frac{OA'}{OA} = k \quad \text{and} \quad \frac{OB'}{OB} = k$$

In $\triangle OA'B'$,

$$\frac{OA'}{OA} = \frac{OB'}{OB}$$

$\therefore AB \parallel A'B'$ (Converse of DST)

Similarly $BC \parallel B'C'$ and $AC \parallel A'C'$.

(b) $\therefore A'B' \parallel AB$
 $\therefore \angle OAB = \angle OA'B'$ (TPT)

$\therefore A'C' \parallel AC$

$\therefore \angle OAC = \angle OA'C'$ (TPT)

$$\angle OAC - \angle OAB = \angle OA'C' - \angle OA'B'$$

$$\angle CAB = \angle C'A'B'$$

Similarly

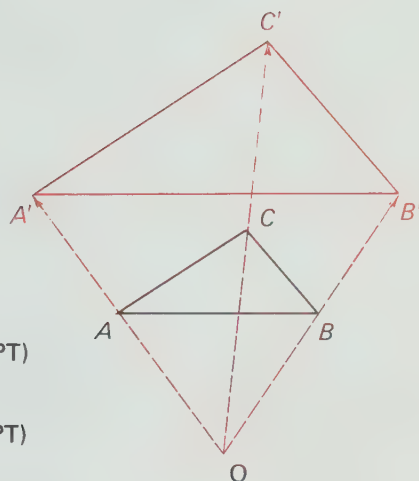
$$\angle ABC = \angle A'B'C'$$

$$\therefore \triangle ABC \sim \triangle A'B'C' \quad (\text{AA} \sim)$$

$$(c) \quad \triangle OAB \sim \triangle OA'B' \quad (\text{AA} \sim)$$

$$\therefore \frac{A'B'}{AB} = \frac{OA'}{OA} = k$$

$$\therefore A'B' = kAB$$



EXAMPLE 1. If $D: P(x, y) \rightarrow P'(x', y')$ is a dilatation with magnification factor k and centre the origin, find the coordinates of P' .

Solution

We draw a diagram to illustrate the case where $x > 0$ and $y > 0$, but similar reasoning gives the same answer for any values of x and y .

Draw the perpendiculars PQ , $P'Q'$ to the x -axis.

In $\triangle OP'Q'$ and $\triangle OPQ$,

$\angle O$ is common

$\angle PQO = \angle P'Q'O$ (right angles)

$\therefore \triangle OP'Q' \sim \triangle OPQ$ (AA~)

$$\therefore \frac{OQ'}{OQ} = \frac{P'Q'}{PQ} = \frac{OP'}{OP}$$

But $OP' = kOP$

$$\therefore \frac{OP'}{OP} = k$$

$$\frac{OQ'}{OQ} = \frac{P'Q'}{PQ} = k$$

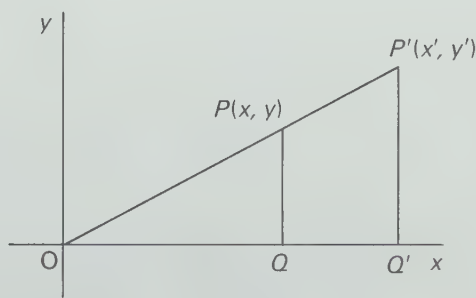
$$OQ' = kOQ$$

$$P'Q' = kPQ$$

$$\text{i.e., } x' = kx$$

$$y' = ky$$

The coordinates of P' are kx, ky .



If D is a dilatation with magnification factor k and centre $O(0, 0)$, then D is given by

$$D:(x, y) \rightarrow (kx, ky).$$

Originally we defined dilatations only for positive magnification factors k , but the expression

$$D:(x, y) \rightarrow (kx, ky)$$

makes sense for negative values of k also, and we use it to define general dilatations. We observe in the next example that if $k < 0$, then P and P' are on opposite sides of O , whereas if $k > 0$, then P and P' are on the same side of O . In general, lengths are multiplied by $|k|$ under a dilatation.

EXAMPLE 2. Find the image of the triangle with vertices $K(-2, 0)$, $L(0, 3)$, $M(-1, 4)$ under the dilatation with centre the origin and magnification factors

(a) $\frac{1}{2}$

(b) -2 .

Solution

(a) Since $(x, y) \rightarrow (\frac{1}{2}x, \frac{1}{2}y)$,

we have $K(-2, 0) \rightarrow K'(-1, 0)$,

$L(0, 3) \rightarrow L'(0, 1.5)$,

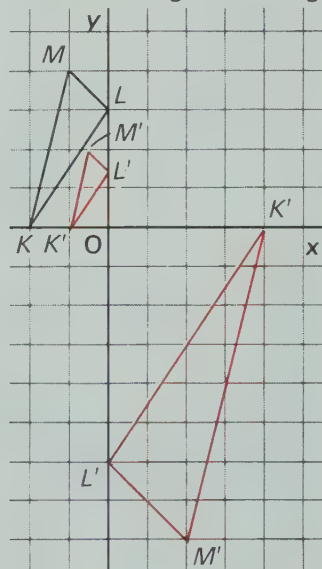
$M(-1, 4) \rightarrow M'(-0.5, 2)$

(b) Since $(x, y) \rightarrow (-2x, -2y)$,

we have $K(-2, 0) \rightarrow K'(4, 0)$,

$L(0, 3) \rightarrow L'(0, -6)$,

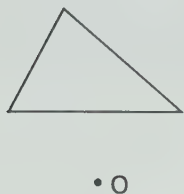
$M(-1, 4) \rightarrow M'(2, -8)$.



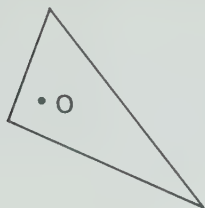
EXERCISE 9-8

- A**
- State the images of the following points under the dilatation $(x, y) \rightarrow (6x, 6y)$.
(a) $(5, 3)$ (b) $(10, -8)$ (c) $(0, 6)$ (d) $(-2, -5)$ (e) $(\frac{1}{2}, 0)$
 - State the images of the above points under the dilatation with centre $O(0, 0)$ and magnification factor $-\frac{1}{5}$.
 - (a) Which points are left unchanged by a dilatation?
(b) Which lines are left unchanged by a dilatation?
- B**
- Copy each of the following figures and construct their images under the dilatation with given centre O and given magnification factor k .

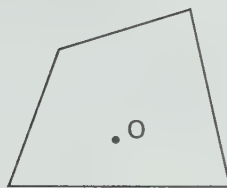
(a) $k=4$



(b) $k=3$



(c) $k=\frac{1}{3}$



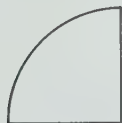
(d) $k=-\frac{1}{2}$



(e) $k=1$



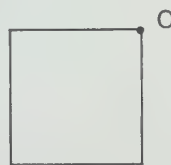
(f) $k=-1$



(g) $k=\frac{3}{2}$



(h) $k=-2$



5. What is the effect on the size of a figure of a dilatation with magnification factor k if

- (a) $k > 1$ (b) $k = 1$ (c) $0 < k < 1$ (d) $-1 < k < 0$
 (e) $k = -1$ (f) $k < -1$

6. Given the triangle with vertices $A(1, 1)$, $B(2, -3)$, $C(-2, 0)$ and the dilatation $D: (x, y) \rightarrow (3x, 3y)$.

- (a) If $D: \triangle ABC \rightarrow \triangle A'B'C'$, find A' , B' , and C' .
 (b) Graph $\triangle ABC$ and $\triangle A'B'C'$.
 (c) Find and compare the lengths of the sides of the triangles.
 (d) Verify that corresponding sides of the triangles are parallel.
 (e) Find the ratio of the areas of triangles $A'B'C'$ and ABC .

7. Given the triangle with vertices $P(2, 0)$, $Q(4, 10)$, $R(6, 5)$ and the dilatation with centre $O(0, 0)$ and magnification factor $-\frac{1}{2}$.

- (a) Find the vertices P' , Q' , R' of the image triangle under the dilatation.
 (b) Graph $\triangle PQR$ and $\triangle P'Q'R'$.
 (c) Find the lengths of the sides of the triangles.
 (d) Verify that $\triangle P'Q'R' \sim \triangle PQR$.

8. A dilatation maps $X \rightarrow X'$, $Y \rightarrow Y'$, $Z \rightarrow Z'$. If $XY=4$, $YZ=6$, $X'Y'=10$, and $X'Z'=18$, find

- (a) the magnification factor, (b) XZ (c) $Y'Z'$.

9. If D is the dilatation with centre $O(0, 0)$ and factor 3,
 T is the translation defined by the vector $\vec{v} = [1, -2]$,
 F is reflection in the y -axis,
 R is rotation through 90° about O ,

find and graph the image of the triangle $A(1, 0)$, $B(3, 2)$, $C(4, 1)$ under the following transformations.

- (a) $T \circ D$ (b) $D \circ T$ (c) $R \circ D$ (d) $D \circ F$

10. If D_1 is a dilatation with centre O and factor k_1 and D_2 is a dilatation with centre O and factor k_2 , describe the transformation $D_2 \circ D_1$.

A rectangular floor is composed of square tiles of the same size, 81 along one side, 63 along the other. If a straight line is drawn diagonally across the floor from corner to corner how many tiles will it cross?

9.9 TRANSFORMATIONS USING MATRICES

Any 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ defines a transformation because if we write a point (x, y) in the plane as a column vector

$$(x, y) = \begin{pmatrix} x \\ y \end{pmatrix} = \vec{v}$$

then the product $A\vec{v}$ of the matrix and the column vector is again a column vector and therefore represents a new point in the plane. The transformation is given by

$$T: \vec{v} \rightarrow A\vec{v}$$

where $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} = (x, y)$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

For instance if $A = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$, then

$$T: \vec{v} \rightarrow A\vec{v}$$

becomes

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &\rightarrow \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} 1 \cdot x + 0 \cdot y \\ 3 \cdot x + 1 \cdot y \end{pmatrix} \\ &= \begin{pmatrix} x \\ 3x + y \end{pmatrix} \end{aligned}$$

$$\therefore (x, y) \rightarrow (x, 3x + y)$$

EXAMPLE 1. What kind of transformation is represented by the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}?$$

Solution

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &\rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} 1 \cdot x + 0 \cdot y \\ 0 \cdot x + (-1)y \end{pmatrix} \\ &= \begin{pmatrix} x \\ -y \end{pmatrix} \end{aligned}$$

The transformation is $(x, y) \rightarrow (x, -y)$ which we recognize as reflection in the x -axis.

EXAMPLE 2. Express the dilatation $D: (x, y) \rightarrow (kx, ky)$ in matrix notation.

Solution

Let the matrix be $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Then
$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

$$\therefore (x, y) \rightarrow (ax + by, cx + dy)$$

Compare this with $(x, y) \rightarrow (kx, ky)$.

We have $kx = ax + by$ and $ky = cx + dy$ which is true if $a = k, b = 0, c = 0, d = k$. Thus $D: v \rightarrow A\vec{v}$ where $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$.

The matrix notation for transformations is especially helpful when we consider the composition of two transformations. If T_1 and T_2 are transformations defined by matrices A_1 and A_2 , i.e.,

$$T_1: \vec{v} \rightarrow A_1\vec{v}, T_2: \vec{v} \rightarrow A_2\vec{v}, \text{ where } \vec{v} = \begin{pmatrix} x \\ y \end{pmatrix},$$

then
$$T_2 \circ T_1: \vec{v} \xrightarrow{T_1} A_1\vec{v} \xrightarrow{T_2} A_2(A_1\vec{v})$$

$$\therefore T_2 \circ T_1: \vec{v} \rightarrow (A_2A_1)\vec{v}$$

Thus composition of transformations corresponds to multiplication of matrices.

EXAMPLE 3. If R is reflection in the x -axis and D is the dilatation $(x, y) \rightarrow (2x, 2y)$, find the transformation $R \circ D$.

Solution

The matrix for D is $A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. (Example 2)

The matrix for R is $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. (Example 1)

The matrix for $R \circ D$ is the product

$$\begin{aligned} A_2A_1 &= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 2 \cdot 1 + 0 \cdot 0 & 2 \cdot 0 + 0(-1) \\ 0 \cdot 1 + 2 \cdot 0 & 0 \cdot 0 + 2(-1) \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \end{aligned}$$

Thus $R \circ D$ is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ -2y \end{pmatrix}$$

or $R \circ D: (x, y) \rightarrow (2x, -2y)$.

Certain pairs of two-digit numbers can have the same product when both numbers are reversed. For example,

$$12 \times 42 = 21 \times 24$$

$$12 \times 63 = 21 \times 36$$

$$12 \times 84 = 21 \times 48$$

$$23 \times 96 = 32 \times 69$$

$$24 \times 63 = 42 \times 36$$

$$24 \times 84 = 42 \times 48$$

$$26 \times 93 = 62 \times 39$$

$$46 \times 96 = 64 \times 69$$

Can you find the only other pairs of this nature? There are just six.

EXERCISE 9-9

- B** 1. Write the transformations defined by the following matrices in the arrow notation $(x, y) \rightarrow (\quad , \quad)$ and identify each transformation.

(a) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

2. Find the matrices which represent the following transformations.

- (a) rotation through 270°
(b) reflection in the line $y = x$

3. Write the transformations represented by the following matrices in the arrow notation $(x, y) \rightarrow (\quad , \quad)$. Graph the square with vertices $(0, 0)$, $(0, 1)$, $(1, 1)$, $(1, 0)$ and its image under each transformation.

(a) $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$

4. Find the matrix for the transformation

$$T: (x, y) \rightarrow (6x + y, 3x - y).$$

5. If D is the dilatation $(x, y) \rightarrow (4x, 4y)$,

R is the rotation about the origin through 90° ,

F is reflection in the y -axis,

find the matrices which represent the following transformations.

- (a) $D \circ R$ (b) $R \circ F$ (c) $F \circ R$ (d) $D \circ F$

Some Common Transformations

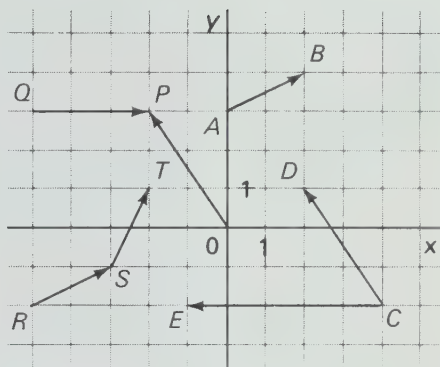
Transformations	Arrow Notation	Matrix Representation
translation	$(x, y) \rightarrow (x + a, y + b)$	
rotation through 90°	$(x, y) \rightarrow (-y, x)$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
rotation through 180°	$(x, y) \rightarrow (-x, -y)$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
rotation through 270°	$(x, y) \rightarrow (y, -x)$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
reflection in x -axis	$(x, y) \rightarrow (x, -y)$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
reflection in y -axis	$(x, y) \rightarrow (-x, y)$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
reflection in $x = y$	$(x, y) \rightarrow (y, x)$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
dilatation	$(x, y) \rightarrow (kx, ky)$	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

Change this into a true statement by altering the position of one digit only.

$$101 - 102 = 1$$

REVIEW EXERCISE

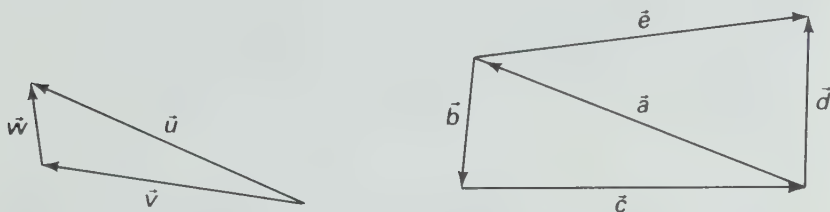
- A** 1. (a) Express each of the following vectors in algebraic form $[a, b]$. $\overrightarrow{AB}$, $\overrightarrow{CD}$, $\overrightarrow{CE}$, $\overrightarrow{OP}$, $\overrightarrow{QP}$, $\overrightarrow{RS}$, $\overrightarrow{ST}$
 (b) Which of the vectors in (a) are equal?



2. Express each of the following vectors in the form $[a, b]$.

- (a) $[5, 3] + [6, -1]$ (b) $[-1, 4] + [3, 3]$
 (c) $[7, 5] - [2, 3]$ (d) $[-2, -3] - [8, 1]$
 (e) $4[3, -8]$ (f) $\frac{1}{2}[6, 9]$
 (g) $[3.6, 9.0] + [6.4, -2.5]$ (h) $-4[2, 1.5]$

3. Express each vector as the sum of two other vectors.



4. What type of transformation is each of the following?

- (a) $(x, y) \rightarrow (-x, y)$ (b) $(x, y) \rightarrow (7x, 7y)$
 (c) $(x, y) \rightarrow (x-3, y-5)$ (d) $(x, y) \rightarrow (y, x)$
 (e) $(x, y) \rightarrow (-x, -y)$ (f) $(x, y) \rightarrow (-y, x)$

5. State the images of the following points under the transformations

- (i) $T: (x, y) \rightarrow (x+3, y-1)$ (ii) $D: (x, y) \rightarrow (5x, 5y)$
 (iii) Reflection in the x -axis (iv) Rotation through 180°
 (a) $(1, 5)$ (b) $(-2, 0)$ (c) $(6, -3)$ (d) $(2, 0.5)$ (e) (a, b)

6. State which points, if any, are left unchanged by the following types of transformations.

- (a) translation (b) rotation (c) reflection (d) dilatation

7. For each of the following figures state the number of lines of symmetry and the order of rotational symmetry.

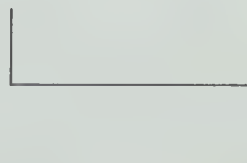
(a)



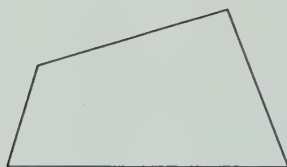
(b)



(c)



(d)



(e)

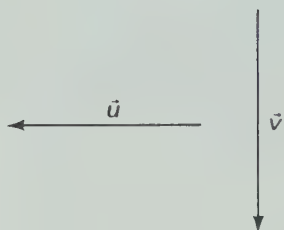


(f)

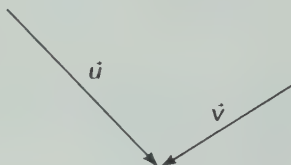


- 8.** Redraw the following pairs of vectors and find their sum $\vec{u} + \vec{v}$ and their difference $\vec{u} - \vec{v}$.

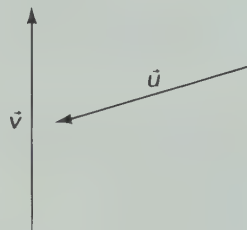
(a)



(b)



(c)

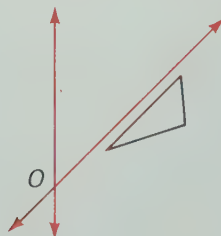
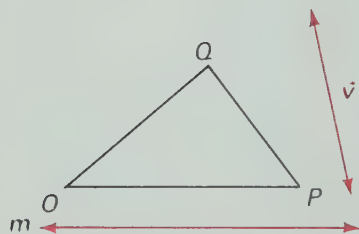


- 9.** Given the vectors $\vec{a} = [2, 3]$, $\vec{b} = [1, -4]$, and $\vec{c} = [-3, 1]$.

- (a) Find $\vec{a} + \vec{b}$, $\vec{b} + \vec{c}$, and $\vec{a} + \vec{c}$ algebraically and geometrically.
 (b) Find the magnitudes $|\vec{a}|$, $|\vec{b}|$, $|\vec{c}|$, $|\vec{a} + \vec{b}|$, $|\vec{b} + \vec{c}|$, and $|\vec{a} + \vec{c}|$.

- 10.** Express each of the following vectors in the form $[a, b]$.

- (a) $9[1, 4] + 2[-1, 7]$ (b) $3[2, -6] - 2[3, -9]$
 (c) $6[2.1, 3.5] + 2[1.2, 9.5]$ (d) $10[1, -2] + [-5, 6] + 3[3, -4]$



- 11.** Copy the figure and draw the images of $\triangle OPQ$ under the following transformations.

- (a) translation defined by the vector $\vec{v}$.
 (b) rotation about O through 120° .
 (c) reflection in m
 (d) dilatation with centre O and magnification factor k .

- 12.** Let F_1 = reflection in m_1 ,
 F_2 = reflection in m_2 ,
 R = rotation through 180° about O ,
 D = dilatation with centre O and factor 2.

Copy the figure and draw the images of the triangle under the following transformations.

- (a) F_1 (b) F_2 (c) R (d) D
 (e) $F_1 \circ F_2$ (f) $F_2 \circ F_1$ (g) $D \circ F_1$ (h) $R \circ F_2$

- 13.** If T is the translation defined by the vector $\vec{v} = [2, 5]$,

R is rotation through 90° about the origin,

F is reflection in the line $y = x$,

D is dilatation with centre the origin and magnification factor $\frac{1}{2}$,

complete the table in your notebook by finding the coordinates of the images of the given points under the given transformations.

	$A(4, 3)$	$B(6, 0)$	$C(-1, -5)$	$P(x, y)$
T				
R				
F				
D				
$R \circ T$				
$T \circ R$				
$D \circ T$				
$F \circ R$				
$F \circ D$				
$D \circ R$				
$T \circ T$				

14. (a) Write the transformations represented by the following matrices in arrow notation $(x, y) \rightarrow (\quad , \quad)$. Graph the square with vertices $(1, 1)$, $(-1, 1)$, $(-1, -1)$, $(1, -1)$ and its image under each transformation.

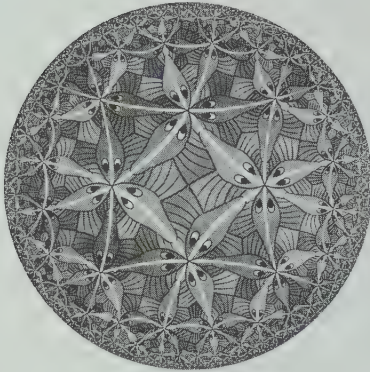
(i) $\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

(b) If T_1 and T_2 are the transformations represented by the matrices (ii) and (iii) respectively, what are the matrices which represent the transformations $T_1 \circ T_2$ and $T_2 \circ T_1$?

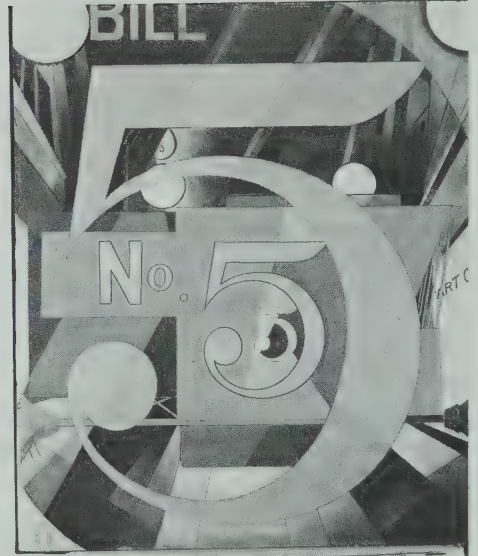
TRANSFORMATIONS IN ART



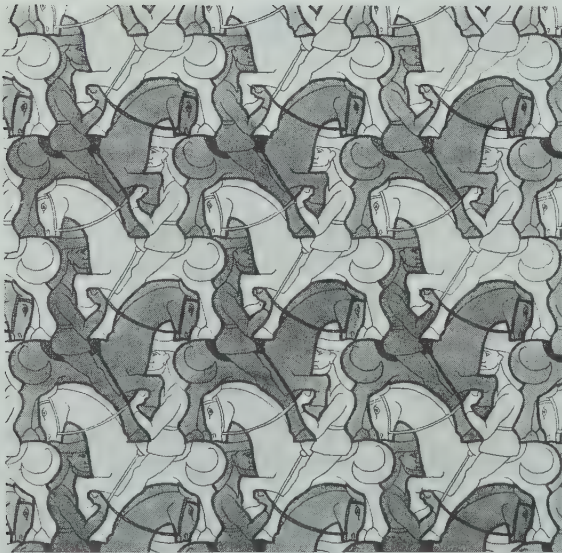
ROTATION Escher: Circle Limit III



REFLECTION Sumerian vase 2700 B.C.

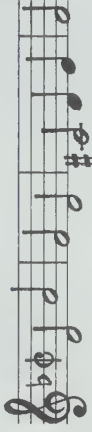


DILATATION Demuth: I Saw the Figure Five in Gold



TRANSLATION Escher: Study of the Regular Division of the Plane with Horsemen

TRANSFORMATIONS IN MUSIC

MATHEMATICAL NAME	MUSICAL NAME	EXAMPLE	OBJECT (MELODY)	IMAGE (TRANSFORMED MELODY)
translation	transposition	The Beatles <i>I Want You</i> (<i>She's So Heavy</i>)		
reflection in y-axis	retrogression	The Beatles <i>Power to the People</i>		
reflection in x-axis	inversion	Bach <i>The Art of the Fugue</i>		
dilatation ($k > 1$)	augmentation	Shostakovich <i>Fifth Symphony</i>		

REVIEW AND PREVIEW TO CHAPTER 10

EXERCISE 1

EQUATION OF A LINE

$$y - y_1 = m(x - x_1) \quad y = mx + b \quad y = m(x - a) \quad \frac{x}{a} + \frac{y}{b} = 1$$

Find the equations of the following lines:

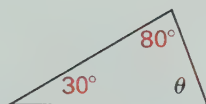
1. through (2, 5), slope 7
2. slope $\frac{2}{3}$, y-intercept 5
3. slope -2, x-intercept -3
4. slope 3, through (0, 0)
5. x-intercept 3, y-intercept -2
6. slope $-\frac{2}{3}$, through (-3, -5)
7. through (0, 4), parallel to $y = 3x - 2$
8. through (1, 3) perpendicular to $y = -x + 1$
9. through (-2, 5), parallel to $3x - y = 5$
10. through (2, -4) and (-5, 3)

EXERCISE 2

ANGLES

Find the value of θ in the following.

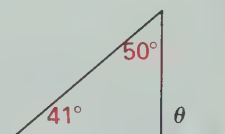
1.



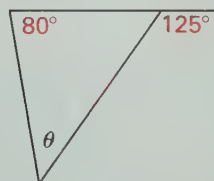
2.



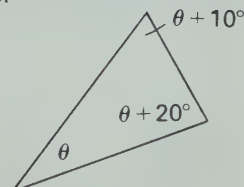
3.



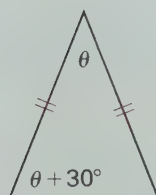
4.



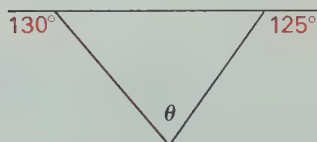
5.



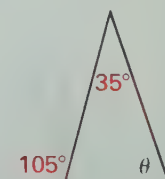
6.



7.



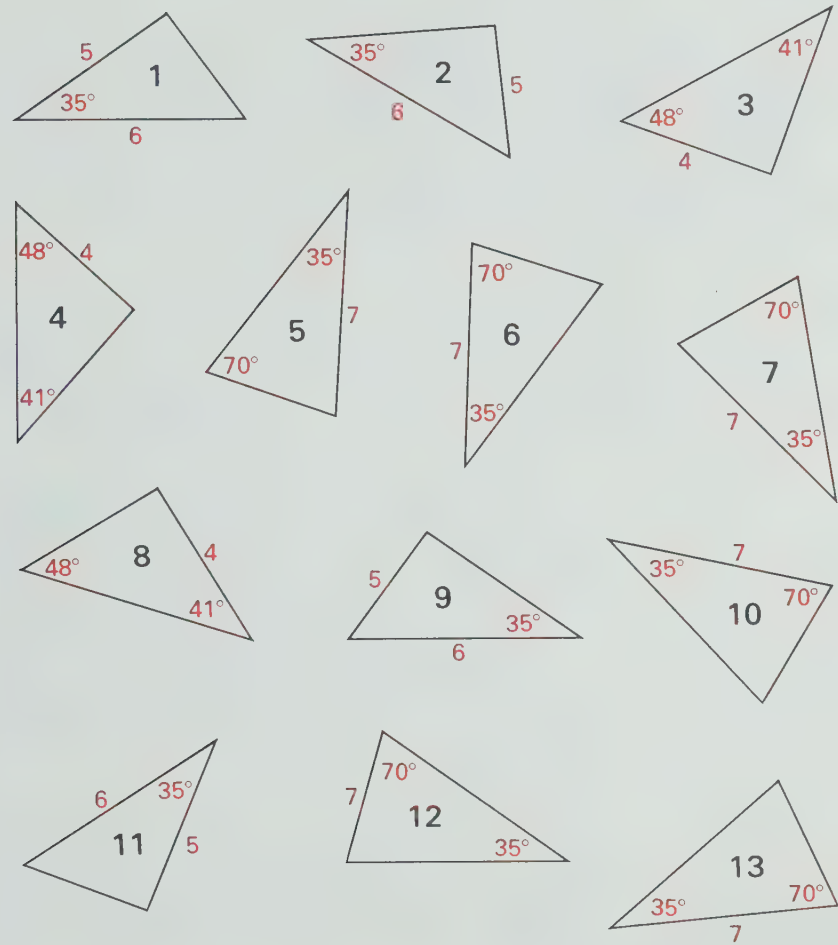
8.



EXERCISE 3

CONGRUENCE

Name pairs of congruent triangles from the following list and give the reason.



The Circle

My joy, my grief, my hope, my love,
Did all within this circle move!

Edmund Waller

Ever since man invented the wheel, the circle has played an important role in human life. Rings, orbits, wheels, gears, and pulleys are a few of the ways that circles are used. But aside from its practical applications, man has also regarded the circle as the most beautiful of geometrical figures because of its complete symmetry. In this chapter we shall explore the basic properties of circles and their tangents.

10.1 THE PARTS OF A CIRCLE

A **circle** is the set of all points in a plane whose distance from a given point O is a given positive number r . The point O is called the **centre** of the circle and r is called the **radius**. The word "radius" also means a line segment joining the centre to a point on the circle.



Figure 10-1

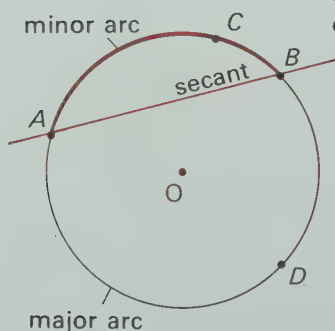


Figure 10-2

Figures 10-1, 10-2, and 10-3 illustrate the various parts of a circle. A **chord** is a line segment that joins two points on the circle. A **secant** is a line that contains a chord. A **diameter** is a chord that passes through the centre. The word "diameter" also means the length d of a diameter, i.e., $d = 2r$.

A secant divides a circle into two parts called **arcs**. The smaller part is called a **minor arc** and the larger part is called a **major arc**. In Figure 10-2, ACB is a minor arc, ADB is a major arc, and both arcs include the end-points A and B .

A **sector** consists of an arc, the radii to the end-points of the arc, and the enclosed points. (An example of a sector is a piece of pie.) The angle formed by the radii at the centre of the circle is called the **central angle** (or **sector angle**). The central angle is said to be **subtended** by the arc.

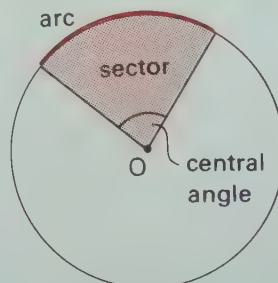
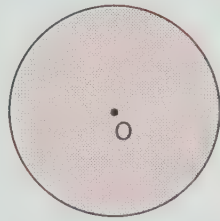


Figure 10-3

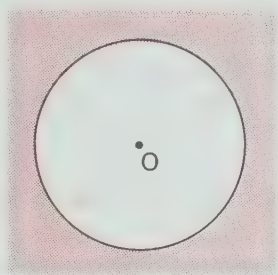
The **interior** of a circle consists of all points inside the circle and the **exterior** consists of all points outside the circle.



circle
 $\{P \mid OP = r\}$



interior
 $\{P \mid OP < r\}$



exterior
 $\{P \mid OP > r\}$

Figure 10-4 illustrates **concentric circles**. This means that the circles all have the same centre O .

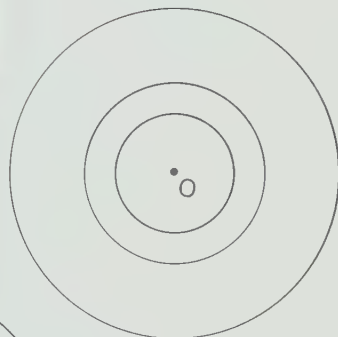


Figure 10-4

EXERCISE 10-1

A 1. In Figure 10-5 O is the centre of the circle.

- What is OB called?
- What is BC called?
- What is CO called?
- What is BD called?
- What is CDB called?
- What is $\angle BOC$ called?
- What kind of triangle is $\triangle OBC$? Why?
- If $\angle B = 45^\circ$, what is $\angle BOC$?

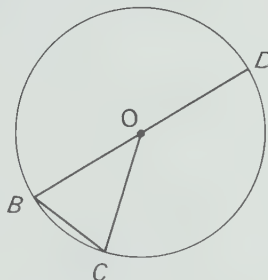
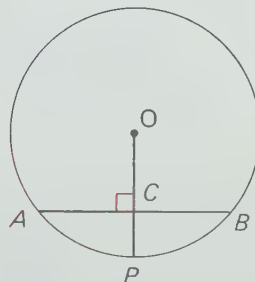


Figure 10-5

B 2. Prove: The perpendicular from the centre of a circle to a chord bisects the chord

Given: Radius OP with $OC \perp AB$

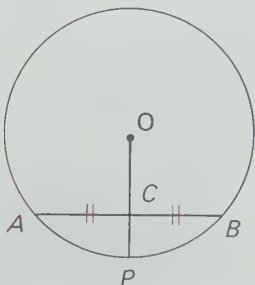
Required: To prove $AC = CB$



3. Prove: A line which joins the centre of a circle to the midpoint of a chord is perpendicular to the chord

Given: Radius OP with $AC = CB$

Required: To prove $OC \perp AB$



Addition

HAM
EGGS
MASH

In questions 4–9, the results of questions 2 and 3 may be used.

4. (a) Prove: **The perpendicular bisector of a chord passes through the centre of the circle**

(b) How can the centre of a circle be constructed, given two non-parallel chords?

(c) Given three non-collinear points, construct a circle which passes through the three points.

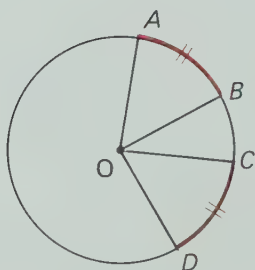
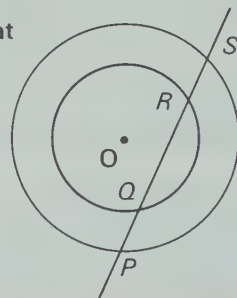
5. A circle has radius 4 cm and a chord has length 6 cm. Find the distance from the centre to the chord.

6. The distance from the centre of a circle to a chord AB is 12 cm. The length of the chord AB is 10 cm. Compute the radius of the circle.

7. Prove that two chords which have the same length must be equidistant from the centre of the circle.

8. If AB and CD are chords of a circle with $AB > CD$, prove that AB is closer to the centre.

9. A secant intersects two concentric circles at points P, Q, R, S as shown. Show that $PQ = RS$.

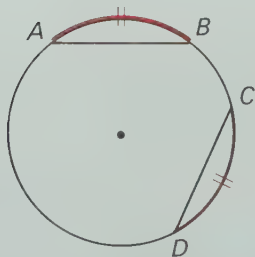


10. Prove: **Equal arcs of a circle subtend equal central angles**

Given: $\text{arc } AB = \text{arc } CD$

Required: To prove $\angle AOB = \angle COD$

Hint: Rotate arc AB about O so that $A \rightarrow C$. Remember that rotations preserve lengths and angles.



11. Prove: **Equal arcs of a circle subtend equal chords**

Given: $\text{arc } AB = \text{arc } CD$

Required: To prove $AB = CD$

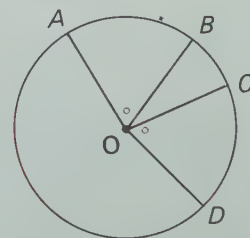
Hint: Use either a rotation or question 10.

12. Prove: **Equal central angles subtend equal arcs**

Given: $\angle AOB = \angle COD$

Required: To prove $\text{arc } AB = \text{arc } CD$

Hint: Use a rotation about O .

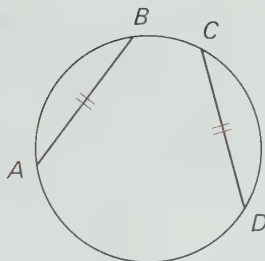


13. Prove: **Equal chords subtend equal arcs.**

Given: $AB = CD$

Required: To prove $\text{arc } AB = \text{arc } CD$

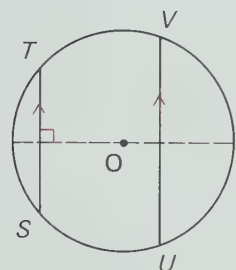
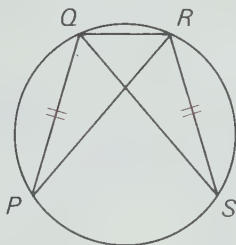
Hint: Use a rotation or question 12.



In the remaining questions the results of questions 10–13 may be used.

14. Given: Equal chords $PQ = RS$

Required: To prove $PR = QS$.

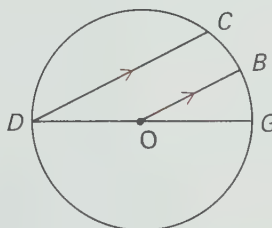


15. Given: Parallel chords ST and UV

Required: To prove $\text{arc } US = \text{arc } VT$.

16. Given: Diameter DOG and chord CD with $CD \parallel OB$

Required: To prove $\text{arc } CB = \text{arc } BG$.



10.2 INSCRIBED ANGLES

An **inscribed angle** is an angle whose vertex lies on the circle and whose arms intersect the circle. We say that the inscribed angle is **subtended** by the arc of the circle which is cut off by the arms of the angle and which does not contain the vertex.

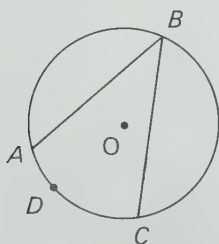


Figure 10-6

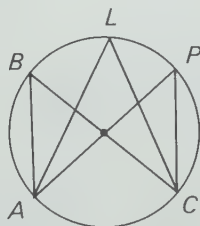


Figure 10-7

In Figure 10-6, $\angle ABC$ is an inscribed angle and the arc ADC subtends $\angle ABC$. A given arc subtends many inscribed angles. Figure 10-7 shows three inscribed angles, $\angle ABC$, $\angle ALC$, and $\angle APC$, all subtended by the same arc AC . Is there any relation between the sizes of these

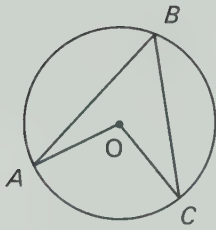


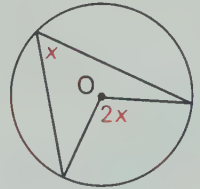
Figure 10-8

three angles? Yes. We shall show in this section that inscribed angles subtended by the same arc are equal.

Figure 10-8 shows two kinds of angles subtended by an arc AC. $\angle AOC$ is a central angle and $\angle ABC$ is an inscribed angle. Is there any relation between the sizes of these angles? Yes. The next theorem shows that the central angle is twice the inscribed angle.

Angles in a Circle Theorem (a)

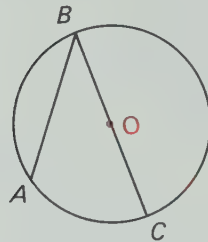
An inscribed angle is equal to one-half of the central angle subtended by the same arc.



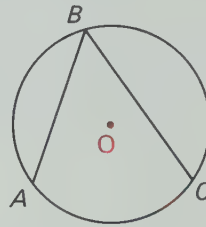
Given: A circle, centre O , with $\angle ABC$ an inscribed angle

Required: To prove $\angle ABC = \frac{1}{2}\angle AOC$.

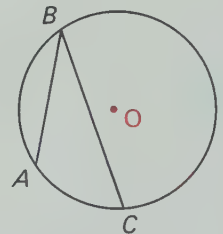
Proof: There are three possibilities.



Case 1



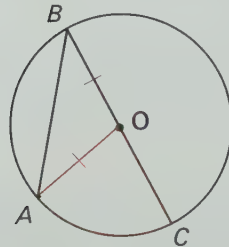
Case 2



Case 3

Case 1: O lies on a side of $\angle ABC$.

Draw the radius OA .



Then $OA = OB$

$$\therefore \angle ABC = \angle OAB \quad (\text{ITT})$$

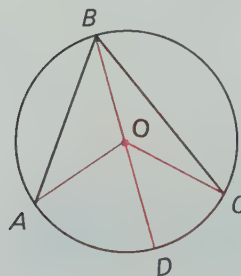
$$\angle AOC = \angle ABC + \angle OAB \quad (\text{EAT})$$

$$= 2\angle ABC$$

$$\angle ABC = \frac{1}{2}\angle AOC.$$

Case 2: O lies inside $\angle ABC$.

Draw the diameter BOD and the radii OA and OC .



$$\angle ABD = \frac{1}{2}\angle AOD \quad (\text{case 1})$$

$$\angle DBC = \frac{1}{2}\angle DOC \quad (\text{case 1})$$

$$\angle ABC = \angle ABD + \angle DBC$$

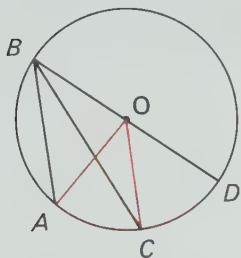
$$= \frac{1}{2}\angle AOD + \frac{1}{2}\angle DOC$$

$$= \frac{1}{2}(\angle AOD + \angle DOC)$$

$$= \frac{1}{2}\angle AOC$$

Case 3: O lies outside $\angle ABC$

Draw the diameter BOD and the radii OA and OC .



$$\angle ABD = \frac{1}{2}\angle AOD \quad (\text{case 1})$$

$$\angle CBD = \frac{1}{2}\angle COD \quad (\text{case 1})$$

$$\begin{aligned} \angle ABC &= \angle ABD - \angle CBD \\ &= \frac{1}{2}\angle AOD - \frac{1}{2}\angle COD \\ &= \frac{1}{2}(\angle AOD - \angle COD) \\ &= \frac{1}{2}\angle AOC. \end{aligned}$$

WORD LADDER

Start with the word "black" and change one letter at a time to form a new word until you reach "white". The best solution has the fewest steps.

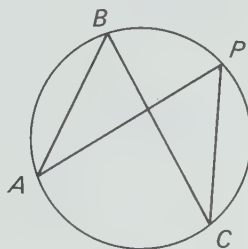
b l a c k

white

Corollary 1: Inscribed angles subtended by the same arc are equal.

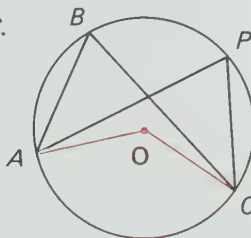
Given: Inscribed angles $\angle ABC$ and $\angle APC$ subtended by the same arc AC .

Required: To prove $\angle ABC = \angle APC$



Proof: Both inscribed angles have the same central angle $\angle AOC$.

$$\therefore \angle ABC = \frac{1}{2}\angle AOC = \angle APC.$$



Corollary 2: An angle inscribed in a semicircle is a right angle.

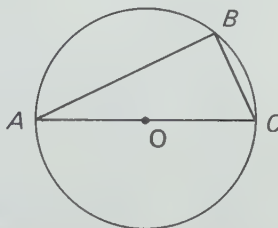
Given: Diameter AC and inscribed angle $\angle ABC$.

Required: To prove $\angle ABC = 90^\circ$

Proof: $\angle ABC = \frac{1}{2}\angle AOC$

$$= \frac{1}{2}(180^\circ)$$

$$= 90^\circ$$



Let us collect the three results of this section.

Angles in a Circle Theorem (ACT)

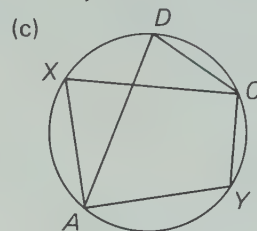
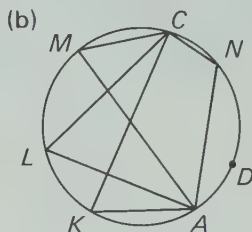
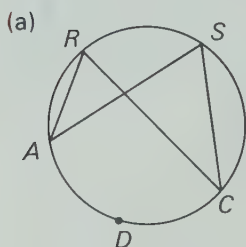
(a) An inscribed angle is equal to one-half the central angle subtended by the same arc.

(b) Inscribed angles subtended by the same arc are equal.

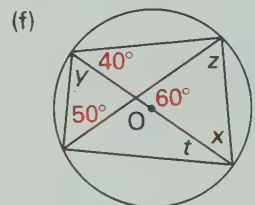
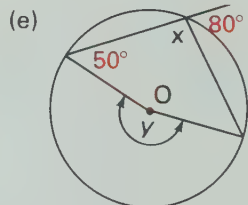
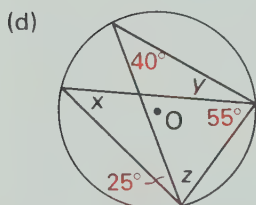
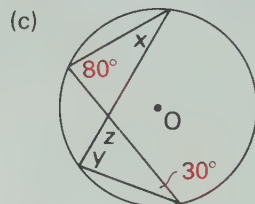
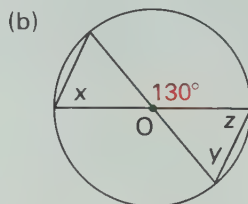
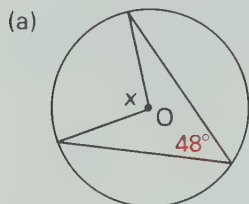
(c) An angle inscribed in a semicircle is a right angle.

EXERCISE 10-2

- A 1.** Name the inscribed angles which are subtended by the arc ADC .



- 2.** Find the values of x , y , z , t in the following diagrams. The centre of each circle is O .



- B 3.** The points P, Q, R lie on a circle with centre O . If $\angle PQR = 43^\circ$ and $\angle QOR = 78^\circ$, find $\angle QPR$ and $\angle QRP$.

- 4.** The points K, L, M , and N lie in order on a circle. If $\angle KLM = 75^\circ$, $\angle LMK = 56^\circ$, and $\angle NKL = 80^\circ$, find $\angle LNK$, $\angle MKL$, $\angle LNM$, and $\angle MNK$.

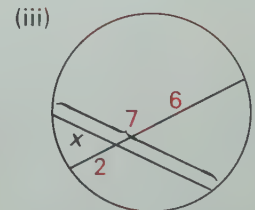
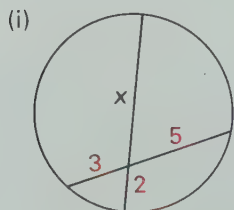
- 5.** Points X, Y, Z lie on a circle, centre O , and $\angle XYZ = 45^\circ$. Show that $XO \perp ZO$.

- 6.** Points D, E, F, G lie in order on a circle. DF is a diameter and $DE = DG$. Prove that $FE = FG$.

- 7.** From a point A outside a circle, two straight lines ABC and ADE are drawn intersecting the circle in B, C and D, E respectively. Prove that $\triangle ABE \sim \triangle ADC$.

- 8.** (a) Prove the Intersecting Chords Theorem: If AB and CD are two chords of a circle which intersect at a point P inside the circle, then $AP \cdot PB = CP \cdot PD$.

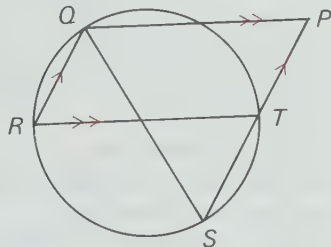
- (b) Find x in each of the following diagrams.



9. Points P, R, A, Y lie in order on a circle and $\angle PAR = \angle ARY$. Prove that $AR \parallel PY$.

10. Given: Parallelogram $PQRT$ and chords QS, RT .

Required: To prove $QS = QP$.



11. Points A, B, C, D lie in order on a circle with centre O . BD is a diameter and $\angle CAB = \angle CAD$. Prove that $BD \perp OC$.

12. Two circles intersect at P and Q . Two lines APB and CPD are drawn through P intersecting the first circle at A and C and the second circle at B and D . Prove that $\angle AQC = \angle BQD$.

13. A, B, C are points on a circle such that $\triangle ABC$ is equilateral. P lies on the minor arc CA . D lies on the chord BP and $BD = PC$. Prove

- $\triangle ABD \cong \triangle APC$
- $\triangle ADP$ is equilateral
- $BP = AP + PC$

Find x if $\frac{x^2}{x+x} = 2$ and $x \neq 0$.

C 14. DG is a diameter of a circle with centre O , and P is any other point on the circle. DP is produced to Q so that $PQ = OP$. Prove that $\angle QOG = 3\angle POQ$.

15. Points A, B, C, D lie in order on a circle and $AB = CD$. Prove that $AC = BD$.

16. AB and CD are perpendicular chords in a circle with centre O . Prove that $\angle BAD = \angle CAO$.

10.3 CYCLIC QUADRILATERALS

A set of points is called **concyclic** if all of the points lie on the same circle. A polygon is called **cyclic** if its vertices form a concyclic set. In Figure 10-9 $\{A, B, C, D, E\}$ is a concyclic set and $ABCDE$ is a cyclic pentagon which is **inscribed** in the circle with centre O .

Every triangle is cyclic because it is always possible to draw a circle through three given noncollinear points. In fact the centre of the circle is the point of intersection of the perpendicular bisectors of two sides of the triangle. (See question 4(c) in Exercise 10-1.)

However not every quadrilateral is cyclic. How can we decide whether or not a given quadrilateral is cyclic? We know that if $ABCD$ is a cyclic quadrilateral then $\angle ACB = \angle ADB$ because they

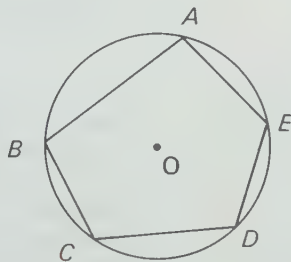
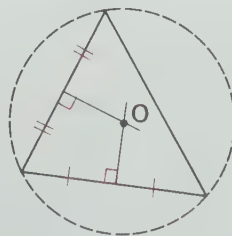
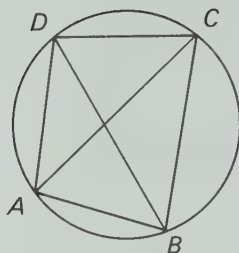
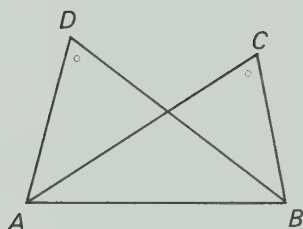


Figure 10-9





are inscribed angles subtended by the same arc AB . It turns out that the converse of this statement is also true and this gives us one method for proving quadrilaterals cyclic.

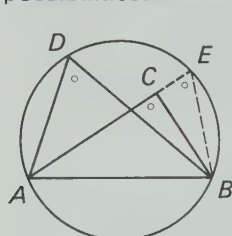


Theorem: If the line segment joining two points subtends two equal angles at two other points on the same side of the segment, then the four points are concyclic.

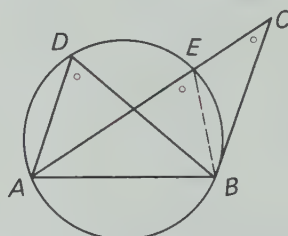
Given: C, D are two points on the same side of AB , and $\angle ACB = \angle ADB$

Required: To prove $ABCD$ is a cyclic quadrilateral

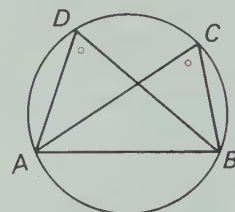
Proof: Construct the circle which passes through A, B , and D . Let E be the point of intersection of the circle and the line AC . There are three possibilities:



Case 1



Case 2



Case 3

In case 1, C lies between A and E . In case 2, E lies between A and C . In case 3, E coincides with C .

In cases 1 and 2, $\angle ADB = \angle AEB$ (ACT)

But $\angle ADB = \angle ACB$ (given)

$\therefore \angle AEB = \angle ACB$

$\therefore EB \parallel CB$ (TPT)

This is impossible because EB and CB intersect at B . Since cases 1 and 2 cannot occur, the only possibility is case 3, i.e., C lies on the circle through A, B , and D . Thus $ABCD$ is a cyclic quadrilateral.

EXAMPLE 1. **Given:** $\triangle ABC$ with $AB = AC$, points P on AB and Q on AC with $AP = AQ$.

Required: To prove quadrilateral $PBCQ$ is cyclic

Proof: Join PC and BQ .

Statements

$PB = QC$

$\angle PBC = \angle QCB$

BC is common

$\therefore \triangle PBC \cong \triangle QCB$

$\therefore \angle BPC = \angle CQB$

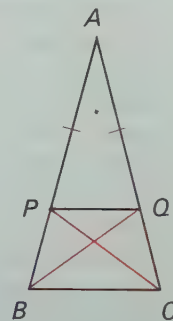
$\therefore PBCQ$ is a cyclic quadrilateral

Reasons

subtraction

ITT

SAS



Analysis

We can use the above theorem to prove $PBCQ$ cyclic if BC subtends equal angles at P and Q . So join PC and BQ , and try to prove $\angle BPC = \angle BQC$ using congruent triangles.

Theorem: The opposite angles of a cyclic quadrilateral are supplementary.

Given: $ABCD$ is a cyclic quadrilateral

Required: To prove (i) $\angle A + \angle C = 180^\circ$

(ii) $\angle B + \angle D = 180^\circ$

Proof: Since $ABCD$ is cyclic, its vertices lie on a circle. Let O be the centre of the circle and join BO and DO . Assume that C lies on the minor arc BD and A on the major arc BD .

$$\angle BAD = \frac{1}{2} \angle BOD \quad (\text{ACT (a)})$$

$$\angle BCD = \frac{1}{2} \text{reflex } \angle BOD$$

$$\begin{aligned} \angle BAD + \angle BCD &= \frac{1}{2} (\angle BOD + \text{reflex } \angle BOD) \\ &= \frac{1}{2} (360^\circ) \\ &= 180^\circ \end{aligned}$$

Similarly, by joining AO and CO , (ii) can be proved.

Corollary: An exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

Given: cyclic quadrilateral $ABCD$ with CD produced to E

Required: To prove $\angle ABC = \angle ADE$

Proof: $\angle ABC + \angle ADC = 180^\circ$

But $\angle ADE + \angle ADC = 180^\circ$

$$\therefore \angle ABC = \angle ADE$$

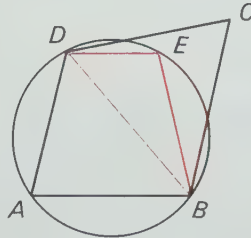
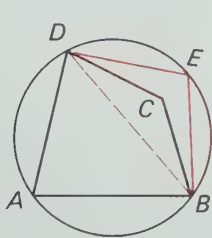
The converse of the above theorem is also true, and this gives us a second method for proving quadrilaterals cyclic.

Theorem: If the opposite angles of a quadrilateral are supplementary then the quadrilateral is cyclic.

Given: quadrilateral $ABCD$ with $\angle A + \angle C = 180^\circ$

Required: To prove $ABCD$ is cyclic

Proof: Construct the circle which passes through A , B , and D . Let E be any point on the circle which lies on the arc BD not containing A .



$ABED$ is a cyclic quadrilateral

$$\therefore \angle A + \angle E = 180^\circ$$

$$\text{But } \angle A + \angle C = 180^\circ \quad (\text{given})$$

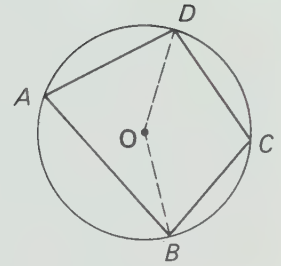
$$\therefore \angle C = \angle E$$

Thus BD subtends equal angles at C and E , which lie on the same side of BD .

$\therefore BCED$ is cyclic

$\therefore C$ lies on the circle which passes through B , E , D , and A ,

$\therefore ABCD$ is cyclic.



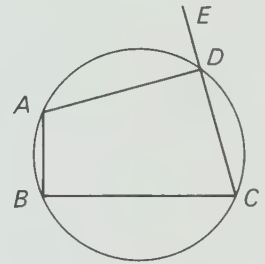
acute



obtuse



reflex



Corollary: If an exterior angle of a quadrilateral is equal to the interior opposite angle, then the quadrilateral is cyclic.

Let us summarize the results of this section.

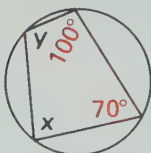
Cyclic Quadrilateral Theorem

- (a) A quadrilateral is cyclic if and only if the segment joining two of the vertices subtends equal angles at the other two vertices on the same side of the segment.
- (b) A quadrilateral is cyclic if and only if the opposite angles are supplementary.
- (c) A quadrilateral is cyclic if and only if an exterior angle is equal to the interior opposite angle.

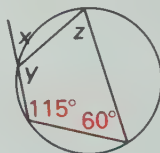
EXERCISE 10-3

- A** 1. Find the values of x , y , z in the following diagrams.

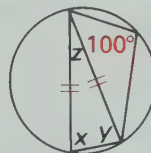
(a)



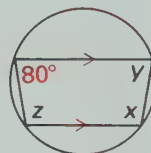
(b)



(c)

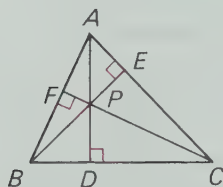


(d)

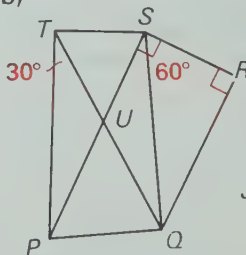


- 2. Are all squares cyclic?
- 3. Are all rectangles cyclic?
- 4. Name the cyclic quadrilaterals in the following diagrams.

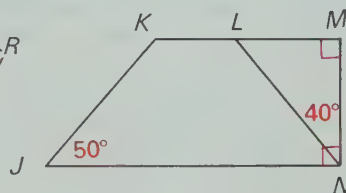
(a)



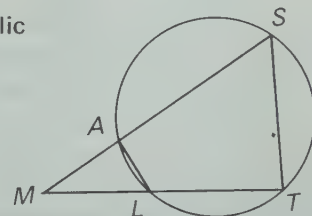
(b)



(c)



- B** 5. Given: Quadrilateral $SALT$ is cyclic
Required: To prove $\triangle MAL \sim \triangle MTS$



- 6. The diagonals of quadrilateral $KLMN$ meet at P such that $\angle KNP = 64^\circ$, $\angle KPL = 64^\circ$, $\angle MNP = 60^\circ$, $\angle KPL = 106^\circ$, and $\angle NLM = 42^\circ$.
 - (a) Prove that $KLMN$ is a cyclic quadrilateral.
 - (b) Find $\angle NLK$.

7. In $\triangle ABC$, $AB = AC$ and the bisectors of $\angle ABC$ and $\angle ACB$ meet AC and AB at X and Y respectively. Prove that B, C, X, Y are concyclic points.

8. AB and CD are equal segments which bisect each other at P . Prove that $ADBC$ is a cyclic quadrilateral.

9. $ABCD$ is a cyclic quadrilateral and X, Y are points on AB and CD respectively such that $XY \parallel BC$. Prove that $AXYD$ is a cyclic quadrilateral.

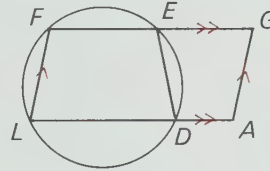
10. AB is a diameter of a circle and C is another point on the circle. F lies on AB , G lies on the chord BC , and $FG \perp AB$. Prove that $\{A, F, G, C\}$ is a concyclic set.

11. Two circles intersect at X and Y . AXB and CYD are two lines which meet the first circle at A and C and the second circle at B and D . Prove that $AC \parallel BD$.

12. Prove that a circle constructed on the hypotenuse of a right triangle as diameter passes through the vertex of the right angle.

13. Given: Parallelogram $FLAG$, cyclic quadrilateral $FLDE$

Required: To prove $AGED$ is cyclic



C 14. A, B, C are points on a circle and X, Y, Z are points on the arcs AB, BC , and CA . Prove that $\angle AXB + \angle BYC + \angle CZA = 360^\circ$.

15. $\triangle ABC$ is inscribed in a circle and $AB > AC$. P is the point on AB with $AP = AC$. The bisector of $\angle A$ meets BC at Q and the circle again at R . Prove that $BPQR$ is a cyclic quadrilateral.

10.4 TANGENTS

If a circle and a line lie in the same plane, there are three possibilities. In Figure 10-10, the line l does not intersect the circle, and the line s (a secant) intersects the circle twice. The line t intersects the circle once and only once and is called a **tangent**. The point P where t intersects the circle is called the **point of contact**.

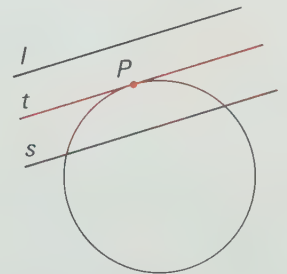
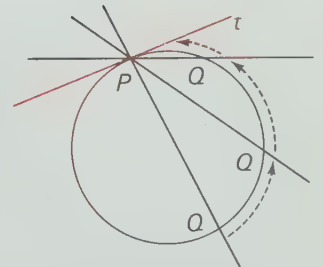


Figure 10-10

A tangent to a circle is a line in the plane of the circle which intersects it in exactly one point.

Notice that the tangent t at P can be regarded as the limiting position of the secant PQ as Q moves around the circumference and approaches P . [For curves which are more complicated than the circle, this property is taken as the definition of a tangent.]



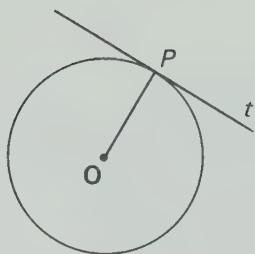
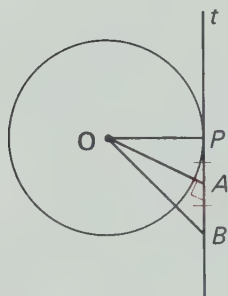


Figure 10-11

In Figure 10-11 t is a tangent to the circle with centre O and P is the point of contact. It appears from the diagram that t may be perpendicular to the radius OP . This guess turns out to be true, as we shall now show.

Tangent Theorem (a)

A tangent to a circle is perpendicular to the radius drawn to the point of contact.



Given: A circle with centre O , tangent t with point of contact P

Required: To prove $t \perp OP$

Proof: We shall use an indirect proof. Suppose that t is *not* perpendicular to OP . Then draw the perpendicular OA from O to t . Let B be the point on t (on the opposite side of A from P) such that $AB = PA$. In $\triangle$'s OAP and OAB ,

$$AP = AB,$$

$$\angle OAP = \angle OAB \quad (\text{right angles})$$

$$OA \text{ is common}$$

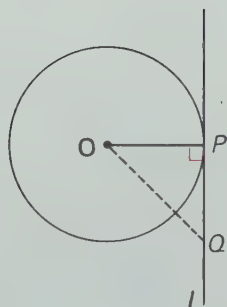
$$\therefore \triangle OAP \cong \triangle OAB \quad (\text{SAS})$$

$$\therefore OB = OP$$

But OP is a radius, and so OB must be a radius. Thus B must lie on the circle, which means that t intersects the circle at two points, P and B . This contradicts our assumption that t is a tangent, since tangents only intersect a circle once. Therefore our assumption that t is not perpendicular to OP is false, and so $t \perp OP$.

The converse of this theorem is also true:

If a line is perpendicular to a radius at its outer endpoint, then the line is a tangent to the circle.



Given: A circle with centre O , a point P on the circle, and a line l through P with $l \perp OP$

Required: To prove l is a tangent to the circle at P

Proof: Let Q be any point on l except for P . Then $\triangle OPQ$ is a right triangle with hypotenuse OQ . Since the hypotenuse is the longest side of a right triangle,

$$\therefore OQ > OP$$

$$\therefore Q \text{ lies in the exterior of the circle.}$$

Thus any point on l , except for P , lies outside the circle.

$$\therefore l \text{ is a tangent to the circle at } P.$$

EXAMPLE 1. **Given:** A circle, centre O , and a point P outside the circle.

Required: To construct a tangent from P to the circle.

Construction: Join OP

Bisect OP and call its midpoint C .

Draw a circle with centre C and radius OC , intersecting the circle at A and B

Join PA, PB .

Then PA and PB are the tangents from P to the circle.

Proof: Join OA .

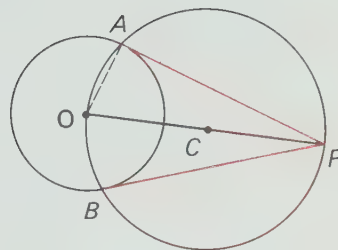
$\angle OAP$ is an angle in a semicircle

$$\therefore \angle OAP = 90^\circ$$

Since OA is a radius and $\angle OAP = 90^\circ$

$\therefore PA$ is a tangent

Similarly PB is a tangent.



In Example 1 we saw that there are two tangents to a circle from a given point P outside the circle. The next theorem shows that the tangent segments PA and PB have equal length.

Tangent Theorem (b)

The tangent segments to a circle from an external point have equal length, the segments subtend equal angles at the centre, and they make equal angles with the line joining the external point to the centre.

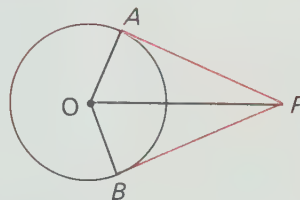
Given: A circle with centre O , and tangent segments PA, PB drawn from an external point P .

Required: To prove (a) $PA = PB$

(b) $\angle AOP = \angle BOP$

(c) $\angle APO = \angle BPO$

Proof:



In $\triangle$'s AOP and BOP ,

$$AO = BO$$

Reasons

radii

OP is common

$\angle OAP$ and $\angle OBP$ are right angles

Tangent Theorem (a)

$$\therefore \triangle AOP \cong \triangle BOP$$

HS

$$\therefore PA = PB,$$

$$\angle AOP = \angle BOP,$$

$$\text{and } \angle APO = \angle BPO.$$

EXERCISE 10-4

A 1. How many tangents can be drawn to a circle

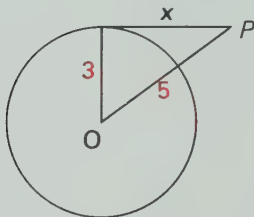
(a) at a point on the circle?

(b) from a point outside the circle?

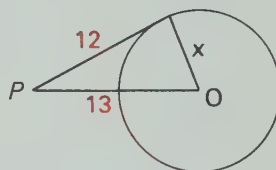
(c) through a point inside the circle?

2. In the diagrams below, tangents have been drawn to the circles from P . Find the values of x , y , z , t

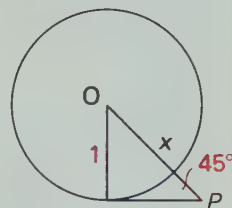
(a)



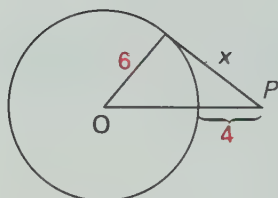
(b)



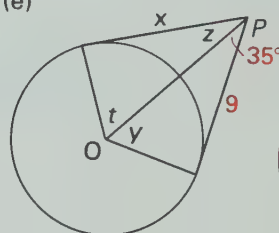
(c)



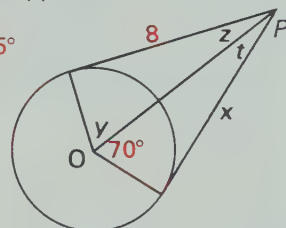
(d)



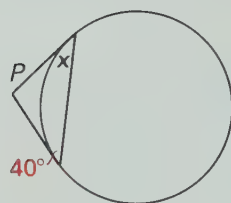
(e)



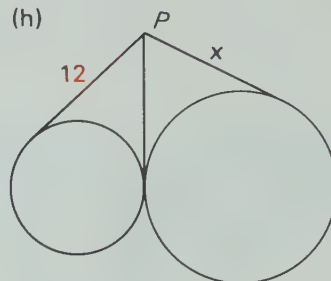
(f)



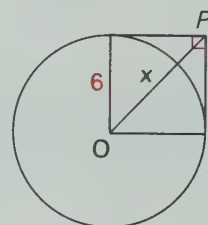
(g)



(h)



(i)



B 3. Draw a circle, mark a point A on it, and construct the tangent to the circle at A .

4. Draw a circle, mark a point P outside the circle, and construct the tangents to the circle from P .

5. Two concentric circles have radii 9 cm and 15 cm. Find the length of a chord of the larger circle that is tangent to the smaller circle.

6. Is it possible for a pair of circles to have

- | | |
|---------------------------|----------------------------|
| (a) no common tangents? | (b) one common tangent? |
| (c) two common tangents? | (d) three common tangents? |
| (e) four common tangents? | (f) five common tangents? |

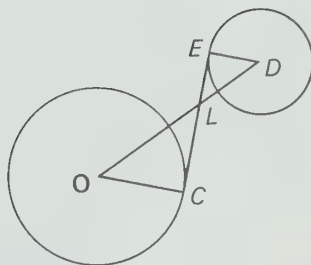
Draw diagrams to illustrate the possible cases.

7. If two tangents from an external point P meet a circle at A and B , prove that $\angle PAB = \angle PBA$.

8. PX and PY are tangents from an external point P which meet a circle with centre O at X and Y . XOZ is a diameter and $\angle PYX = 53^\circ$. Find $\angle YOZ$.

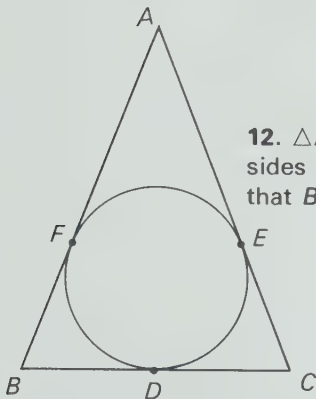
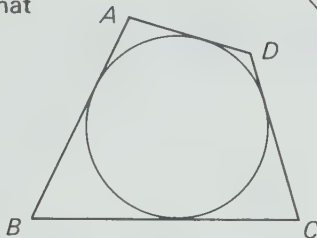
9. PA and PB are tangent segments to a circle, centre O , from a point P outside the circle. Prove that $PAOB$ is a cyclic quadrilateral.

10. Given: CE is a tangent to two circles, OD joins the centres of the circles
 Required: To prove $\angle O = \angle D$



11. Quadrilateral $ABCD$ is circumscribed about a circle. (This means that each side of the quadrilateral is tangent to the circle.) Prove that

$$AB + CD = AD + BC.$$



12. $\triangle ABC$ is circumscribed about a circle and its sides meet the circle at D, E, F . If $AB = AC$, prove that $BD = DC$.

What is a GOOGOL?

13. A point X is 26 cm from the centre of a circle with radius 10 cm. A and B are the points of contact of the tangents drawn to the circle from X . Find the length of the chord AB .

10.5 THE TANGENT CHORD THEOREM

Tangent Chord Theorem

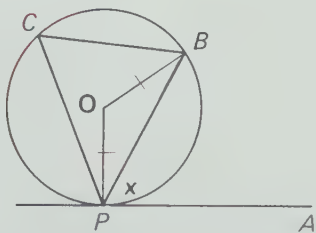
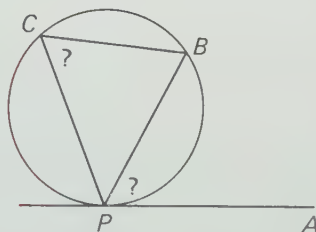
If a chord is drawn from the point of contact of a tangent of a circle, then the angle between the tangent and the chord is equal to the angle inscribed in the circle on the opposite side of the chord.

Given: A tangent PA to a circle, a chord PB , and C any point on the arc of the circle opposite A .

Required: To prove $\angle BPA = \angle PCB$.

Proof: If O is the centre of the circle, join OP, OB .
 Let $\angle BPA = x$.

Statements	Reasons
$\angle OPA = 90^\circ$	Tangent Theorem (a)
$\therefore \angle OPB = 90^\circ - x$	
$\therefore \angle PBO = 90^\circ - x$	ITT



$$\begin{aligned}
 \therefore \angle POB &= 180^\circ - 2(90^\circ - x) \\
 &= 180^\circ - 180^\circ + 2x \\
 &= 2x
 \end{aligned}$$

$$\therefore \angle PCB = x$$

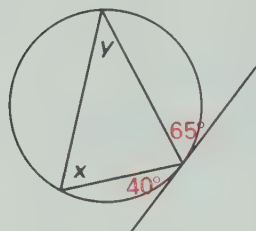
ACT

$$\therefore \angle PCB = \angle BPA$$

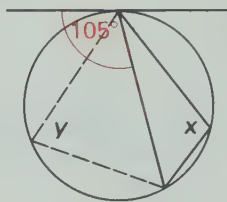
EXERCISE 10-5

A 1. Find the values of x , y , z , t , u , v in the following diagrams.

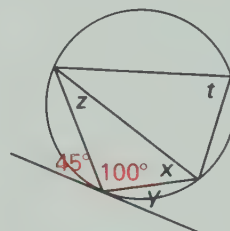
(a)



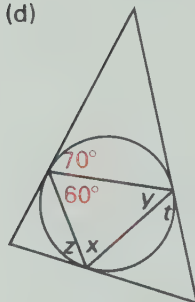
(b)



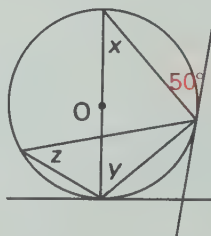
(c)



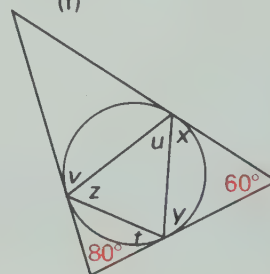
(d)



(e)



(f)



B 2. A, B, C are points on a circle and $\angle ABC = 74^\circ$, $\angle BCA = 46^\circ$. Find the size of the angles of the triangle which is formed when tangents are drawn to the circle at A, B , and C .

3. $FARM$ is a quadrilateral inscribed in a circle. If $\angle FRA = 43^\circ$ and the angle between MF and the tangent at M is 58° , find $\angle MFA$.

4. A, B, C are three points on a circle. If the tangent at A is parallel to the chord BC , prove that $\triangle ABC$ is isosceles.

5. Points A, B, C are situated on a circle in such a way that if DA is the tangent at A , then BA bisects $\angle DAC$. Prove that $AB = BC$.

6. PQ is a diameter of a circle and PR is a chord. SR is the tangent at R and $PS \perp SR$. Prove that PR bisects $\angle SPQ$.

7. Two circles lie on opposite sides of a common tangent and have a common point of contact P . XPY and UPV are straight lines. X and U lie on the first circle, V and Y lie on the second circle. Prove that $XU \parallel VY$.

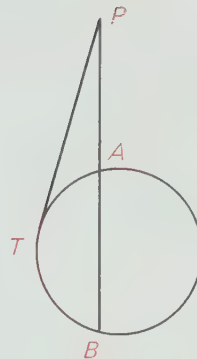
8. Two circles intersect at A and N . One of their common tangents has points of contact P and T . Prove that $\angle PAT$ and $\angle PNT$ are supplementary.

9. Prove the following theorem using similar triangles.

Tangent Secant Theorem

If a tangent from an external point P intersects a circle at a point T and a secant from P intersects the same circle at A and B , then

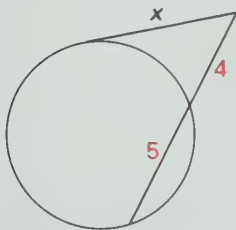
$$PT^2 = PA \cdot PB.$$



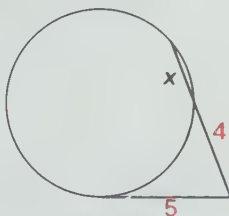
In questions 10-13, the result of question 9 may be used.

10. Find x in each of the following diagrams.

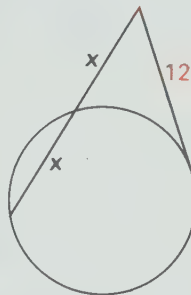
(a)



(b)



(c)



11. A circle has a chord XY of length 12 cm. A tangent segment from an external point P has length 8 cm and XY produced meets the tangent segment at P . Find the length of PY .

12. (a) If two secants from an external point P intersect a circle at A, B and C, D respectively, prove that

$$PA \cdot PB = PC \cdot PD.$$

[This can also be proved using question 7 in Exercise 10-2.]

(b) SA and YT are two chords of a circle. SA produced and YT produced meet at L . If $SA = 5$, $AL = 3$, and $TL = 4$, find YT .

13. Two circles intersect at A and B . One of their common tangents has points of contact P and Q . Prove that the line AB bisects the segment PQ .

14. A, B, C, D are points in order on a circle. AC and BD meet at X . Let t be the tangent at X to the circle which passes through A, B , and X . Prove that $t \parallel CD$.

10.6 EQUATIONS OF TANGENTS

In Chapter 6 you learned that an equation of a circle with centre the origin and radius r is

$$x^2 + y^2 = r^2.$$

In this section we shall use this equation, together with the properties of tangents discovered in this chapter, to find equations of tangents and lengths of tangent segments.

Put the numbers from 1 to 9 in the spaces to make the statements true.

$$\square \times \square - \square = 2$$

$$\square + \square \div \square = 2$$

$$\square - \square + \square = 2$$

EXAMPLE 1. (a) Show that the point $P(3, -2)$ lies on the circle with equation $x^2 + y^2 = 13$.

(b) Find the slope of the radius OP and use it to find the slope of the tangent to the given circle at P .

(c) Find an equation of the tangent to the given circle at P .

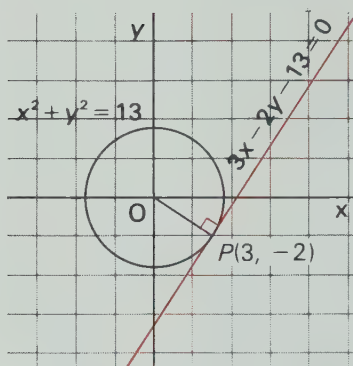
Solution

(a) $3^2 + (-2)^2 = 9 + 4 = 13$

$\therefore P(3, -2)$ lies on the circle $x^2 + y^2 = 13$

(b) $m_{OP} = \frac{-2-0}{3-0} = -\frac{2}{3}$

The tangent at P is perpendicular to the radius OP , by Tangent Theorem (a).



$$\begin{aligned} \therefore \text{slope of tangent} &= -\frac{1}{m_{OP}} \\ &= -\frac{1}{(-\frac{2}{3})} \\ &= \frac{3}{2} \end{aligned}$$

(c) Using the slope-point form of a line, $y - y_1 = m(x - x_1)$, we find that the equation of the tangent to $x^2 + y^2 = 13$ at $P(3, -2)$ is

$$y - (-2) = \frac{3}{2}(x - 3)$$

$$2(y + 2) = 3(x - 3)$$

$$3x - 2y - 13 = 0.$$

EXAMPLE 2. (a) Show that $P(7, 8)$ is an external point of the circle $x^2 + y^2 = 36$.

(b) Find the length of a tangent segment from P to the circle.

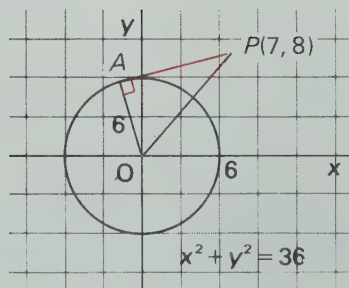
Solution

(a) We know from section 6.5 that $P(x, y)$ is an external point of the circle $x^2 + y^2 = 36$ if $x^2 + y^2 > 36$. For $P(7, 8)$ we have

$$7^2 + 8^2 = 49 + 64$$

$$= 113 > 36.$$

$\therefore P(7, 8)$ lies outside the circle $x^2 + y^2 = 36$.



(b) If A is the point of contact of a tangent from P , then $OA \perp AP$.

$$OP^2 = 7^2 + 8^2 = 113 \text{ from (a)}$$

$$OA^2 = 6^2 = 36$$

Using the Pythagorean Theorem, we have

$$AP^2 = OP^2 - OA^2$$

$$= 113 - 36$$

$$= 77$$

Therefore the length of a tangent segment from P is

$$AP = \sqrt{77}.$$

EXERCISE 10-6

B 1. Use the method of Example 1 to find an equation of the tangent to the given circle at the given point.

(a) $x^2 + y^2 = 25$, $A(3, 4)$

(b) $x^2 + y^2 = 1$, $B\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

(c) $x^2 + y^2 = 65$, $C(-4, -7)$

(d) $2x^2 + 2y^2 = 5$, $D\left(\sqrt{2}, -\frac{1}{\sqrt{2}}\right)$

(e) $x^2 + y^2 = 10$, $E(3, 1)$

(f) $x^2 + y^2 = r^2$, $P(x_1, y_1)$

2. Use the method of Example 2 to find the length of a tangent segment from the given point to the given circle.

(a) $A(2, 2)$, $x^2 + y^2 = 1$

(b) $B(0, \sqrt{5})$, $x^2 + y^2 = 1$

(c) $C(-5, 5\sqrt{3})$, $x^2 + y^2 = 36$

(d) $D(3, -7)$, $x^2 + y^2 = 14$

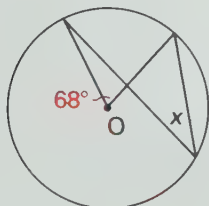
(e) $E(1, 1)$, $3x^2 + 3y^2 = 1$

(f) $P(x_1, y_1)$, $x^2 + y^2 = r^2$

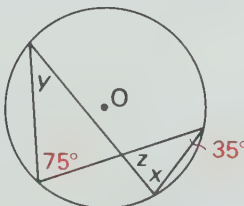
REVIEW EXERCISE

A 1. Find the values of x , y , z , t in the following diagrams. The centre of each circle is O .

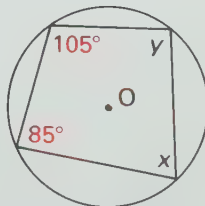
(a)



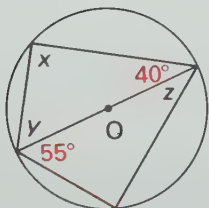
(b)



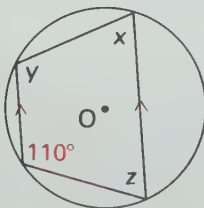
(c)



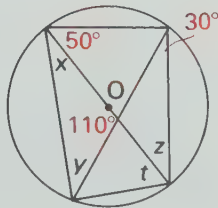
(d)



(e)

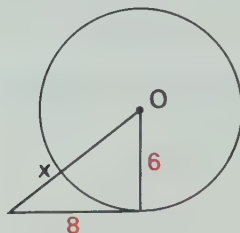


(f)

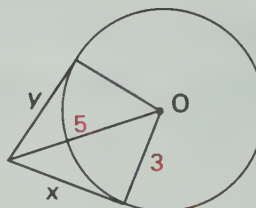


2. In the diagrams below tangents have been drawn to the circles. Find the values of x , y , z , t .

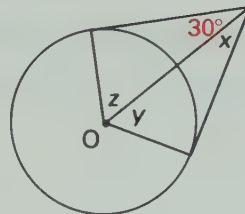
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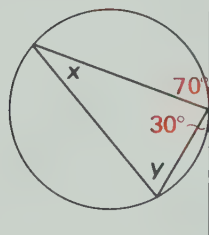
(b)



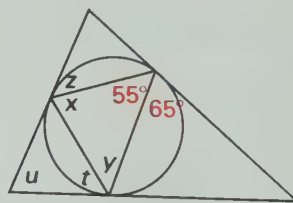
(c)



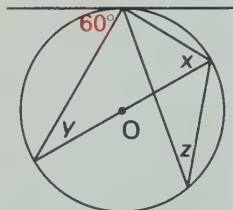
(d)



(e)



(f)



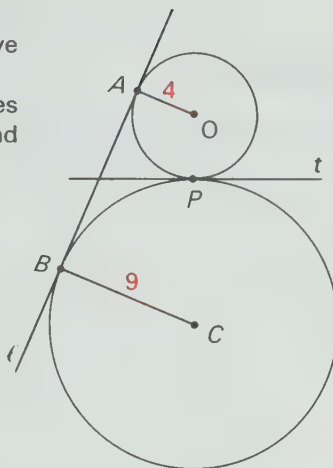
- B** 3. A circle with centre O has radius 6 cm. The distance from O to a chord is 3 cm. Find the length of the chord.
4. If AB and CD are chords of a circle and AB is closer to the centre than CD , prove that $AB > CD$.
5. O is the centre of a circle which passes through points A, B, C . If $\angle BOC = 84^\circ$ and $\angle ABC = 53^\circ$, find $\angle CAB$ and $\angle BCA$.
6. If $ABCD$ is a cyclic quadrilateral and $\angle ABC = 97^\circ$, $\angle CAD = 29^\circ$, $\angle ACB = 37^\circ$, find $\angle DBC$, $\angle ADC$, $\angle ADB$, $\angle CDB$, and $\angle ACD$.
7. In $\triangle ABC$, $AB = AC$. A circle with diameter AB intersects BC at C . Prove that D is the midpoint of BC .
8. AB and CD are perpendicular chords of a circle with centre O . They intersect at a point inside the circle. Prove that $\angle AOD$ and $\angle BOC$ are supplementary.
9. R, A, H are points on a circle with $RA = RH$. S is any point on the arc AH which is opposite to the arc ARH . Prove that $\angle ASR = \angle HSR$.
10. AB is a diameter of a circle and C is a third point on the circle. X lies on the line segment BC and Y lies on AB so that $XY \perp AB$. Prove that $AYXC$ is a cyclic quadrilateral.
11. $\triangle PSL$ is inscribed in a circle. G and U are points on PS and PL respectively so that GU is parallel to the tangent at P . Prove that $SLUG$ is a cyclic quadrilateral.
12. Two circles intersect at A and B . CAT and HAM are two lines through A which meet the first circle at C and H and the second circle at T and M . Prove that $\angle HBM = \angle CBT$.
13. If AB is the diameter of a circle, prove that the tangents at A and B are parallel.

14. Two circles lie on opposite sides of a common tangent t and have a common point of contact P .

(a) If the centres of the circles are O and C , prove that O, P, C are collinear.

(b) A second common tangent l meets the circles at A and B . If the radii of the circles are 4 cm and 9 cm, find the distance AB .

[Hint: Draw the perpendicular from O to BC .]



15. Tangents are drawn from a point P outside a circle with centre O meeting the circle at A and B .

(a) Prove that PO is the right bisector of the chord AB .

(b) Prove that $\angle BAO = \frac{1}{2}\angle APB$.

16. Points A, B, C, D lie in order on a circle such that $AD \parallel BC$. The tangent at A intersects the secant CB at X , and XD intersects the circle again at Y . Prove that $\angle XAY = \angle CXD$.

17. Two circles intersect at A and B . From a point P on one circle lines PAR and PBS are drawn meeting the other circle at R and S . Prove that the tangent at P is parallel to RS .

18. PR and KN are two chords of a circle. PR produced and KN produced meet at A . $PA = 8$ cm, $PR = 3$ cm, and $AN = 6$ cm.

(a) Find the length of the chord KN .

(b) Find the length of a tangent segment from A to the circle.

19. Find an equation of the tangent to the circle $x^2 + y^2 = 40$ at the point $P(6, -2)$.

20. Find the length of the tangent segment from the point $P(6, -2)$ to the circle $x^2 + y^2 = 16$.

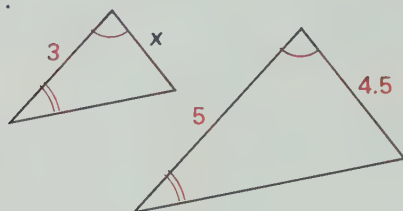
REVIEW AND PREVIEW TO CHAPTER 11

EXERCISE 1

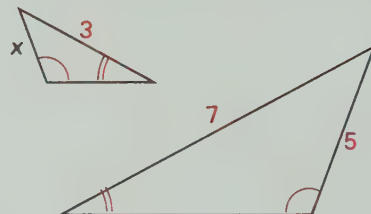
SIMILAR TRIANGLES

Find the length of the indicated side.

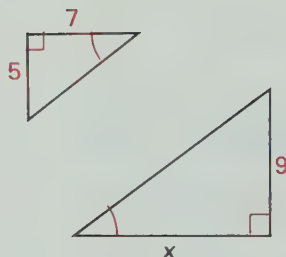
1.



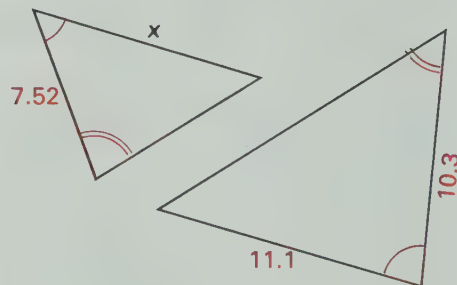
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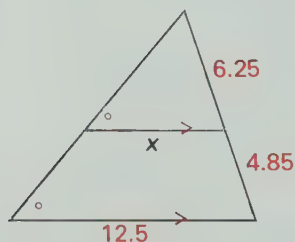
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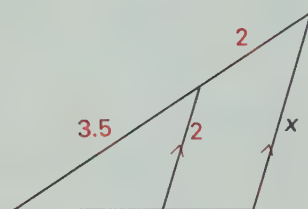
4.



5.



6.

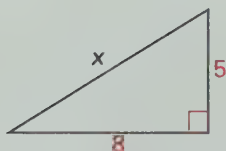


EXERCISE 2

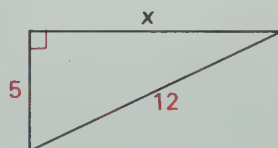
THE PYTHAGOREAN THEOREM

Find the length of the indicated side in the following.

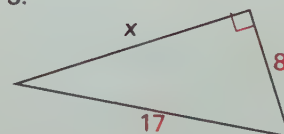
1.



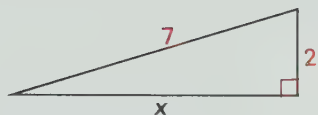
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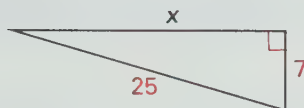
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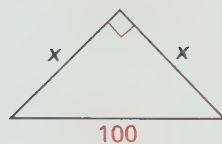
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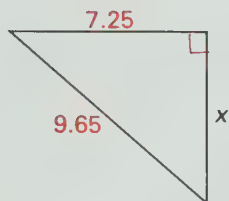
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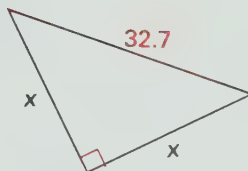
6.



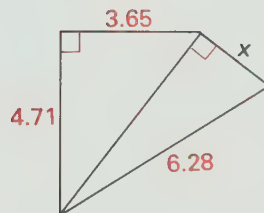
7.



8.



9.



EXERCISE 3

EQUATIONS

Solve the following equations.

1. $\frac{x}{7} = \frac{11}{35}$

2. $\frac{3x}{11} = \frac{25}{18}$

3. $\frac{5x}{38} = \frac{21}{18}$

4. $\frac{35}{11} = \frac{x}{5}$

5. $\frac{2}{3} = \frac{7}{x}$

6. $\frac{51}{5} = \frac{17}{3x}$

7. $\frac{x}{4.25} = \frac{16.8}{7.35}$

8. $\frac{1.2x}{5.37} = \frac{21.6}{18.5}$

9. $\frac{x}{1.25 \times 1.74} = \frac{5.28}{6.75}$

Trigonometry

During the second century B.C., the first trigonometric table apparently was compiled by the astronomer Hipparchus of Nicaea, who thus earned the right to be known as 'the father of trigonometry'.

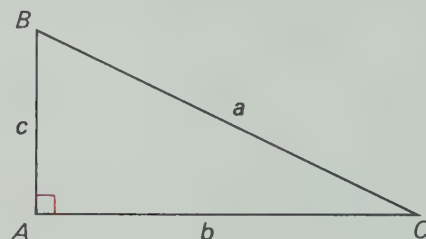
Carl Boyer

We have proved the following relationships for any right triangle.

In $\triangle ABC$, where $\angle A = 90^\circ$

$$(1) a^2 = b^2 + c^2$$

$$(2) \angle A = \angle B + \angle C$$



In *trigonometry* (from the Greek "trigonon" meaning triangle, and "metrikos" meaning measure), we relate the angles of a triangle to the sides using a new set of formulas.

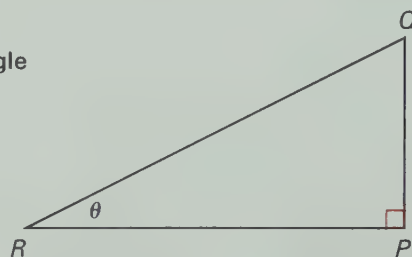
11.1 TRIGONOMETRIC RATIOS OF AN ACUTE ANGLE

If we have a triangle, with $\angle P = 90^\circ$, and we designate $\angle QRP = \theta$, then with reference to θ , the sides of $\triangle PQR$ are described as follows:

RQ is the *hypotenuse* of the triangle

QP is the *opposite* side to θ

RP is the *adjacent* side to θ



Naming the sides of the right triangle in this manner, we define the primary trigonometric ratios:

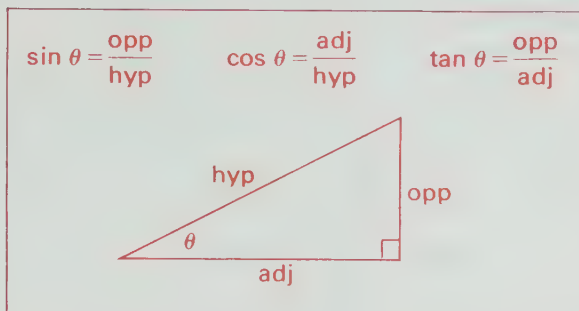
$$\text{sine of } \theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\text{cosine of } \theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\text{tangent of } \theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

Three men wish to cross a river in a boat that can only carry 150 kg. The men have masses of 50 kg, 75 kg, and 100 kg. How do they cross without exceeding the 150 kg capacity of the boat?

These definitions are abbreviated to:



We also define the *reciprocal* trigonometric ratios as follows:

$$\text{cosecant of } \theta = \frac{\text{hypotenuse}}{\text{opposite side}}$$

$$\text{secant of } \theta = \frac{\text{hypotenuse}}{\text{adjacent side}}$$

$$\text{cotangent of } \theta = \frac{\text{adjacent side}}{\text{opposite side}}$$

We abbreviate these to

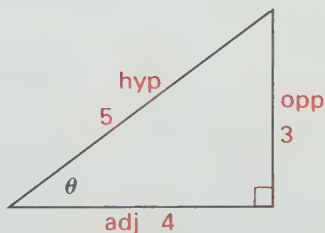
$$\csc \theta = \frac{\text{hyp}}{\text{opp}} \quad \sec \theta = \frac{\text{hyp}}{\text{adj}} \quad \cot \theta = \frac{\text{adj}}{\text{opp}}$$

EXAMPLE 1. Find the six trigonometric ratios of θ in the given triangle.

Solution

$$\sin \theta = \frac{3}{5} \quad \cos \theta = \frac{4}{5} \quad \tan \theta = \frac{3}{4}$$

$$\csc \theta = \frac{5}{3} \quad \sec \theta = \frac{5}{4} \quad \cot \theta = \frac{4}{3}$$



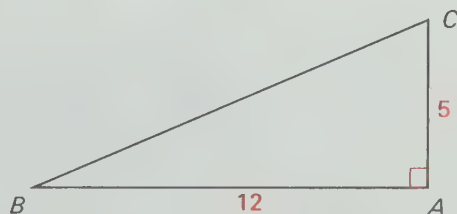
EXAMPLE 2. In $\triangle ABC$, $\angle A = 90^\circ$, $AB = 12$ cm, and $AC = 5$ cm. Find the six trigonometric ratios of the acute angles.

Solution

Using the Pythagorean theorem,

$$\begin{aligned} BC^2 &= AB^2 + AC^2 \\ &= 12^2 + 5^2 \\ &= 144 + 25 \\ &= 169 \end{aligned}$$

$$\therefore BC = 13$$



Taking $\angle B$ as the designated angle,

AC is the opposite side

AB is the adjacent side

$$\sin B = \frac{5}{13} \qquad \csc B = \frac{13}{5}$$

$$\cos B = \frac{12}{13} \qquad \sec B = \frac{13}{12}$$

$$\tan B = \frac{5}{12} \qquad \cot B = \frac{12}{5}$$

Taking $\angle C$ as the designated angle,

AB is the opposite side,

AC is the adjacent side.

$$\sin C = \frac{12}{13} \qquad \csc C = \frac{13}{12}$$

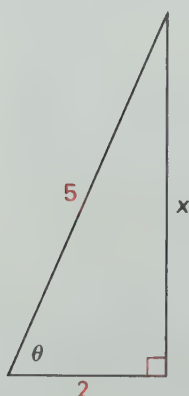
$$\cos C = \frac{5}{13} \qquad \sec C = \frac{13}{5}$$

$$\tan C = \frac{12}{5} \qquad \cot C = \frac{5}{12}$$

EXAMPLE 3. If $\cos \theta = \frac{2}{5}$, find the other five trigonometric ratios of the given angle.

Solution

We let the hypotenuse be 5 units in length, the adjacent side 2 units and the opposite side x units. Using the Pythagorean theorem,



$$x^2 + 2^2 = 5^2$$

$$x^2 = 21$$

$$x = \sqrt{21}$$

$$\sin \theta = \frac{\sqrt{21}}{5} \qquad \csc \theta = \frac{5}{\sqrt{21}}$$

$$\tan \theta = \frac{\sqrt{21}}{2} \qquad \sec \theta = \frac{5}{2}$$

$$\cot \theta = \frac{2}{\sqrt{21}}$$

The trigonometric ratios of an angle are independent of the lengths of the sides of the triangle.

Given $\triangle ABC$ with $\angle B = 90^\circ$, and $B'C' \parallel BC$ from $\triangle ABC$, we have defined

$$\sin A = \frac{BC}{AC}$$

and from $\triangle AB'C'$

$$\sin A = \frac{B'C'}{AC'}$$

Let us check to see if our definition is consistent.

In $\triangle AB'C'$ and $\triangle ABC$,

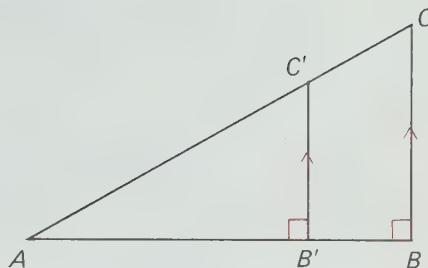
$$\angle AB'C' = \angle ABC = 90^\circ$$

$$\angle A \text{ is common}$$

$$\therefore \triangle AB'C' \sim \triangle ABC \quad (AA \sim)$$

$$\frac{B'C'}{AC'} = \frac{BC}{AC}$$

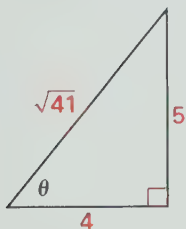
Hence $\sin A$ is independent of the lengths of the sides of the triangles. In a similar manner we can prove that the other five ratios are independent of the sides of the triangle.



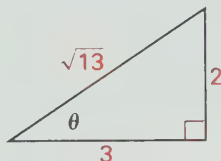
EXERCISE 11-1

A 1. State the six trigonometric ratios for each of the indicated angles:

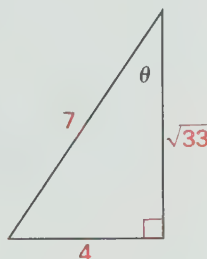
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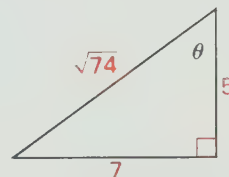
(b)



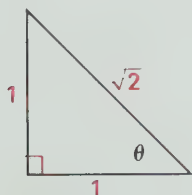
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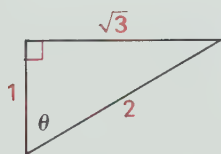
(d)



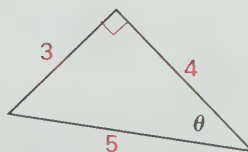
(e)



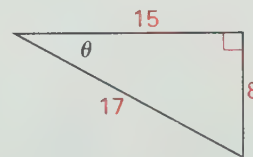
(f)



(g)

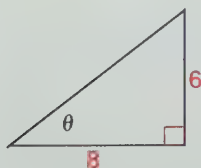


(h)

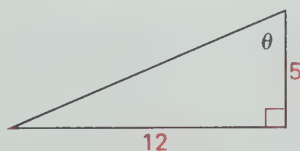


B 2. Find the length of the unknown side and then state the six trigonometric ratios of the indicated angle.

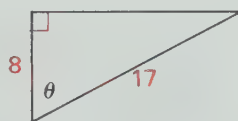
(a)



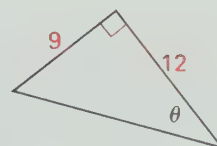
(b)



(c)



(d)

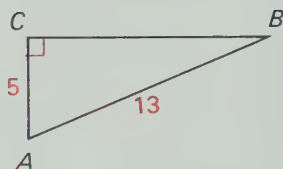


3. Find the length of the unknown side and state the six trigonometric ratios of each of the acute angles.

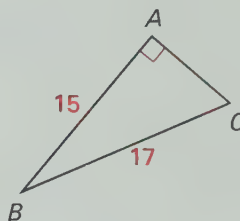
(a)



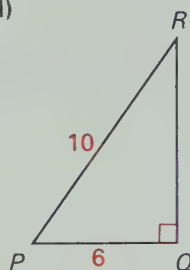
(b)



(c)



(d)



4. If $\sin \theta = \frac{1}{2}$, and $\angle \theta$ is acute, make a diagram, and find the other five ratios.

5. If $\sec \theta = \frac{13}{5}$, and $\angle \theta$ is acute, make a diagram, and find the other five ratios.

6. If $\tan \theta = 2$, and $\angle \theta$ is acute, make a diagram, and find the other five ratios.

7. If $\cos \theta = \frac{\sqrt{3}}{2}$, and $\angle \theta$ is acute, make a diagram, and find the other five ratios.

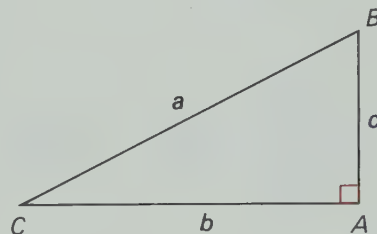
C 8. (a) Given $\triangle ABC$ with sides having lengths a, b, c as shown, find the six trigonometric ratios of $\angle B$ and $\angle C$ in terms of a, b and c .

(b) Noting that $\angle B + \angle C = 90^\circ$ find

(i) a ratio of $\angle B$ equal to $\sin C$.

(ii) a ratio of $\angle B$ equal to $\cos C$.

(iii) a ratio of $\angle B$ equal to $\tan C$.



$$a^2 = b^2 + c^2$$

9. In $\triangle ABC$, $\angle C = 90^\circ$, $a = 5$, $c = \sqrt{26}$.

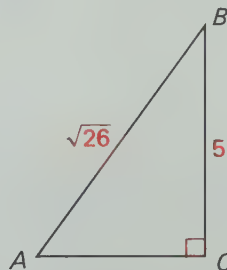
(a) Find the six trigonometric ratios of each of the acute angles A and B .

(b) Prove the following

(i) $\sin^2 A + \cos^2 A = 1$

(ii) $\sin^2 B + \cos^2 B = 1$

(iii) $\frac{\sin A}{\cos A} = \tan A$



10. The trigonometric ratios of an acute angle of a right triangle are independent of the lengths of the sides of the triangle. Prove this statement for

(a) $\cos \theta$

(b) $\tan \theta$

$$\sin^2 A = (\sin A)^2$$

11-2 TRIGONOMETRIC RATIOS OF SPECIAL ANGLES

If we consider the following special triangles we can find the trigonometric ratios of 30° , 45° and 60° . Since the trigonometric ratios of the acute angles of a right triangle are independent of the lengths of the sides, we choose convenient values as follows.

In an isosceles right triangle, $\triangle ABC$, let $AC = 1$, $BC = 1$ and $\angle C = 90^\circ$. From our work in geometry we can find

$$AB = \sqrt{2}, \quad \angle A = 45^\circ, \quad \angle B = 45^\circ$$

Then

$\sin 45^\circ = \frac{1}{\sqrt{2}}$	$\csc 45^\circ = \sqrt{2}$
$\cos 45^\circ = \frac{1}{\sqrt{2}}$	$\sec 45^\circ = \sqrt{2}$
$\tan 45^\circ = 1$	$\cot 45^\circ = 1$

In $\triangle ABD$, $AB = BD = DA = 2$. $AC \perp BD$, so that C bisects BD , making $BC = 1$, $\angle B = 60^\circ$, and $\angle BAC = 30^\circ$. We find $AC = \sqrt{3}$ using the Pythagorean Theorem.

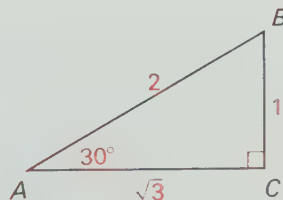
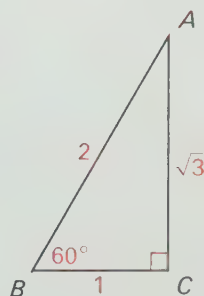
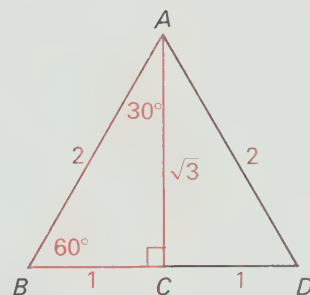
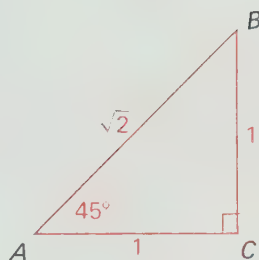
Using $\triangle ABC$, and $\angle ABC$:

Then

$\sin 60^\circ = \frac{\sqrt{3}}{2}$	$\csc 60^\circ = \frac{2}{\sqrt{3}}$
$\cos 60^\circ = \frac{1}{2}$	$\sec 60^\circ = \frac{2}{1}$
$\tan 60^\circ = \frac{\sqrt{3}}{1}$	$\cot 60^\circ = \frac{1}{\sqrt{3}}$

For $\angle BAC$:

$\sin 30^\circ = \frac{1}{2}$	$\csc 30^\circ = \frac{2}{1}$
$\cos 30^\circ = \frac{\sqrt{3}}{2}$	$\sec 30^\circ = \frac{2}{\sqrt{3}}$
$\tan 30^\circ = \frac{1}{\sqrt{3}}$	$\cot 30^\circ = \frac{\sqrt{3}}{1}$



EXAMPLE 1. Find the value of $\sin^2 45^\circ + 2 \sin 30^\circ \sec 60^\circ$

Solution

$$\begin{aligned}\sin^2 45^\circ + 2 \sin 30^\circ \sec 60^\circ &= \left(\frac{1}{\sqrt{2}}\right)^2 + 2\left(\frac{1}{2}\right)\left(\frac{2}{\sqrt{3}}\right) \\ &= \frac{1}{2} + \frac{2}{\sqrt{3}} \\ &= \frac{1}{2} + \frac{2\sqrt{3}}{3} \\ &= \frac{3+4\sqrt{3}}{6}\end{aligned}$$

Put the numbers from 1 to 9 in the spaces to make the statements true.

$$\square - \square + \square = 6$$

$$\square - \square - \square = 6$$

$$\square \div \square \times \square = 6$$

EXERCISE 11-2

B Find the value of

1. $\sin 30^\circ \sin 45^\circ \sin 60^\circ$

2. $\sin^2 30^\circ + \cos^2 30^\circ$

3. $\sin 30^\circ \cos 30^\circ + \sin 60^\circ \cos 60^\circ$

4. $\sin 60^\circ \cos 30^\circ + \sin 30^\circ \cos 60^\circ$

5. $2 \sin 30^\circ \cos 30^\circ$

6. $5 \sec 30^\circ \tan 60^\circ$

C Prove

7. $\sin^2 30^\circ + \cos^2 30^\circ = \sin^2 60^\circ + \cos^2 60^\circ$

8. $1 + \tan^2 45^\circ = \sec^2 45^\circ$

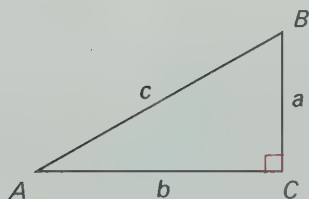
9. $\csc^2 60^\circ = 1 + \cot^2 60^\circ$

10. $\cos 60^\circ \sec 30^\circ = \tan 30^\circ$

11-3 TRIGONOMETRIC RATIOS OF ANY ACUTE ANGLE

In the previous section, we found ratios of 30° , 45° , and 60° using special triangles. We can find ratios of other acute angles by the accurate construction of many triangles, or from mathematical tables as found on pages 426–427. Tables of trigonometric ratios have been computed using formulas from more advanced mathematics. These same formulas are used in those calculators equipped with the trigonometric functions.

A calculator with the trigonometric functions is often limited to the primary trigonometric ratios. In order to use the machine for all ratios we must first establish the following relationships.



$$\frac{1}{\sin A} = \frac{1}{\frac{a}{c}} = \frac{c}{a} = \csc A$$

$$\frac{1}{\cos A} = \frac{1}{\frac{b}{c}} = \frac{c}{b} = \sec A$$

$$\frac{1}{\tan A} = \frac{1}{\frac{a}{b}} = \frac{b}{a} = \cot A$$

Hence we can find the reciprocal ratios using the following relationships

$$\csc \theta = \frac{1}{\sin \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\cot \theta = \frac{1}{\tan \theta}$$

EXERCISE 11-3

A 1. Read the following from tables on pages 426–427

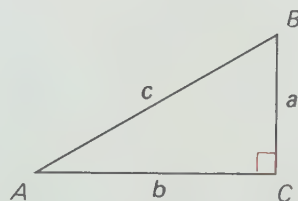
- | | | | |
|---------------------|---------------------|---------------------|---------------------|
| (a) $\sin 25^\circ$ | (b) $\cot 15^\circ$ | (c) $\tan 45^\circ$ | (d) $\sec 62^\circ$ |
| (e) $\csc 21^\circ$ | (f) $\cos 42^\circ$ | (g) $\csc 81^\circ$ | (h) $\cos 54^\circ$ |
| (i) $\sec 28^\circ$ | (j) $\sin 15^\circ$ | (k) $\tan 72^\circ$ | (l) $\cos 18^\circ$ |
| (m) $\cot 53^\circ$ | (n) $\csc 54^\circ$ | (o) $\sin 37^\circ$ | (p) $\tan 58^\circ$ |

2. Find the measure of the angle:

- | | | |
|----------------------------|----------------------------|----------------------------|
| (a) $\sin \theta = 0.9063$ | (b) $\csc \theta = 1.1547$ | (c) $\tan \theta = 2.246$ |
| (d) $\sin \theta = 0.4226$ | (e) $\cot \theta = 1.2349$ | (f) $\sec \theta = 1.0403$ |
| (g) $\cos \theta = 0.4226$ | (h) $\sec \theta = 5.2408$ | (i) $\cos \theta = 0.8192$ |
| (j) $\tan \theta = 0.7002$ | (k) $\csc \theta = 2.3662$ | (l) $\cot \theta = 0.8391$ |

11-4 SOLUTION OF RIGHT TRIANGLES

We say that a triangle is solved when we are able to state the measure of the three angles and the three sides. We facilitate this work by agreeing to label triangles so that the small letter a represents the length of the side opposite $\angle A$.



EXAMPLE 1. Given $\triangle ABC$, where $\angle A = 90^\circ$, $b = 5.2$ and $c = 4.7$, find the measure of the angles.

Solution

From the diagram

$$\begin{aligned}\tan B &= \frac{5.2}{4.7} \\ &\doteq 1.1064\end{aligned}$$

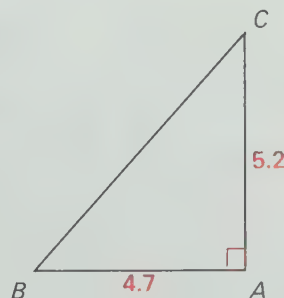
$$\angle B \doteq 48^\circ$$

$$\angle C = 90^\circ - \angle B$$

$$\doteq 90^\circ - 48^\circ$$

$$\doteq 42^\circ$$

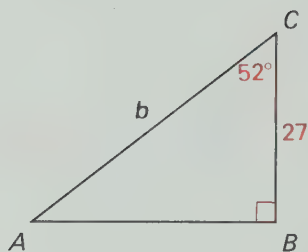
$$\therefore \angle B \doteq 48^\circ \text{ and } \angle C \doteq 42^\circ.$$



EXAMPLE 2. Given $\triangle ABC$, where $\angle B = 90^\circ$, $a = 27$ and $\angle C = 52^\circ$, find b .

Solution

From the diagram



$$\begin{aligned}\frac{b}{27} &= \sec 52^\circ \\ b &= 27 \sec 52^\circ \\ &\doteq 27(1.6243) \\ &\doteq 43.9 \\ \therefore b &\doteq 43.9\end{aligned}$$

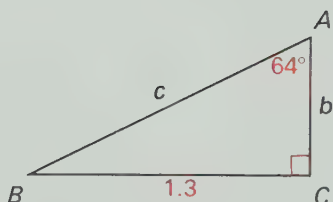
The length of a side is found using the relationship

$$\frac{\text{unknown}}{\text{known}} = \text{trigonometric ratio}$$

EXAMPLE 3. Solve $\triangle ABC$ if $\angle C = 90^\circ$, $a = 1.3$, and $\angle A = 64^\circ$.

Solution

From the diagram,



(i)

$$\begin{aligned}\angle B &= 90^\circ - \angle A \\ &= 90^\circ - 64^\circ \\ &= 26^\circ\end{aligned}$$

(ii)

$$\begin{aligned}\frac{b}{1.3} &= \cot 64^\circ \\ b &= 1.3 \cot 64^\circ \\ &\doteq 1.3(0.4877) \\ &\doteq 0.634\end{aligned}$$

(iii)

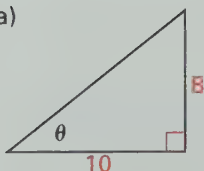
$$\begin{aligned}\frac{c}{1.3} &= \csc 64^\circ \\ c &= 1.3 \csc 64^\circ \\ &\doteq 1.3(1.1126) \\ &\doteq 1.446\end{aligned}$$

$$\therefore \angle B = 26^\circ, \quad b \doteq 0.634, \quad \text{and} \quad c \doteq 1.45$$

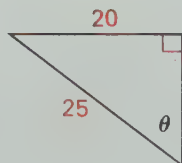
EXERCISE 11-4

B 1. Find the measure of the indicated angle.

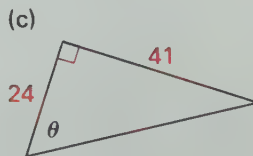
(a)



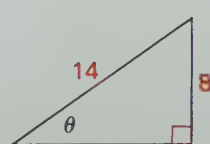
(b)



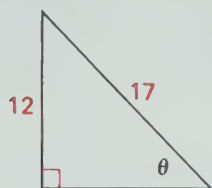
(c)



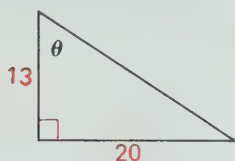
(d)



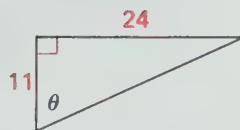
(e)



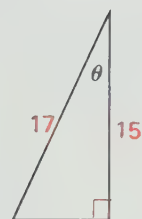
(f)



(g)

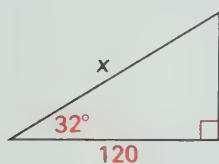


(h)

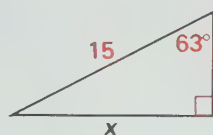


2. Find the length of the indicated side.

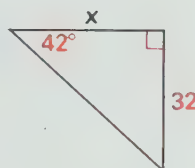
(a)



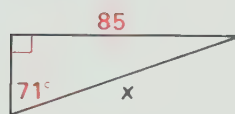
(b)



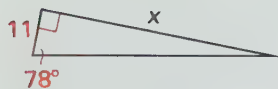
(c)



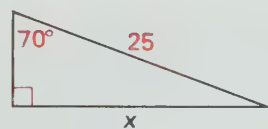
(d)



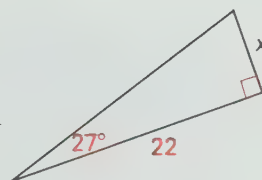
(e)



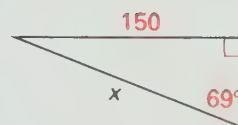
(f)



(g)

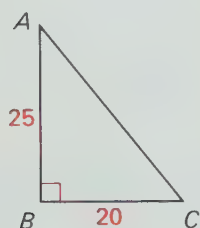


(h)

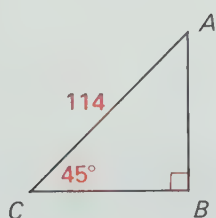


3. Solve the following triangles.

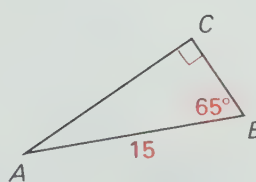
(a)



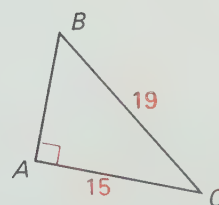
(b)



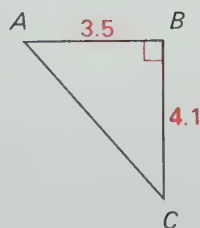
(c)



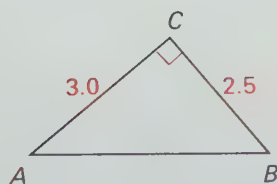
(d)



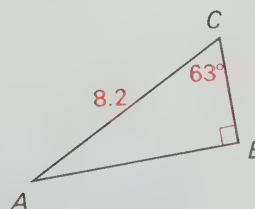
(e)



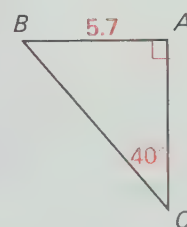
(f)



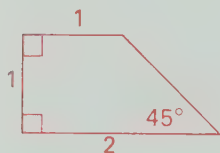
(g)



(h)



Divide this figure into four equal pieces that have the same shapes as the original figure.



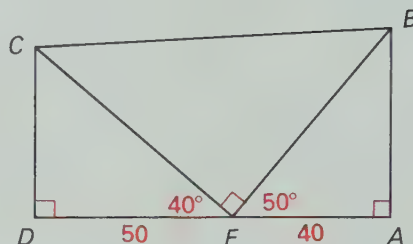
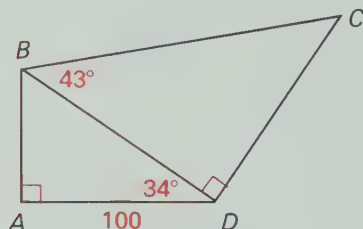
4. Solve the following triangles.

- (a) $\triangle ABC$, $\angle A = 90^\circ$, $a = 15$, $b = 11$
- (b) $\triangle DEF$, $\angle E = 90^\circ$, $\angle D = 35^\circ$, $f = 100$.
- (c) $\triangle GHI$, $\angle I = 90^\circ$, $\angle H = 80^\circ$, $i = 120$.
- (d) $\triangle JKL$, $\angle K = 90^\circ$, $j = 50$, $k = 60$.

5. Solve the following triangles.

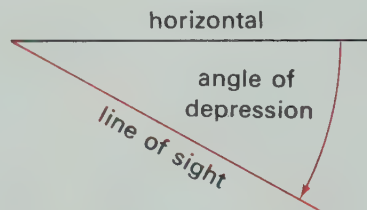
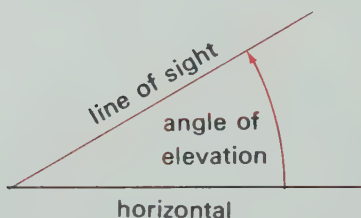
- (a) $\triangle ABC$, $\angle A = 90^\circ$, $a = 3.75$, $b = 2.15$
- (b) $\triangle ABC$, $\angle B = 90^\circ$, $a = 21.6$, $c = 32.7$
- (c) $\triangle ABC$, $\angle A = 90^\circ$, $\angle B = 38^\circ$, $c = 4.35$
- (d) $\triangle ABC$, $\angle C = 90^\circ$, $\angle A = 67^\circ$, $c = 10.7$

C 6. Find the length of BC .



11.5 APPLICATIONS OF RIGHT TRIANGLES

In this section we will apply some of the computational skills developed earlier to real world situations. It is important to make a reasonably accurate diagram, identify the proper angles and use the correct ratios. The following diagrams help to describe some of the terminology that will be used.



EXAMPLE 1. Find the height of a tree that casts a shadow 25.7 m long when the angle of elevation of the sun is 51° .

Solution

Let the height of the tree be h metres.

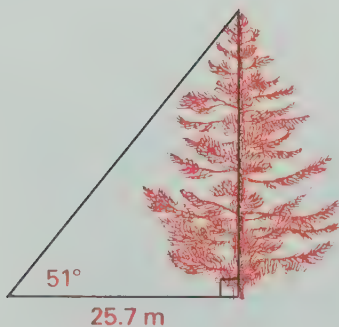
$$\frac{h}{25.7} = \tan 51^\circ$$

$$h = 25.7 \tan 51^\circ$$

$$\approx 25.7(1.235)$$

$$\approx 31.7$$

$\therefore$ the height of the tree is 31.7 m.



EXAMPLE 2. From opposite ends of the Simplon Tunnel (started in 1898) between Italy and Switzerland the angles of elevation of the peak are 14° and 28° . If the peak is 2130 m above the tunnel, find the length of the tunnel.

Solution

The length of the tunnel is given by $(x + y)$ metres as shown in the diagram.

In $\triangle ABD$,

$$\frac{x}{2130} = \cot 14^\circ$$

$$x = 2130 \cot 14^\circ$$

$$\doteq 2130(4.011)$$

$$\doteq 8543$$

$$x + y \doteq 8543 + 4007$$

$$\doteq 12\,550 \text{ m.}$$

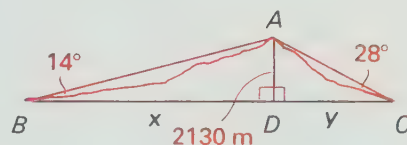
In $\triangle ACD$,

$$\frac{y}{2130} = \cot 28^\circ$$

$$y = 2130 \cot 28^\circ$$

$$= 2130(1.881)$$

$$= 4007$$



$\therefore$ the length of the tunnel is approximately 12 550 m.

EXAMPLE 3. From two tracking stations, A and B, 450 km apart a satellite is sighted at C, directly above the line joining A to B. At the moment of sighting, $\angle CAB = 39^\circ$ and $\angle CBA = 45^\circ$. Find the height of the satellite.

Solution

Let the height of the satellite be h km.

From the diagram

In $\triangle CAD$,

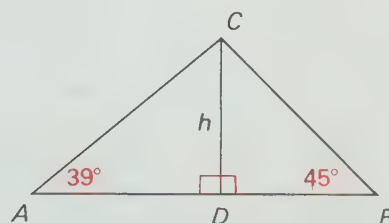
$$\frac{AD}{h} = \cot 39^\circ$$

$$AD = h \cot 39^\circ$$

In $\triangle CBD$,

$$\frac{BD}{h} = \cot 45^\circ$$

$$BD = h \cot 45^\circ$$



Adding

$$AD + BD = h \cot 39^\circ + h \cot 45^\circ$$

$$450 = h(\cot 39^\circ + \cot 45^\circ)$$

$$\doteq h(1.235 + 1.000)$$

$$\doteq h(2.235)$$

$$\frac{450}{2.235} \doteq h$$

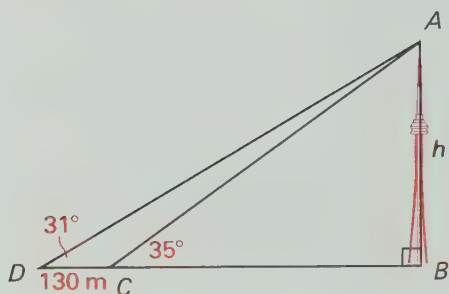
$$201 \doteq h$$

$\therefore$ the height of the satellite is approximately 201 km.

EXAMPLE 4. From a point on level ground the angle of elevation of the top of the CN tower is measured and found to be 31° . From a point 130 m closer the angle of elevation is 35° . Find the height of the tower.

Solution

Let the height of the tower be h metres.
Label the diagram as shown.



Copyright
CN Tower Ltd.

In $\triangle ABD$,

$$\frac{BD}{h} = \cot 31^\circ$$

$$BD = h \cot 31^\circ$$

In $\triangle ABC$,

$$\frac{BC}{h} = \cot 35^\circ$$

$$BC = h \cot 35^\circ$$

$$CD = BD - BC$$

$$130 = h \cot 31^\circ - h \cot 35^\circ$$

$$= h(\cot 31^\circ - \cot 35^\circ)$$

$$\doteq h(1.664 - 1.428)$$

$$\doteq h(0.236)$$

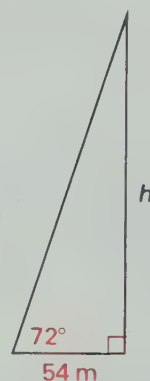
$$\frac{130}{0.236} \doteq h$$

$$551 \doteq h$$

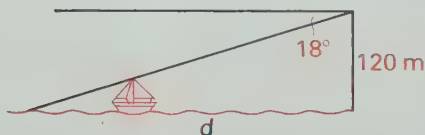
$\therefore$ the CN tower is approximately 551 m high.

EXERCISE 11-5

- B** 1. The angle of elevation of the top of a building is 72° from a point 54 m from the foot of the building. Find the height of the building.



2. A tower 115 m high casts a shadow 24 m long. Find the angle of elevation of the sun.



3. From the top of a cliff 120 m above the water, the angle of depression of a boat on the water is 18° . How far is the boat from the cliff?

4. A 150 m tower is to be secured using four guy wires each making an angle of 55° with the ground. Find the length of wire required for each line, allowing 4 m for fastening.

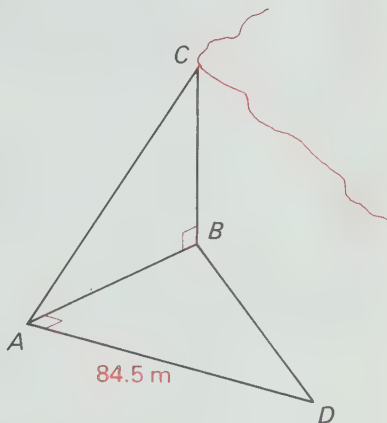
5. Kelly lets out 175 m of kite string, then estimates that the angle of elevation of the kite is 30° . What is the height of the kite, if Kelly holds the string 1.2 m above the ground?



6. A railroad underpass drops 6 m and a safe angle of descent is 6° . Find the length of the approach.

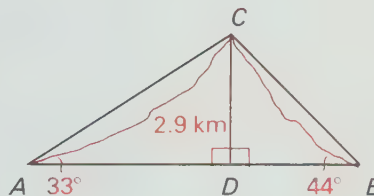
7. What angle will a 72 m guy wire make with the ground if it secures a tower from a point 45 m up the tower?
8. A tree casts a 23 m shadow when the angle of elevation of the sun is 52° .
- Find the height of the tree.
 - Find the length of the shadow when the angle of elevation of the sun is 38° .

9. A surveyor wishes to find the height of an inaccessible cliff BC . To do this he sets up his transit at A , measures $\angle CAB$, lays off a baseline AD perpendicular to AB , then measures $\angle ADB$. He records the following data: $\angle CAB = 62^\circ$, $\angle ADB = 51^\circ$, $AD = 84.5$ m. Find the height of the cliff.

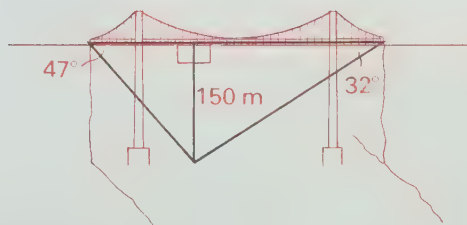


10. From a window of one building, the angles of elevation of the top and depression of the bottom are respectively 38° and 51° . Find the height of the second building if the buildings are 42 m apart.

11. The peak of Mont Blanc, the highest of the Alps, is 4.8 km above sea level. If the angles of elevation from points A and B are 33° and 44° as in the diagram, calculate the length of AB the distance through the mountain at an elevation of 1.9 km.



12. A bridge is 150 m above the water. From the ends of the bridge, the angles of depression of a buoy moored in the water directly below the bridge are 32° and 47° . Find the length of the bridge.



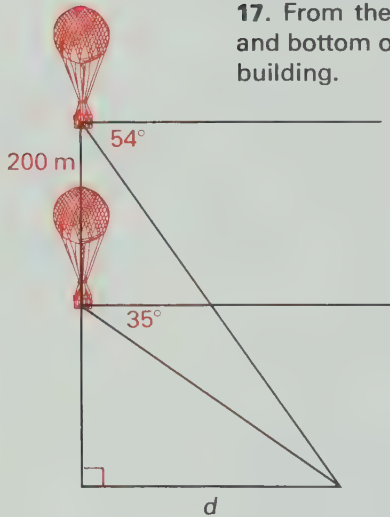
13. From a point 120 m from the foot of a building, the angles of elevation of the top and bottom of the flagpole are 49° and 46° respectively. Find the height of
- the building
 - the flagpole.

14. From two tracking stations A and B , 350 km apart, a UFO is sighted at C above AB , making $\angle CAB = 32^\circ$, $\angle CBA = 54^\circ$. Find the height of the UFO.

15. From a boat on the water the angle of elevation of the top of a cliff is 31° . From a point 300 m closer to the cliff, the angle of elevation is 33° . Find the height of the cliff.

16. From the top of a cliff 185 m high, the angles of depression of two channel buoys in the same line of sight on the water are 13° and 15° . How far apart are the buoys?

17. From the roof of a building, the angles of depression of the top and bottom of a 10 m utility pole are 33° and 52° . Find the height of the building.



18. From a balloon, the angle of depression of a ground marker is 35° . From a point 200 m higher, the angle of depression is 54° . Find the distance from the marker to the point on the ground directly beneath the balloon.

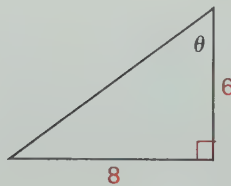
19. A person in a balloon finds the angle of depression of a bonfire is 29° , when the balloon is 160 m above the level ground. After further vertical ascent, the angle of depression is 34° . How much higher is the second position than the first?

20. From a window of one building, the angle of elevation of the top of another building is 38° and the angle of depression of the base of the building is 49° . Find the distance between the buildings if the second building is 154 m high.

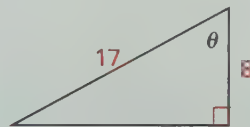
REVIEW EXERCISE

1. Find the six trigonometric ratios of the indicated angle.

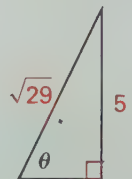
(a)



(b)



(c)



2. If $\csc \theta = \frac{13}{12}$, and $\angle \theta$ is acute, make a diagram and find the other five ratios.

3. Without using tables, find the six trigonometric ratios of
(i) 45° (ii) 30° (iii) 60°

then evaluate

(a) $\sin^2 45^\circ + \cos^2 45^\circ$

(b) $\sin^2 30^\circ + \sin^2 60^\circ$

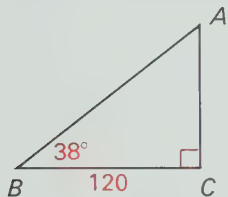
(c) $\frac{\sin 30^\circ + \sin 60^\circ}{\sin 45^\circ}$

(d) $2 \tan 30^\circ \cot 60^\circ$

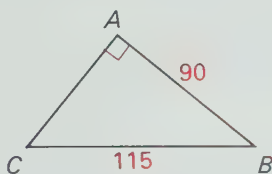
(e) $\sin 30^\circ \cdot \cos 60^\circ \cdot \tan 45^\circ$

4. Solve the following triangles.

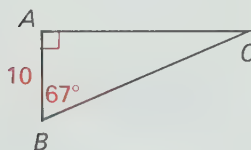
(a)



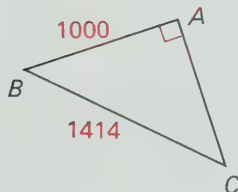
(b)



(c)



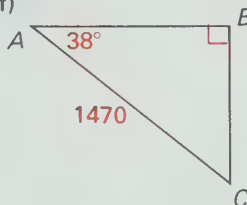
(d)



(e)



(f)



5. A tree casts a shadow 120 m long when the angle of elevation of the sun is 34° . Find the height of the tree.

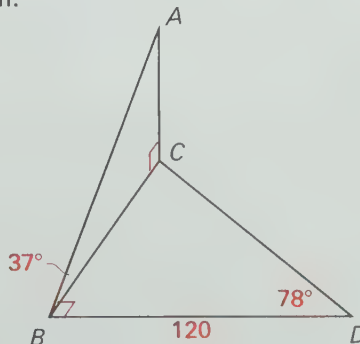
6. The angle of elevation of the top of a 125 m cliff from a boat on the water is 18° . How far is the boat from the foot of the cliff?

7. From opposite ends of a tunnel through a mountain, the angles of elevation of the peak are 37° and 49° . Find the length of the tunnel if the peak is 2750 m above the tunnel.

8. From the top of a cliff 200 m high, the angles of depression of two boats on the water are 20° and 28° . How far apart are the boats?

9. From a point on the water, the angle of elevation of the top of a cliff is measured and found to be 30° . From a point 350 m further out, the angle of elevation is 27° . Find the height of the cliff.

10. In $\triangle ABC$, $\angle ACB = 90^\circ$ and $\angle ABC = 37^\circ$. In $\triangle BCD$, $\angle CBD = 90^\circ$, $BD = 120$ m and $\angle BDC = 78^\circ$. Find the length of AC .



REVIEW AND PREVIEW TO CHAPTER 12

$$d = \sqrt{x^2 + y^2}$$

EXERCISE 1

DISTANCE FROM THE ORIGIN

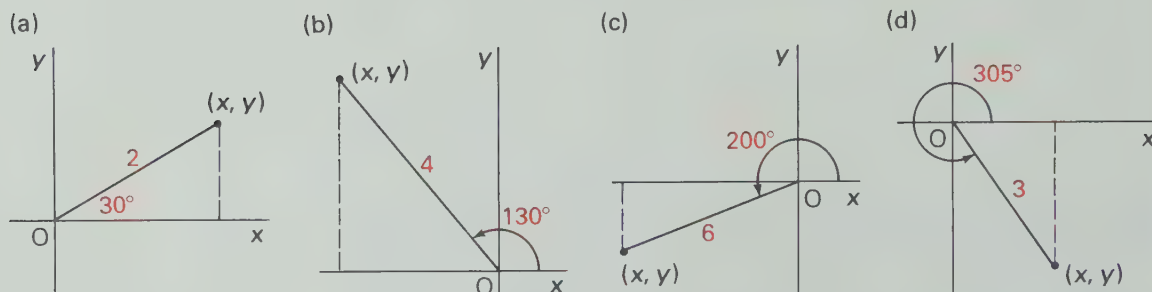
Find the distance from the origin to the given point.

- | | | | |
|------------|-------------|-------------|--------------|
| 1. (3, 0) | 2. (-6, 8) | 3. (5, -12) | 4. (-7, -24) |
| 5. (-3, 7) | 6. (5, -2) | 7. (8, -15) | 8. (5, -7) |
| 9. (-7, 5) | 10. (3, 11) | | |

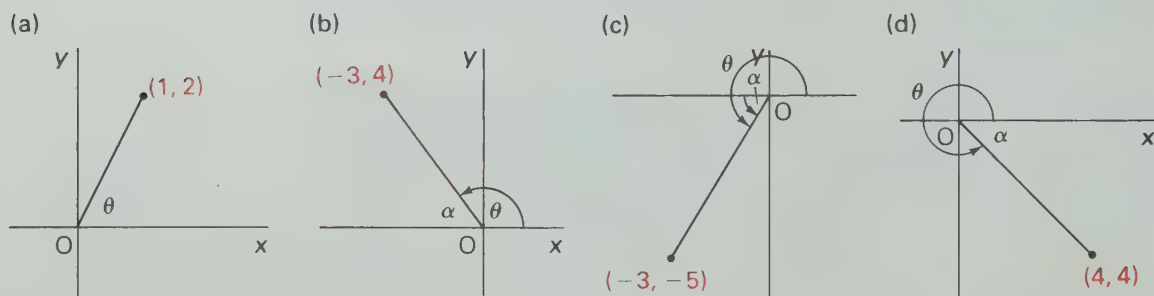
EXERCISE 2

PRIMARY TRIGONOMETRIC RATIOS

1. Find the coordinates of each point indicated.



2. Find the angles.



Trigonometric Functions

Trigonometry contains the science of continually undulating magnitude: meaning magnitude which becomes alternately greater and less.

Augustus De Morgan

12.1 TRIGONOMETRIC RATIOS OF ANY ANGLE

An angle is in *standard position* if it is related to a pair of coordinate axes so that its vertex is at the origin and the initial arm lies along the positive x-axis as in Figure 12-1.

$P(x, y)$ is any point on the terminal arm of $\angle POX$ in standard position, and $OP = r$. If we draw $PM \perp OX$, then $\triangle POM$ is formed with $\angle OMP = 90^\circ$. The length OP is found using

$$r = \sqrt{x^2 + y^2}$$

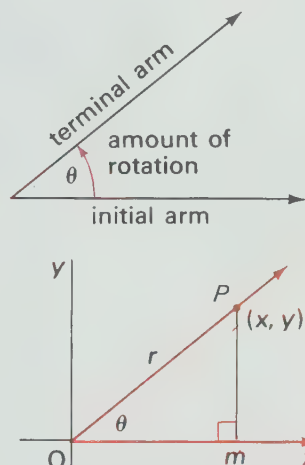


Figure 12-1

While x and y may be positive or negative, r is always positive. For an angle θ in standard position the trigonometric ratios can be defined in terms of x , y , and r .

$$\begin{aligned} \sin \theta &= \frac{y}{r} & \cos \theta &= \frac{x}{r} & \tan \theta &= \frac{y}{x} & x \neq 0 \\ \csc \theta &= \frac{r}{y}, \quad y \neq 0, & \sec \theta &= \frac{r}{x}, \quad x \neq 0, & \cot \theta &= \frac{x}{y}, \quad y \neq 0 \end{aligned}$$

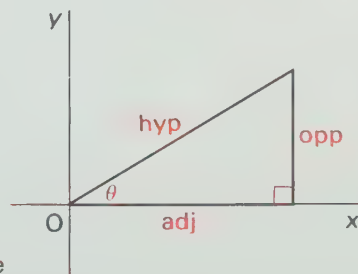


Figure 12-2

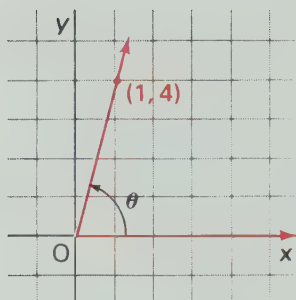
These definitions are consistent with those in chapter 11, for acute angles as shown in Figure 12-2.

EXAMPLE 1. The point $(1, 4)$ lies on the terminal arm of θ in standard position. Find the six trigonometric ratios.

Solution

$$x = 1, y = 4$$

$$r = \sqrt{x^2 + y^2}$$



$$r = \sqrt{1^2 + 4^2}$$

$$= \sqrt{17}$$

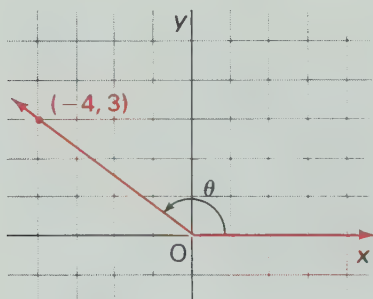
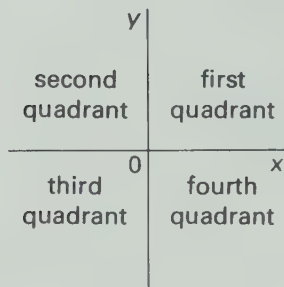
$$\sin \theta = \frac{4}{\sqrt{17}} \quad \cos \theta = \frac{1}{\sqrt{17}} \quad \tan \theta = \frac{4}{1}$$

$$\csc \theta = \frac{\sqrt{17}}{4} \quad \sec \theta = \frac{\sqrt{17}}{1} \quad \cot \theta = \frac{1}{4}$$

EXAMPLE 2. If θ is a second quadrant angle and $\tan \theta = -\frac{3}{4}$, find the other five trigonometric ratios.

Solution

Since θ is a second quadrant angle and $\tan \theta = -\frac{3}{4}$, $x = -4$ and $y = 3$ is one pair of values for x and y .



$$r = \sqrt{x^2 + y^2}$$

$$= \sqrt{(-4)^2 + 3^2}$$

$$= \sqrt{16 + 9}$$

$$= \sqrt{25}$$

$$= 5$$

$$\sin \theta = \frac{3}{5} \quad \csc \theta = \frac{5}{3}$$

$$\cos \theta = -\frac{4}{5} \quad \sec \theta = -\frac{5}{4}$$

$$\cot \theta = -\frac{4}{3}$$

EXAMPLE 3. If $\sin \theta = -\frac{3}{\sqrt{34}}$, find the other five ratios.

Solution:

Since $\sin \theta = -\frac{3}{\sqrt{34}}$, $y = -3$ and $r = \sqrt{34}$ is one set of values.

$$r = \sqrt{x^2 + y^2}$$

$$\sqrt{34} = \sqrt{x^2 + (-3)^2}$$

$$34 = x^2 + (-3)^2$$

$$34 = x^2 + 9$$

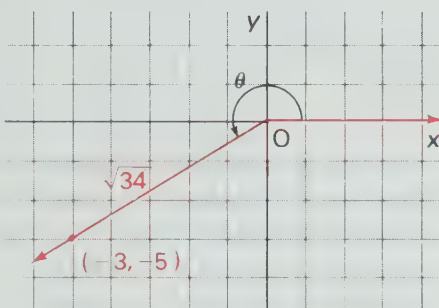
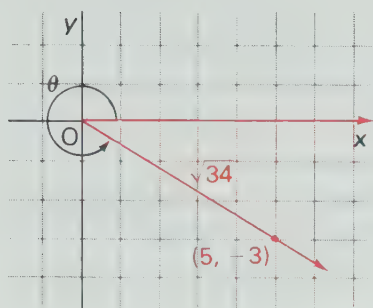
$$25 = x^2$$

$$x = 5 \quad \text{or} \quad x = -5$$

Since it is not known whether x is positive or negative, we continue the example by considering both cases.

Case 1 $x = 5$

Case 2 $x = -5$



$$\cos \theta = \frac{5}{\sqrt{34}}$$

$$\tan \theta = -\frac{3}{5}$$

$$\csc \theta = -\frac{\sqrt{34}}{3}$$

$$\sec \theta = \frac{\sqrt{34}}{5}$$

$$\cot \theta = -\frac{5}{3}$$

$$\cos \theta = -\frac{5}{\sqrt{34}}$$

$$\tan \theta = \frac{3}{5}$$

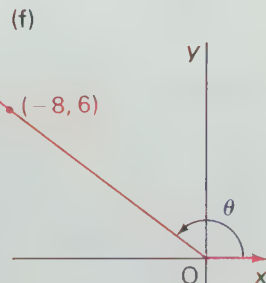
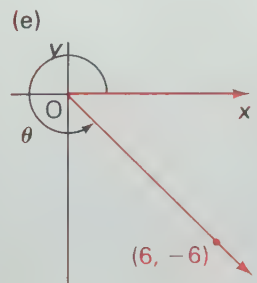
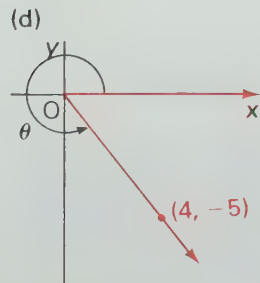
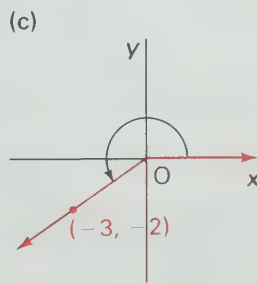
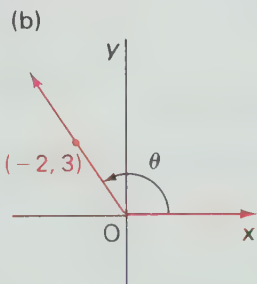
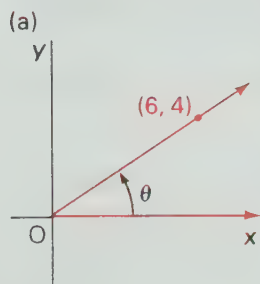
$$\csc \theta = -\frac{\sqrt{34}}{3}$$

$$\sec \theta = -\frac{\sqrt{34}}{5}$$

$$\cot \theta = \frac{5}{3}$$

EXERCISE 12-1

A 1. State the value of r and the six trigonometric ratios for each of the following.



WORD LADDER

Start with the word
"wheat" and change one
letter at a time to form a
new word until you reach
"bread". The best solution
has the fewest steps.

w h e a t

- - - - -

- - - - -

- - - - -

- - - - -

- - - - -

- - - - -

b r e a d

- B** 2. Make a diagram, calculate the value of r , and state the six trigonometric ratios of the angle in standard position if the terminal arm passes through:

- (a) (6, 8) (b) (-5, 12) (c) (-15, -8) (d) (24, -7)
(e) (5, 9) (f) (3, -7) (g) (-4, 7) (h) (-5, -2)

3. If θ is a third quadrant angle, and $\sin \theta = -\frac{3}{5}$, state the coordinates of a point on the terminal arm and find $\tan \theta$.

4. If θ is a fourth quadrant angle and $\tan \theta = -1$, state the coordinates of a point on the terminal arm, then find $\sin \theta$ and $\sec \theta$.

5. If $\sin \theta = \frac{5}{13}$ where θ is an angle in standard position,

- (a) calculate two possible values of x
(b) make a diagram for each possibility
(c) state the other trigonometric ratios.

6. The point (3, 4) lies on the terminal arm of angle θ in standard position.

- (a) State the coordinates of three other points on the terminal arm.
(b) Calculate r for each of the points.
(c) Complete the following table.

(x, y)	A(3, 4)	B(6, ■)	C(■, 12)	D(■, ■)
r	$\sqrt{3^2 + 4^2} = 5$			
$\sin \theta$				
$\cos \theta$				
$\tan \theta$				
$\csc \theta$				
$\sec \theta$				
$\cot \theta$				

Note that the values of the trigonometric ratios do not depend upon the point used on the terminal arm.

- C** 7. If $\tan \theta = \frac{b}{a}$ and θ is a first quadrant angle, find:

- (a) $\sin \theta$ and $\cos \theta$
(b) $\sin^2 \theta + \cos^2 \theta$
(c) $\frac{\sin \theta}{\cos \theta}$

12.2 USE OF TABLES FOR ANGLES GREATER THAN 90°

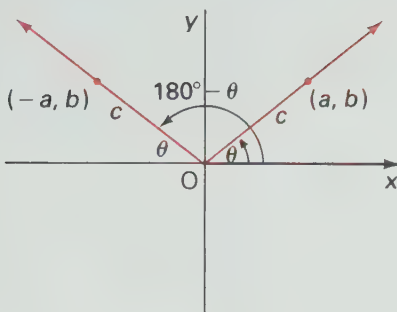
Most tables, including those in this book, contain values of the trigonometric ratios for angles from 0° to 90°. In this section, we shall relate the ratios of angles greater than 90° to those of the acute angles in the tables.

$$\frac{b}{c} = \sin \theta \quad \frac{a}{c} = \cos \theta \quad \frac{b}{a} = \tan \theta$$

$$\sin(180^\circ - \theta) = \frac{b}{c} = \sin \theta$$

$$\cos(180^\circ - \theta) = -\frac{a}{c} = -\cos \theta$$

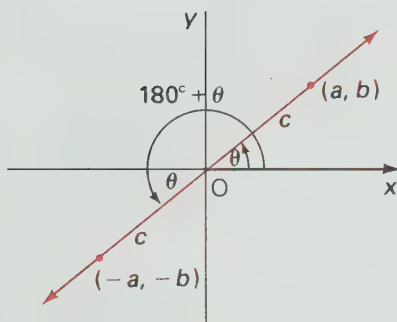
$$\tan(180^\circ - \theta) = -\frac{b}{a} = -\tan \theta$$



$$\sin(180^\circ + \theta) = -\frac{b}{c} = -\sin \theta$$

$$\cos(180^\circ + \theta) = -\frac{a}{c} = -\cos \theta$$

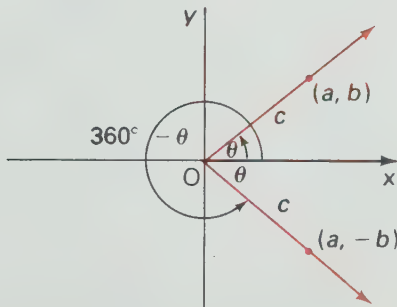
$$\tan(180^\circ + \theta) = \frac{b}{a} = \tan \theta$$



$$\sin(360^\circ - \theta) = -\frac{b}{c} = -\sin \theta$$

$$\cos(360^\circ - \theta) = \frac{a}{c} = \cos \theta$$

$$\tan(360^\circ - \theta) = -\frac{b}{a} = -\tan \theta$$



EXAMPLE 1. Evaluate the following using tables.

- | | | |
|----------------------|----------------------|----------------------|
| (a) $\sin 127^\circ$ | (b) $\cos 154^\circ$ | (c) $\tan 128^\circ$ |
| (d) $\tan 245^\circ$ | (e) $\cos 251^\circ$ | (f) $\cos 322^\circ$ |
| (g) $\sin 215^\circ$ | | |

Solution

Using the relationships described above:

- $\sin 127^\circ = \sin(180^\circ - 127^\circ) = \sin 53^\circ \doteq 0.7986$
- $\cos 154^\circ = -\cos(180^\circ - 154^\circ) = -\cos 26^\circ \doteq -0.8988$
- $\tan 128^\circ = -\tan(180^\circ - 128^\circ) = -\tan 52^\circ \doteq -1.2799$
- $\tan 245^\circ = \tan(180^\circ + 65^\circ) = \tan 65^\circ \doteq 2.1445$
- $\cos 251^\circ = \cos(180^\circ + 71^\circ) = -\cos 71^\circ \doteq -0.3256$
- $\cos 322^\circ = \cos(360^\circ - 322^\circ) = \cos 38^\circ \doteq 0.7880$
- $\sin 215^\circ = \sin(180^\circ + 35^\circ) = -\sin 35^\circ \doteq -0.5736$

For angles in the first quadrant, all trigonometric ratios are positive. The sine is the only primary trigonometric ratio that is positive in the second quadrant. In the third quadrant the tangent is positive, and the cosine is positive in the fourth quadrant. These results are helpful in checking the sign (+or-) of the ratio for angles greater than 90° , and are remembered using the CAST rule.

2	1
3	4

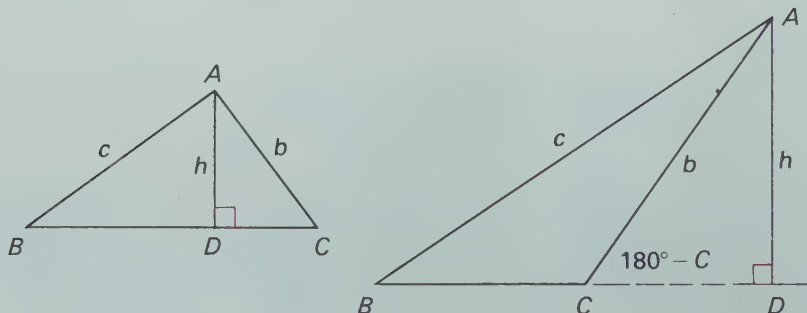
S sine is positive	A all ratios positive
T tangent is positive	C cosine is positive

EXERCISE 12-2

- B** 1. Find the value of
- | | | | |
|----------------------|----------------------|----------------------|----------------------|
| (a) $\sin 150^\circ$ | (b) $\tan 95^\circ$ | (c) $\sin 110^\circ$ | (d) $\cos 160^\circ$ |
| (e) $\cos 130^\circ$ | (f) $\cos 260^\circ$ | (g) $\tan 125^\circ$ | (h) $\sin 155^\circ$ |
| (i) $\cos 120^\circ$ | (j) $\sin 170^\circ$ | (k) $\tan 145^\circ$ | (l) $\cos 100^\circ$ |
| (m) $\tan 100^\circ$ | (n) $\cos 140^\circ$ | (o) $\cos 135^\circ$ | |
2. Find
- | | | | |
|----------------------|----------------------|----------------------|----------------------|
| (a) $\cos 132^\circ$ | (b) $\sin 215^\circ$ | (c) $\cos 247^\circ$ | (d) $\tan 315^\circ$ |
| (e) $\cos 173^\circ$ | (f) $\tan 318^\circ$ | (g) $\sin 218^\circ$ | (h) $\cos 220^\circ$ |
| (i) $\tan 225^\circ$ | (j) $\cos 210^\circ$ | (k) $\sin 245^\circ$ | (l) $\sin 290^\circ$ |
| (m) $\tan 253^\circ$ | (n) $\cos 318^\circ$ | (o) $\sin 340^\circ$ | |
- C** 3. Find two values for θ in each of the following, $0^\circ \leq \theta \leq 360^\circ$.
- | | | |
|-----------------------------|-----------------------------|-----------------------------|
| (a) $\sin \theta = 0.8746$ | (b) $\tan \theta = 1.1917$ | (c) $\sin \theta = -0.3420$ |
| (d) $\cos \theta = -0.6018$ | (e) $\tan \theta = -1.0000$ | (f) $\cos \theta = 0.8660$ |
| (g) $\sin \theta = 0.9903$ | (h) $\tan \theta = -5.6713$ | (i) $\cos \theta = -0.7771$ |
| (j) $\sin \theta = -0.7071$ | | |

12.3 THE LAW OF SINES

Triangles that do not contain a right angle are called oblique triangles. In solving right triangles, one side and two other parts of a triangle were given. The remaining three parts were found using geometry and the trigonometric ratios. In this section we shall solve oblique triangles using the *law of sines*.



In $\triangle ABC$, we draw AD perpendicular to BC , or BC extended, representing an altitude of $\triangle ABC$. For the case where $\triangle ABC$ is obtuse we use the relationship

$$\sin(180^\circ - \theta) = \sin \theta$$

to derive and use the law for both acute and obtuse triangles.

In $\triangle ABD$,

$$\frac{h}{c} = \sin B$$

$$h = c \sin B$$

In $\triangle ACD$,

$$\frac{h}{b} = \sin C \quad \text{or} \quad \frac{h}{b} = \sin(180^\circ - C)$$

$$h = b \sin C$$

$$h = b \sin(180^\circ - C)$$

$$= b \sin C$$

Equating the expressions for the altitudes

$$c \sin B = b \sin C$$

Dividing both sides by bc

$$\frac{c \sin B}{bc} = \frac{b \sin C}{bc}$$

which simplifies to

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

We generalize this result to

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

This relationship is the *first form* of the Law of Sines, and is used to find an angle. We can find sides of triangles by using the *second form* of the law found by taking the reciprocals.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

EXAMPLE 1. **AAS** In $\triangle ABC$, $\angle A = 75^\circ$, $\angle B = 57^\circ$ and $a = 6.75$. Find c .

Solution

$$\begin{aligned} \angle C &= 180^\circ - (\angle A + \angle B) \\ &= 180^\circ - (75^\circ + 57^\circ) \\ &= 48^\circ \end{aligned}$$

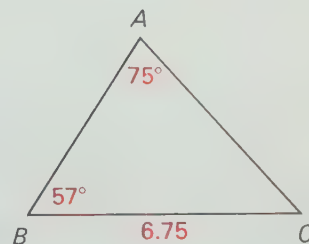
$$\begin{aligned} \frac{c}{\sin C} &= \frac{a}{\sin A} \\ c &= \frac{a \sin C}{\sin A} \end{aligned}$$

Put the numbers from 1 to 9 in the spaces to make the statements true.

$$\square \div \square + \square = 5$$

$$\square + \square - \square = 5$$

$$\square \times \square - \square = 5$$



We pick two parts of the law of sines so that the required quantity is in the numerator on the left side, and the other three parts are known.

$$c = \frac{6.75 \sin 48^\circ}{\sin 75^\circ}$$

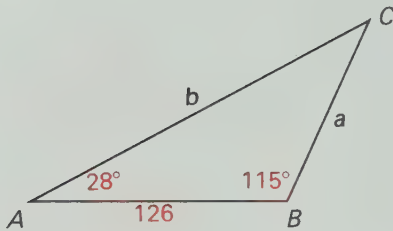
$$\doteq \frac{6.75(0.7431)}{0.9659}$$

$$\doteq 5.19$$

$\therefore$ the length of c is 5.19

EXAMPLE 2. ASA In $\triangle ABC$, $\angle A = 28^\circ$, $\angle B = 115^\circ$ and $c = 126$ cm. Solve the triangle.

Solution



$$\begin{aligned}\sin 115^\circ &= \sin(180^\circ - 115^\circ) \\ &= \sin 65^\circ\end{aligned}$$

$$\begin{aligned}\angle C &= 180^\circ - (28^\circ + 115^\circ) \\ &= 37^\circ\end{aligned}$$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$a = \frac{c \sin A}{\sin C}$$

$$a = \frac{126 \sin 28^\circ}{\sin 37^\circ}$$

$$\doteq \frac{126(0.4695)}{0.6018}$$

$$\doteq 98.3$$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$b = \frac{c \sin B}{\sin C}$$

$$b = \frac{126 \sin 115^\circ}{\sin 37^\circ}$$

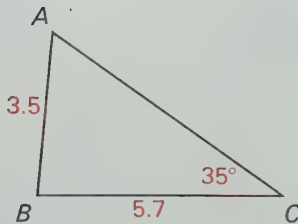
$$\doteq \frac{126(0.9063)}{0.6018}$$

$$\doteq 189.8$$

$\therefore$ in $\triangle ABC$, $a = 98.3$, $b = 190$, and $\angle C = 37^\circ$.

EXAMPLE 3. SSA In $\triangle ABC$, $a = 5.7$ cm, $c = 3.5$ cm and $\angle C = 35^\circ$. Find $\angle A$, given that $\triangle ABC$ is acute.

Solution



$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

$$\sin A = \frac{a \sin C}{c}$$

$$\sin A = \frac{5.7 \sin 35^\circ}{3.5}$$

$$= \frac{5.7(0.5736)}{3.5}$$

$$= 0.9341$$

from tables
 $\sin 69^\circ = 0.9336$

$$\begin{aligned}\therefore \angle A &\doteq 69^\circ \quad \text{or} \quad \angle A \doteq 180^\circ - 69^\circ \\ &\doteq 111^\circ\end{aligned}$$

The value $\angle A = 111^\circ$ comes from the relationship

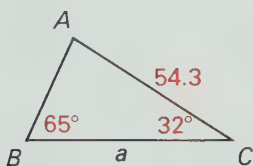
$$\sin(180^\circ - \theta) = \sin \theta.$$

Since it was stated that $\triangle ABC$ is acute the value $\angle A = 111^\circ$ is inadmissible so that the solution is $\angle A = 69^\circ$.

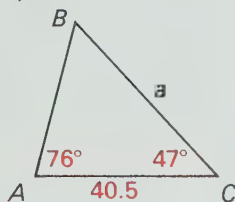
EXERCISE 12-3

- B** 1. Calculate the value of the indicated quantity in each of the following.

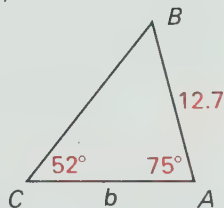
(a)



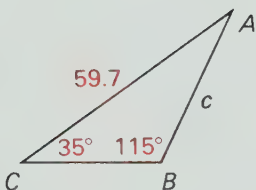
(b)



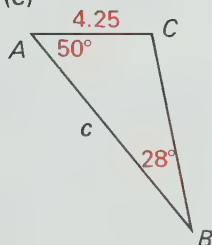
(c)



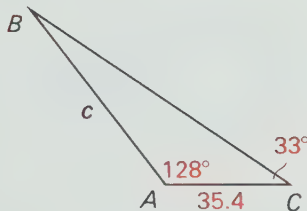
(d)



(e)



(f)

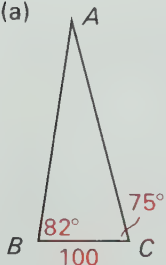


2. Find the indicated quantity.

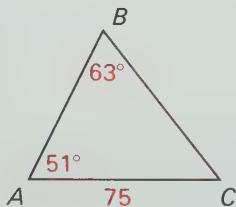
- (a) $\triangle ABC$, $\angle A = 25^\circ$, $\angle C = 80^\circ$, $a = 24.7$. Find b .
 (b) $\triangle ABC$, $\angle B = 27^\circ$, $\angle C = 74^\circ$, $b = 8.25$. Find a .
 (c) $\triangle ABC$, $\angle A = 125^\circ$, $\angle B = 27^\circ$, $a = 72.7$. Find b .
 (d) $\triangle ABC$, $\angle A = 45^\circ$, $\angle B = 25^\circ$, $c = 125$. Find a and b .
 (e) $\triangle ABC$, $AB = 3.6$, $BC = 3.9$, $\angle C = 43^\circ$. Find $\angle A$.

3. Solve the following triangles.

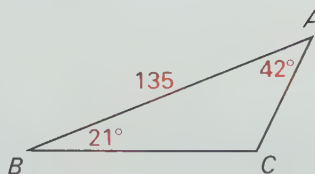
(a)



(b)



(c)



- (d) $\triangle ABC$, $a = 12.8$, $\angle B = 54^\circ$, $\angle C = 63^\circ$.
 (e) $\triangle ABC$, $\angle A = 105^\circ$, $b = 1.18$, $\angle C = 27^\circ$.
 (f) $\triangle ABC$, $AB = 11.5$, $BC = 15.3$, $\angle A = 110^\circ$.

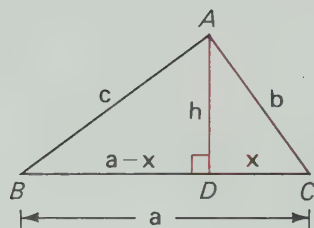
12.4 THE LAW OF COSINES

When an oblique triangle has two sides and the included angle given (SAS) we can find the third side using the *law of cosines*.

In the derivation of the law of cosines which follows, we shall use the relationship

$$\cos(180^\circ - \theta) = -\cos \theta$$

to derive and use the law in both acute and obtuse triangles.



$$\text{In } \triangle ADC, \frac{x}{b} = \cos C$$

$$x = b \cos C$$

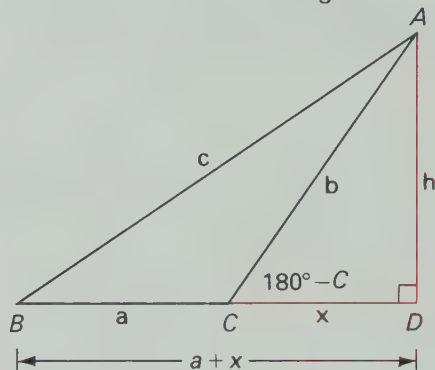
$$\text{and } b^2 = h^2 + x^2$$

$$\text{In } \triangle ABD, c^2 = h^2 + (a-x)^2$$

$$= h^2 + a^2 - 2ax + x^2$$

$$= a^2 + (h^2 + x^2) - 2ax$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$



$$\text{In } \triangle ADC, \frac{x}{b} = \cos(180^\circ - C)$$

$$x = b \cos(180^\circ - C)$$

$$= -b \cos C$$

$$\text{and } b^2 = h^2 + x^2$$

$$\text{In } \triangle ABD, c^2 = h^2 + (a+x)^2$$

$$= h^2 + a^2 + 2ax + x^2$$

$$= a^2 + (h^2 + x^2) + 2ax$$

$$= a^2 + b^2 + 2a(-b \cos C)$$

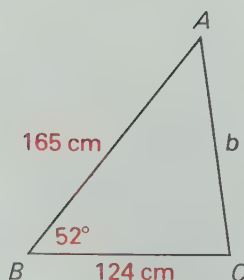
$$c^2 = a^2 + b^2 - 2ab \cos C$$

The three forms of the law of cosines are

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$



EXAMPLE 1. In $\triangle ABC$, $a = 124$ cm, $\angle B = 52^\circ$, and $c = 165$ cm. Solve the triangle.

Solution

To find side b :

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$\begin{aligned}
 b^2 &\doteq 124^2 + 165^2 - 2(124)(165)(0.6157) \\
 &\doteq 15\,376 + 27\,225 - 25\,194 \\
 &\doteq 17\,407 \\
 b &\doteq 132
 \end{aligned}$$

To find the angles:

$$\begin{aligned}
 \frac{\sin A}{a} &= \frac{\sin B}{b} \\
 \sin A &= \frac{a \sin B}{b} \\
 \sin A &= \frac{124 \sin 52^\circ}{132} \\
 &\doteq \frac{124(0.7880)}{132} \\
 &\doteq 0.7402 \\
 \angle A &\doteq 48^\circ \\
 \angle C &\doteq 180^\circ - (48^\circ + 52^\circ) \\
 &\doteq 80^\circ
 \end{aligned}$$

$$\therefore \angle A = 48^\circ, b = 132 \text{ cm}, \angle C = 80^\circ.$$

When given the lengths of the three sides of a triangle (SSS) we can find the measure of the angles using the law of cosines.

$$\begin{aligned}
 a^2 &= b^2 + c^2 - 2bc \cos A \\
 2bc \cos A &= b^2 + c^2 - a^2 \\
 \cos A &= \frac{b^2 + c^2 - a^2}{2bc}
 \end{aligned}$$

Similarly

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} \quad \text{and} \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

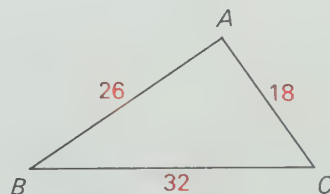
EXAMPLE 2. Find the measure of the smallest angle in $\triangle ABC$, where $a = 32$, $b = 18$, $c = 26$.

Solution

From the geometry of the diagram we see that $\angle B$ is the smallest angle.

$$\begin{aligned}
 \cos B &= \frac{a^2 + c^2 - b^2}{2ac} \\
 \cos B &= \frac{32^2 + 26^2 - 18^2}{2(32)(26)} \\
 &= \frac{1024 + 676 - 324}{1664} \\
 &\doteq 0.8269 \\
 \angle B &\doteq 34^\circ
 \end{aligned}$$

A man has to take a wolf, a goat, and some lettuce across a river. The rowboat has room for the man plus either the wolf or the goat or the lettuce. If he takes the lettuce with him, the wolf will eat the goat. If he takes the wolf, the goat will eat the lettuce. Only when the man is present are the goat and lettuce safe from their enemies. How does the man get the wolf, goat and lettuce across the river?

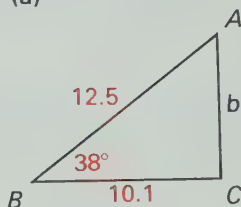


EXERCISE 12-4

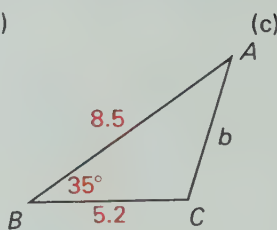
$$\cos(180^\circ - \theta) = -\cos \theta$$

- B** 1. Find the third side in the following.

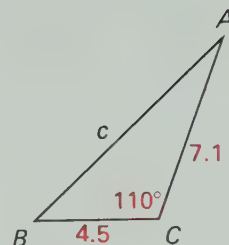
(a)



(b)

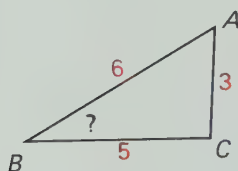


(c)

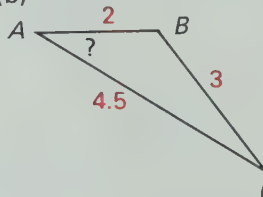


2. Find the measure of the indicated angles.

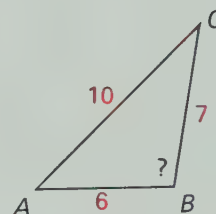
(a)



(b)



(c)



3. Use the law of cosines in the following.

(a) In $\triangle ABC$, $a = 25.6$, $c = 32.7$, $\angle B = 55^\circ$. Find b .

(b) In $\triangle ABC$, $a = 4.75$, $b = 2.25$, $\angle C = 40^\circ$. Find c .

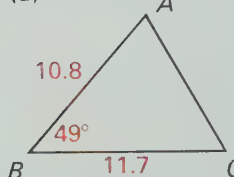
(c) In $\triangle ABC$, $b = 250$, $c = 375$, $\angle A = 78^\circ$. Find a .

(d) In $\triangle ABC$, $a = 25$, $b = 30$, $c = 44$. Find $\angle B$.

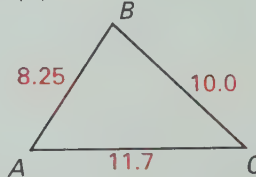
(e) In $\triangle ABC$, $a = 3.5$, $b = 2.7$, $c = 4.9$. Find $\angle C$.

4. Solve the triangles.

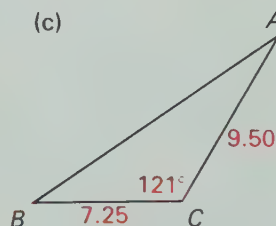
(a)



(b)



(c)



(d) $\triangle ABC$, $a = 15.7$, $\angle B = 68^\circ$, $c = 18.5$.

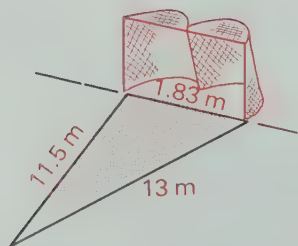
(e) $\triangle ABC$, $a = 11.3$, $b = 7.25$, $c = 12.6$.

12.5 APPLICATIONS OF OBLIQUE TRIANGLES

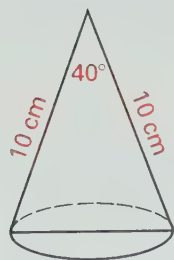
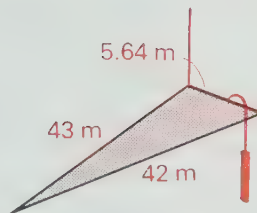
In this section we shall extend the applications of right triangles in section 11.5 to situations involving oblique triangles. All problems should start with a reasonably accurate well-labelled diagram.

EXERCISE 12-5

1. A hockey net is 1.83 m wide. A player shoots from a point where the puck is 13 m from one goal post and 11.5 m from the other. Within what angle must he make his shot to score?

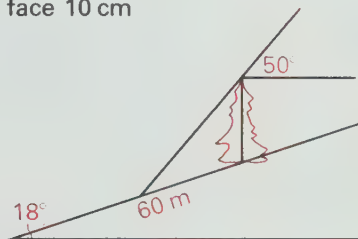


2. Football goalposts are 5.64 m apart and the cross-bar is 3 m above the ground. A player is to attempt a field goal from a point where the ball is 43 m and 42 m from the ends of the cross-bar. Within what angle must he kick the ball?

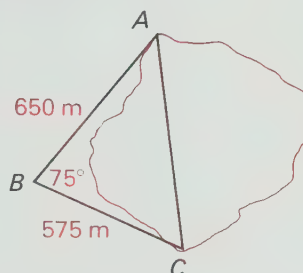


3. The vertical angle of a cone is 40° . Find the diameter of the cone at a point on the face 10 cm from the vertex.

4. A tree on a hillside with constant slope casts a shadow of 60 m down the hill. If the angle of elevation of the sun is 50° and the angle of inclination of the hill is 18° , find the height of the tree.



5. Find the width of a small lake if from point B an angle of 75° is contained by the lengths 650 m and 575 m.

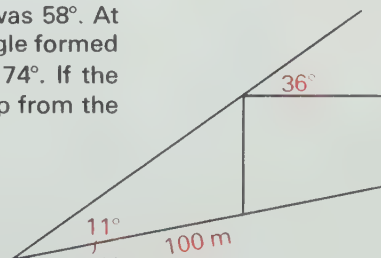


6. Sudbury North Bay and Kirkland Lake form the vertices of a triangle on a map. If Sudbury and North Bay are 113 km apart and the angles at Sudbury and North Bay are 80° and 68° , respectively, calculate the distance from

- (a) North Bay to Kirkland Lake
(b) Sudbury to Kirkland Lake.

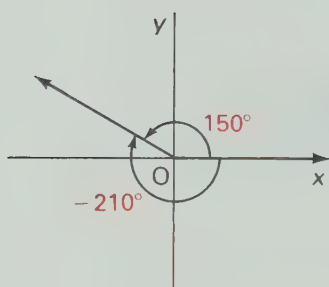
7. On a space flight, astronaut Neil Armstrong reports that the angle formed by his lines of sight to the earth and to the moon was 58° . At the same time, the observer on the earth reports that the angle formed by his lines of sight to the spaceship and to the moon is 74° . If the moon is 382 400 km from the earth, how far is the spaceship from the tracking station?

8. There is a CB tower on a slope inclined at 11° to the horizontal. The tower casts a shadow 100 m down the hill when the angle of elevation of the sun is 36° . How high is the tower?



9. A grandfather clock has a pendulum 92.7 cm long. From one end of the swing to the other, the straight-line separation is 18 cm. Find the angle through which the pendulum swings.

10. The arms of a "jaws" machine used for prying open crushed vehicles in accidents are 75 cm long. How far apart are the tips of the jaws when the angle formed by the arms is 55° ?



12.6 RADIAN MEASURE

We have described the measure of an angle as an amount of rotation, where one complete revolution was divided into 360 equal parts called *degrees of rotation* or simply *degrees*. If the rotation is counterclockwise, the degree measure is positive. If the rotation is clockwise the measure is negative.

$$1 \text{ revolution} = 360^\circ$$

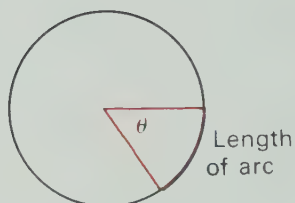
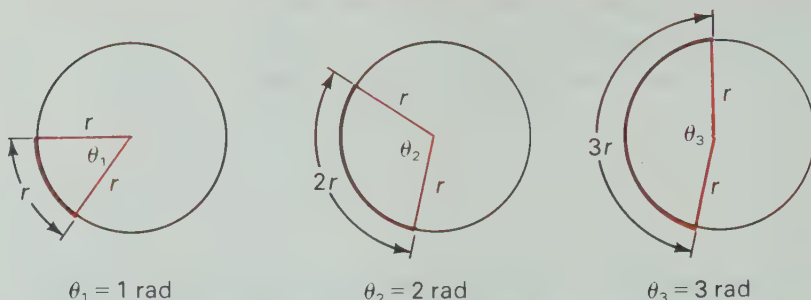


Figure 12-3

An angle subtended at the centre of a circle by an arc equal in length to the radius has a measure of one *radian*.



These diagrams suggest

$$\theta_1 = \frac{r}{r} = 1 \quad \theta_2 = \frac{2r}{r} = 2 \quad \theta_3 = \frac{3r}{r} = 3$$

which leads to the generalization

$$\text{number of radians} = \frac{\text{length of arc}}{\text{radius}}$$

Since $\theta = \frac{a}{r}$, it follows that

$$a = r\theta, \quad \theta > 0$$

where a , r , and θ are the measures of the arc length, radius, and radian measure of the angle, respectively. While the above expression holds for $\theta > 0$, this equation can be generalized to

$$a = r|\theta|, \quad \theta \in \mathbb{R}$$

where $|\theta|$ is the absolute value or magnitude of θ .

In order to convert from degree measure to radian measure and vice versa, we must establish the relationship between degrees and radians. We have stated that one revolution is 360° in degree measure. We now find the same angle in radian measure.

In Figure 12-4 the measure of θ is one revolution (360°) and the arc length a is the circumference of the circle, $2\pi r$. Hence the radian measure of θ is given by

$$\theta = \frac{a}{r} = \frac{2\pi r}{r} = 2\pi$$

The relationship between degrees and radians is given by

$$2\pi \text{ rad} = 360^\circ$$

which simplifies to

$$\pi \text{ rad} = 180^\circ$$

EXAMPLE 1. Calculate the degree measure of the angles whose radian measures are

- (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{2}$ (c) $\frac{5\pi}{4}$ (d) $-\frac{3\pi}{2}$ (e) 3π

Solution

Since $\pi \text{ rad} = 180^\circ$

- (a) $\frac{\pi}{6} \text{ rad} = \frac{180^\circ}{6} = 30^\circ$
 (b) $\frac{\pi}{2} \text{ rad} = \frac{180^\circ}{2} = 90^\circ$
 (c) $\frac{5\pi}{4} \text{ rad} = \frac{5(180^\circ)}{4} = 225^\circ$
 (d) $-\frac{3\pi}{2} = -\frac{3(180^\circ)}{2} = -270^\circ$
 (e) $3\pi = 3(180^\circ) = 540^\circ$

EXAMPLE 2. Calculate the radian measure of the angles whose measures are

- (a) 45° (b) 315° (c) -210°

Solution

Since $180^\circ = \pi \text{ rad}$

$$1^\circ = \frac{\pi}{180} \text{ rad}$$

- (a) $45^\circ = 45 \left(\frac{\pi}{180} \right) = \frac{\pi}{4} \text{ rad}$
 (b) $315^\circ = 315 \left(\frac{\pi}{180} \right) = \frac{7\pi}{4} \text{ rad}$
 (c) $-210^\circ = -210 \left(\frac{\pi}{180} \right) = -\frac{7\pi}{6} \text{ rad}$

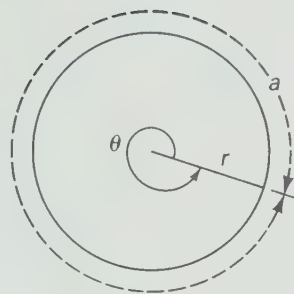
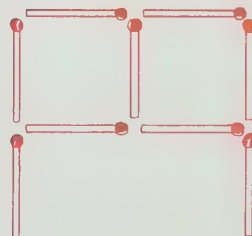


Figure 12-4

Move two matches to make 3 identical squares.



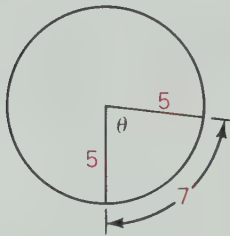
EXAMPLE 3. Calculate the number of degrees in 1 radian.

Solution

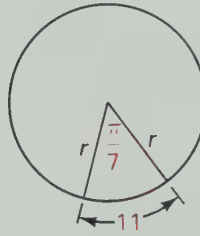
$$\begin{aligned}\pi \text{ rad} &= 180^\circ \\ 1 \text{ rad} &= \left(\frac{180^\circ}{\pi} \right) \\ &\doteq \frac{180^\circ}{3.14159} \\ &\doteq 57.2958^\circ\end{aligned}$$

EXAMPLE 4. Find the unknown quantity in each of the following.

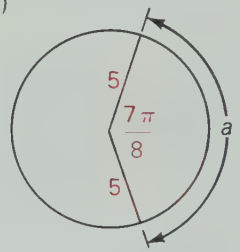
(a)



(b)



(c)



Solution

Since $a = r\theta$

$$\begin{aligned}\text{(a)} \quad \theta &= \frac{a}{r} \\ \theta &= \frac{7}{5} \\ &= 1.4 \text{ rad}\end{aligned}$$

$$\begin{aligned}\text{(b)} \quad r &= \frac{a}{\theta} \\ r &= \frac{11}{\frac{\pi}{7}} \\ &= \frac{77}{\pi}\end{aligned}$$

$$\begin{aligned}\text{(c)} \quad a &= r\theta \\ a &= 5 \left(\frac{7\pi}{8} \right) \\ &= \frac{35\pi}{8}\end{aligned}$$

EXERCISE 12-6

B 1. Convert the following radian measures to degree measure.

$$\text{(a)} \frac{3\pi}{4} \quad \text{(b)} \frac{2\pi}{3} \quad \text{(c)} \frac{4\pi}{3} \quad \text{(d)} \frac{5\pi}{6} \quad \text{(e)} \frac{3\pi}{2}$$

2. Convert the following degree measures to radian measure in terms of π .

$$\text{(a)} 60^\circ \quad \text{(b)} 150^\circ \quad \text{(c)} 90^\circ \quad \text{(d)} 330^\circ \quad \text{(e)} -240^\circ$$

3. Calculate the radian measure of an angle whose measure is 1° .

4. Convert the following radian measures to degree measure. (Use $\pi \doteq 3.14$)

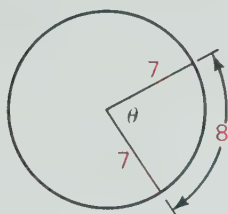
$$\text{(a)} 5 \quad \text{(b)} 2.4 \quad \text{(c)} 3.7 \quad \text{(d)} 0.145 \quad \text{(e)} 213$$

5. Convert the following degree measures to radian measure. (Use $\pi \doteq 3.14$)

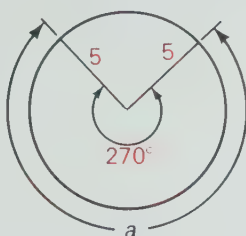
$$\text{(a)} 35^\circ \quad \text{(b)} 70^\circ \quad \text{(c)} 170^\circ \quad \text{(d)} 220^\circ \quad \text{(e)} 475^\circ$$

6. Find the unknown quantity in each of the following.

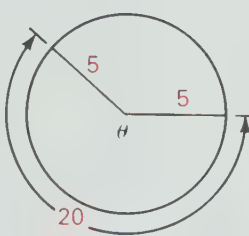
(a)



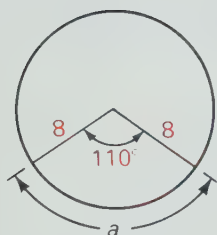
(b)



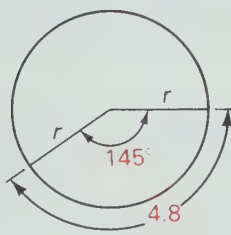
(c)



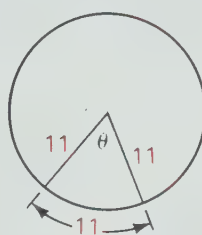
(d)



(e)



(f)



- C 7. How much longer is the rope that circles the earth 1 m above the surface than the rope that circles the earth on the surface?

12.7 GRAPHS OF THE TRIGONOMETRIC FUNCTIONS

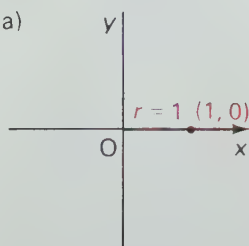
In this section we shall use the ratios of special angles to draw the graphs of trigonometric functions.

EXAMPLE 1. Find (a) $\cos 0^\circ$ (b) $\sin 90^\circ$ (c) $\cos 180^\circ$
(d) $\tan 270^\circ$ (e) $\sin 360^\circ$

Solution

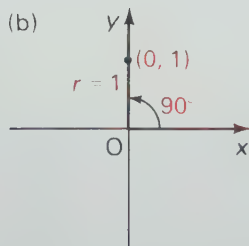
In each case, we choose a convenient point on the terminal arm of the angle in standard position.

(a)



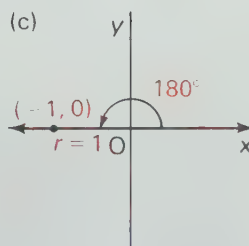
$$\cos 0^\circ = \frac{1}{1} = 1$$

(b)

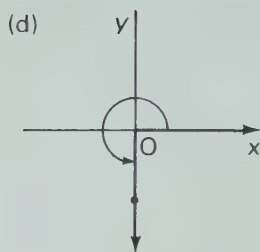


$$\sin 90^\circ = \frac{1}{1} = 1$$

(c)

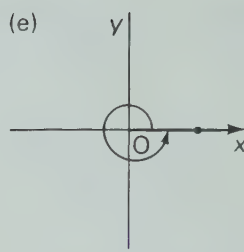


$$\cos 180^\circ = \frac{-1}{1} = -1$$



$$\tan 270^\circ = \frac{-1}{0}$$

which is undefined.



$$\sin 360^\circ = \frac{0}{1} = 0$$

When we write $\sin 10^\circ$ we mean that the measure of the angle is 10° while when we write $\sin 10$ we mean that the measure of the angle is 10 radians.

EXAMPLE 2. Find (a) $\sin \frac{2\pi}{3}$ (b) $\cos \frac{5\pi}{4}$ (c) $\tan \frac{11\pi}{6}$

Solution

We write the formulas of section 12.2 using radian measure.

$\sin(\pi - \theta) = \sin \theta$ $\cos(\pi - \theta) = -\cos \theta$ $\tan(\pi - \theta) = -\tan \theta$	$\sin(\pi + \theta) = -\sin \theta$ $\cos(\pi + \theta) = -\cos \theta$ $\tan(\pi + \theta) = \tan \theta$	$\sin(2\pi - \theta) = -\sin \theta$ $\cos(2\pi - \theta) = \cos \theta$ $\tan(2\pi - \theta) = -\tan \theta$
--	--	---

$$(a) \sin \frac{2\pi}{3} = \sin \left(\pi - \frac{\pi}{3} \right) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$(b) \cos \frac{5\pi}{4} = \cos \left(\pi + \frac{\pi}{4} \right) = -\cos \frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$(c) \tan \frac{11\pi}{6} = \tan \left(2\pi - \frac{\pi}{6} \right) = -\tan \frac{\pi}{6} = -\frac{1}{\sqrt{3}}$$

INVESTIGATION 12-7

1. Complete the following table in your notebook.

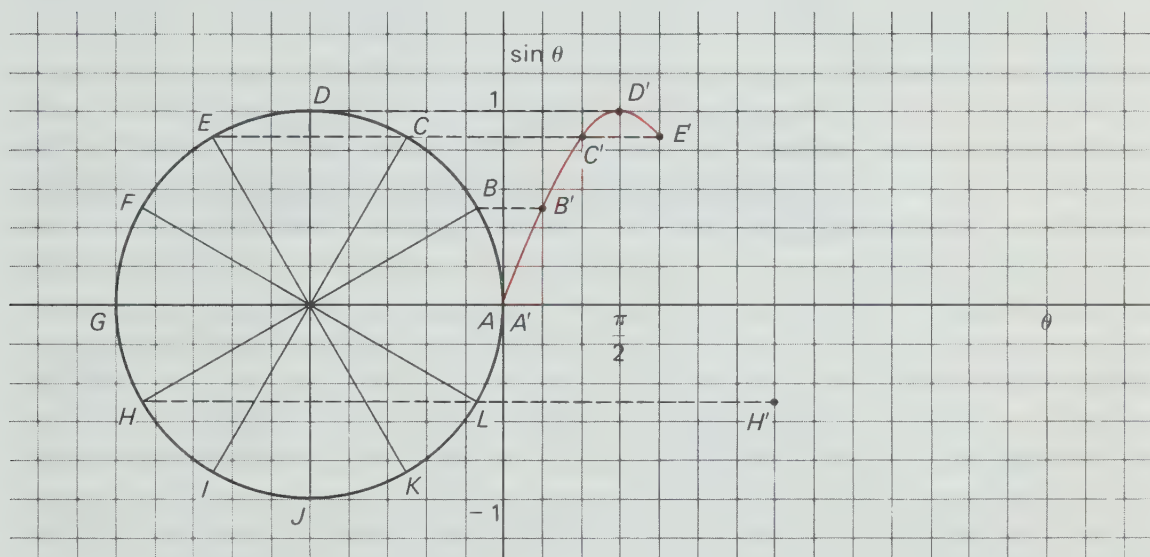
Degrees	0°	30°	45°	60°	90°	120°	
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	
$\sin \theta$							
$\cos \theta$							
$\tan \theta$							

2. The graph of $f(\theta) = \sin \theta$

- Using θ and $f(\theta)$ axes, plot the values of $\sin \theta$ from the table in question 1, and draw the curve for $0 \leq \theta \leq 2\pi$.
- State the range of $f(\theta) = \sin \theta$ for the domain $0 \leq \theta \leq 2\pi$.
- From your graph of $f(\theta) = \sin \theta$, $0 \leq \theta \leq 2\pi$, state:
 - the θ -intercepts
 - a maximum point
 - a minimum point.

Since every value of θ determines a unique value for $\sin \theta$, the graph of the ordered pairs $(\theta, \sin \theta)$ is the graph of a function—the **sine function**. Check this using the vertical line test.

- We can generate the sine curve geometrically by plotting the vertical displacement of a point on the circumference of a circle against the angle as shown in the diagram which follows. Complete the drawing in your notebook.

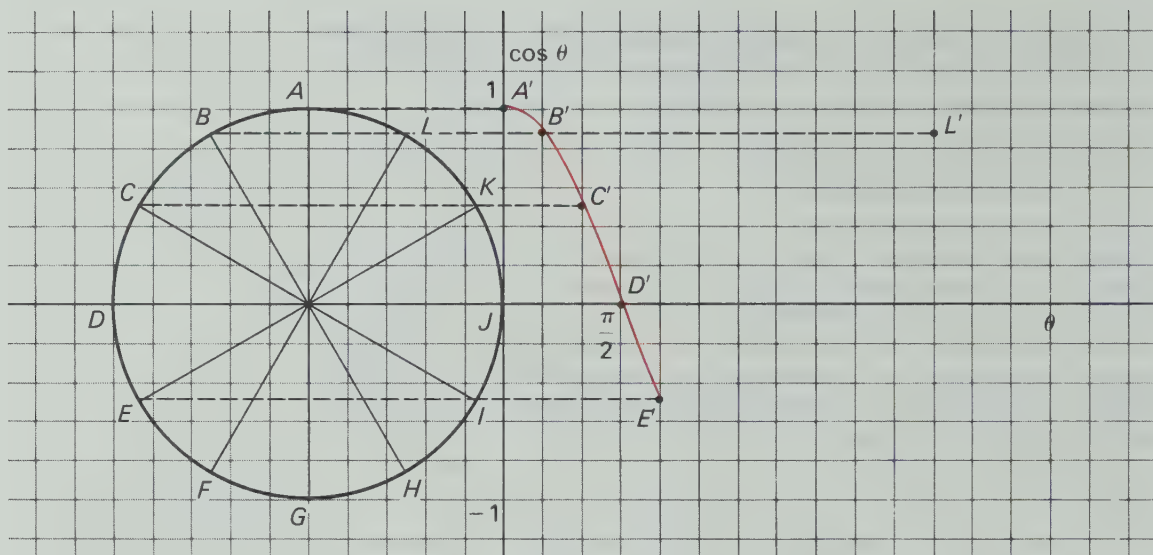


3. The graph of $f(\theta) = \cos \theta$

- Using θ and $f(\theta)$ axes, plot the values of $\cos \theta$ from the table in question 1, and draw the curve for $0 \leq \theta \leq 2\pi$.
- State the range of $f(\theta) = \cos \theta$ for the domain $0 \leq \theta \leq 2\pi$.
- From your graph of $f(\theta) = \cos \theta$, $0 \leq \theta \leq 2\pi$, state
 - the θ -intercepts
 - two maximum points
 - a minimum point.

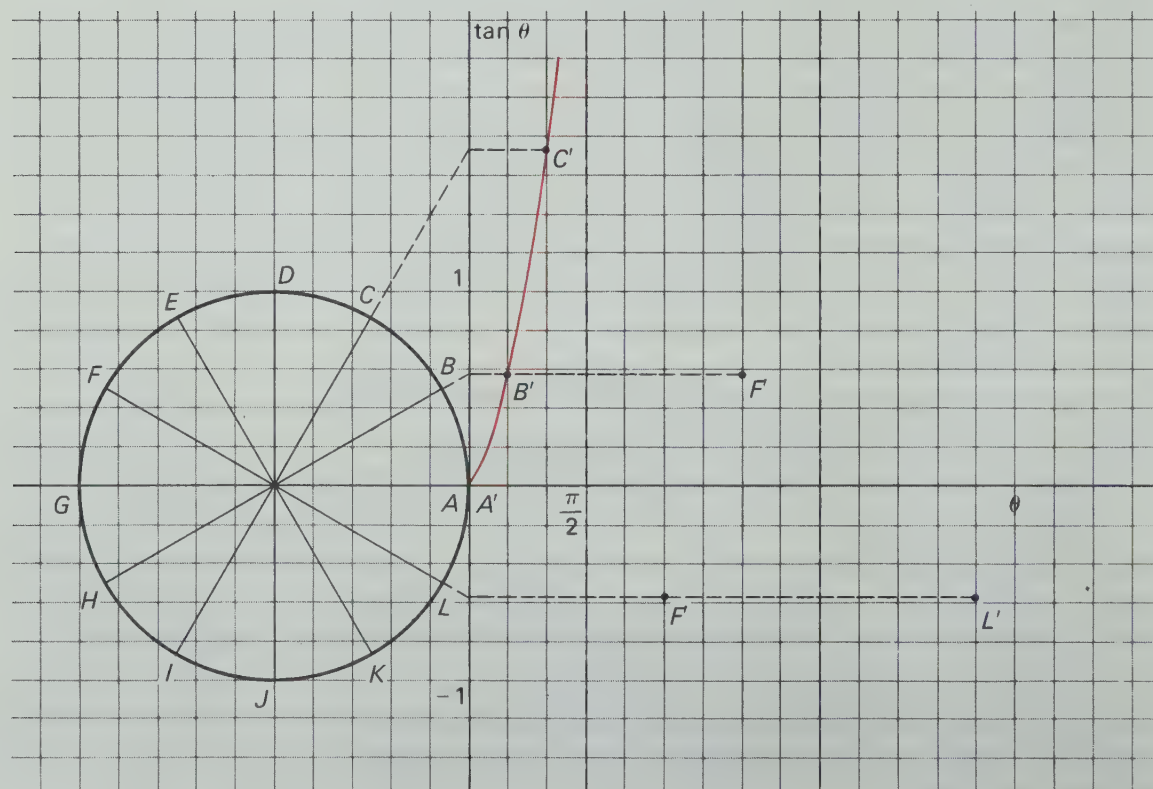
Since every value of θ determines a unique value for $\cos \theta$, the graph of the ordered pairs $(\theta, \cos \theta)$ is the graph of a function—the **cosine function**.

- We can generate the cosine curve geometrically in the same manner that we drew the sine curve in 2(d), starting at the vertical position to plot the vertical displacements against the angle. Complete the following drawing in your notebook.



4. The graph of $f(\theta) = \tan \theta$

(a) On θ and $f(\theta)$ axes, draw dotted lines parallel to the $f(\theta)$ -axis at the values of θ for which $\tan \theta$ is undefined. Since $\tan \theta$ is not defined for these values of θ , no point on the graph of $f(\theta) = \tan \theta$ can lie on these dotted lines.



(b) Plot the ordered pairs $(\theta, \tan \theta)$ from the table in question 1 and draw the curve for $0 \leq \theta \leq 2\pi$.

(c) From your graph of $f(\theta) = \tan \theta$, $0 \leq \theta \leq 2\pi$, state

(i) the θ -intercepts

(ii) whether there are maximum or minimum points.

(d) We can generate the graph of the tangent function geometrically by plotting the projection of a point of the circle on a vertical axis against the angle as shown. Complete the drawing in your notebook.

5. Use the graphs of questions 2, 3 and 4 to answer the following.

(a) Describe the graphs of $\sin \theta$, $\cos \theta$, and $\tan \theta$ as θ increases from

(i) 0 to $\frac{\pi}{2}$ (ii) $\frac{\pi}{2}$ to π (iii) π to $\frac{3\pi}{2}$ (iv) $\frac{3\pi}{2}$ to 2π .

(b) State the intervals in (a) for which the values of each of the primary functions are positive.

(c) Describe the graphs of $f(\theta) = \sin \theta$, and $f(\theta) = \cos \theta$ as θ increases beyond 2π .

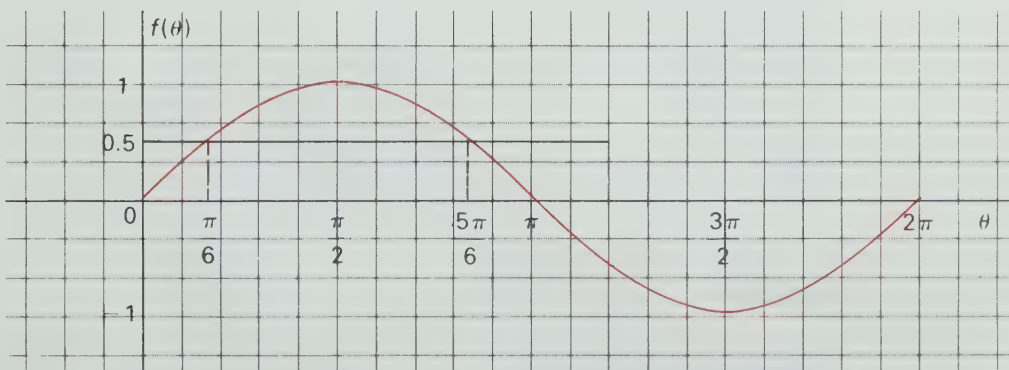
12.8 TRIGONOMETRIC EQUATIONS

Equations such as $\sin \theta = 0.5$ are called trigonometric equations. To solve these we must find the value of θ when given the value of one of the trigonometric functions.

EXAMPLE 1. Solve $\sin \theta = 0.5$, $0 \leq \theta \leq 2\pi$.

Solution

Take the graph of $f(\theta) = \sin \theta$, $0 \leq \theta \leq 2\pi$ and draw a horizontal line at $f(\theta) = 0.5$



The horizontal line $f(\theta) = 0.5$ cuts the graph at $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$.

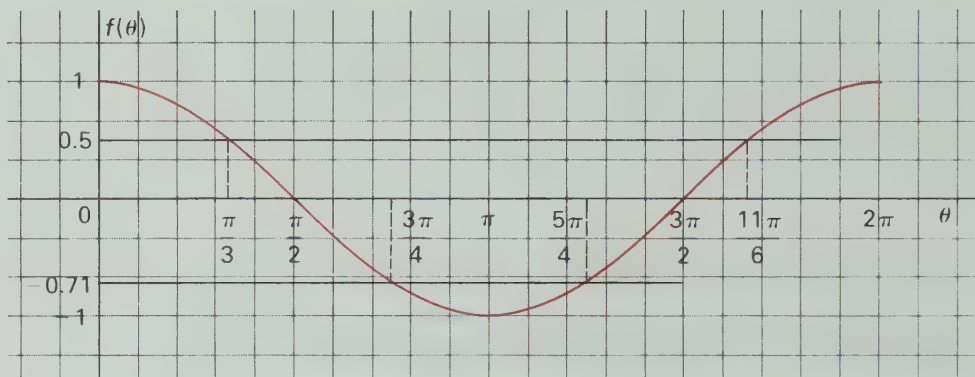
$\therefore$ the solution set is $\left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$.

Example 1 suggests a graphical method of solving trigonometric equations. Some equations, however, require solving algebraically before the graphical method can be used.

EXAMPLE 2. Solve $(2 \cos \theta - 1)(\sqrt{2} \cos \theta + 1) = 0, 0 \leq \theta \leq 2\pi$.

Solution $(2 \cos \theta - 1)(\sqrt{2} \cos \theta + 1) = 0$
 $2 \cos \theta - 1 = 0$ or $\sqrt{2} \cos \theta + 1 = 0$
 $\cos \theta = \frac{1}{2}$ $\cos \theta = -\frac{1}{\sqrt{2}} = -0.71$

Taking the graph of $f(\theta) = \cos \theta, 0 \leq \theta \leq 2\pi$, we draw two horizontal lines $f(\theta) = 0.5$ and $f(\theta) = -0.71$



From the graph, the solution set of the equation in the given domain is

$$\left\{ \frac{\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{11\pi}{6} \right\}$$

EXERCISE 12-8

B Solve for $0 \leq \theta \leq 2\pi$

- | | |
|--------------------------|---|
| 1. $\sin \theta = 0.866$ | 6. $\sin \theta = 0$ |
| 2. $\cos \theta = 0.5$ | 7. $(\cos \theta + 0.5)(\cos \theta - 0.5) = 0$ |
| 3. $\sin \theta = -0.5$ | 8. $(\sin \theta - 0.5)(\sin \theta + 0.5) = 0$ |
| 4. $\tan \theta = 1$ | 9. $\sin \theta(\sin \theta + 1) = 0$ |
| 5. $\tan \theta = -1$ | 10. $\sin^2 \theta - 1 = 0$ |

REVIEW EXERCISE

1. The point $(12, -5)$ lies on the terminal arm of $\angle \theta$ in standard position. State

- (a) $\cos \theta$ (b) $\tan \theta$ (c) $\cot \theta$ (d) $\csc \theta$

2. Find two values for $\sec \theta$, given $\sin \theta = \frac{3}{5}$.

3. Use the special triangles to find

- (a) $\sin 150^\circ$ (b) $\csc 135^\circ$ (c) $\cot 315^\circ$ (d) $\tan 225^\circ$

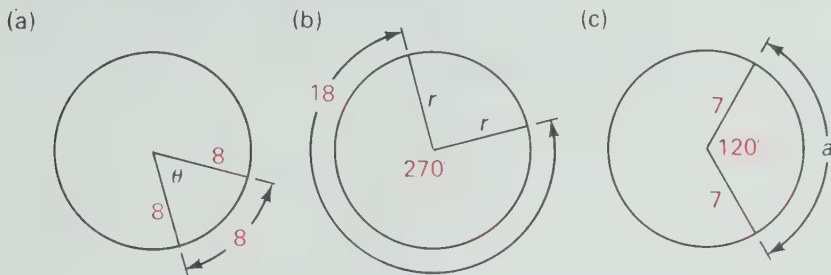
4. Change the following to radian measure.

- (a) 60° (b) 135° (c) 75° (d) 28° (e) 570°

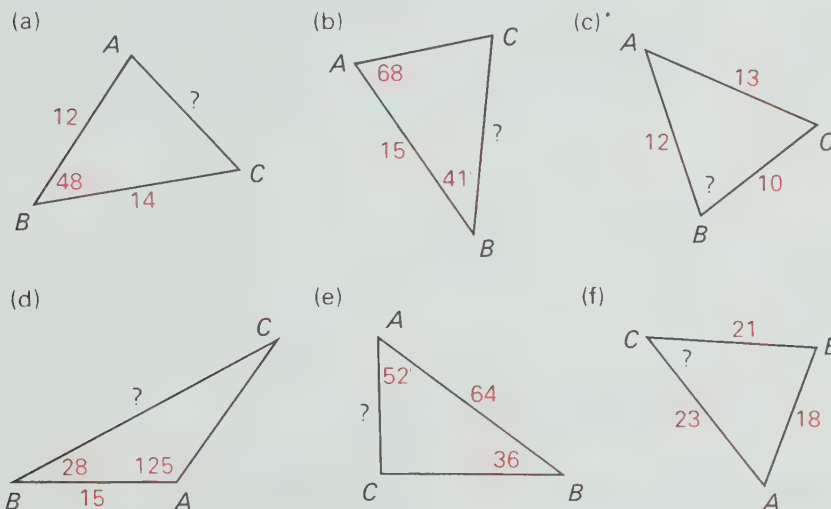
5. Change the following radian measures to degree measures.

- (a) 2 (b) 3.5 (c) 4π (d) $\frac{11\pi}{6}$ (e) $\frac{7\pi}{2}$

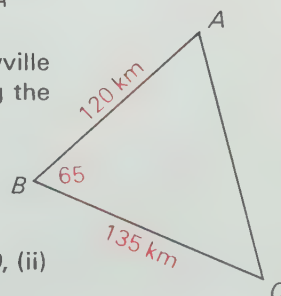
6. Find the unknown quantity in the following.



7. Calculate the value of the indicated quantity in each of the following using the law of sines, or the law of cosines.



8. The distance from Bowville to Armton is 120 km, and from Bowville to Canton is 135 km. The angle at B formed by the lines joining the three towns is 65° . Find the distance from Armton to Canton.



9. (a) On the same set of axes, sketch the graphs of (i) $f(\theta) = \sin \theta$, (ii) $f(\theta) = \cos \theta$, for $0 \leq \theta \leq 2\pi$.
 (b) State the values of θ for which $\sin \theta = \cos \theta$.

10. Solve the following for $0 \leq \theta \leq 2\pi$.

- (a) $\tan \theta = 0$
 (b) $(\sin \theta - 1)(\cos \theta - 1) = 0$
 (c) $4 \cos^2 \theta - 1 = 0$
 (d) $\sin^2 \theta - \sin \theta = 0$

REVIEW AND PREVIEW TO CHAPTER 13

Calculator exercises.

1. Simplify

(a) $\frac{421^2}{417} - 532$

(c) $\frac{7.62 \times 4.98}{8.34} + 11.7$

(e) $\frac{81.8 \times 7.93}{5.24 + 8.6}$

(g) $\frac{(43.7 \times 8.14)^2}{56.2 \div 4.9}$

(i) $\frac{21.6 \times 3.9 - 11.7}{6.8 \times 0.25}$

(b) $\frac{3^4 + 8^2 + 7^3}{45}$

(d) $\frac{7^5 + 8^3}{42.3} - 407.5$

(f) $\frac{0.082 - 0.076}{5.76 + 8.94} - 6$

(h) $\left[\frac{6^4 - 3.7}{8.1 \div 5.2} - 4.6 \right] \times 3$

(j) $\frac{(4.62 \times 0.53)^3}{8.1 - 5.8} - 6.6$

2. Find the value of each fraction in decimal form.

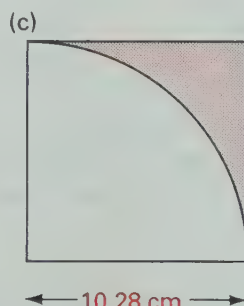
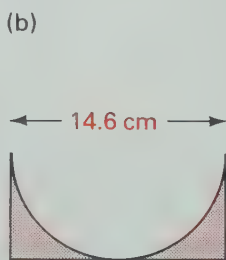
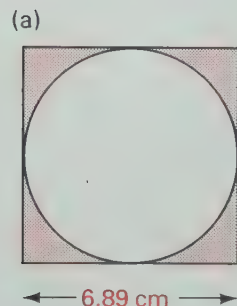
(a) $\frac{1}{3 + \frac{1}{2 + \frac{1}{5}}}$

(b) $12 + \frac{1}{6 + \frac{1}{5 + \frac{1}{6}}}$

(c) $322 + \frac{1}{4 + \frac{1}{13 + \frac{1}{7 + \frac{1}{6}}}}$

(d) $63 + \frac{1}{5 + \frac{1}{7 + \frac{1}{13 + \frac{1}{26}}}}$

3. Find the area of the shaded regions (use $\pi = 3.14$).



4. The formula to calculate the surface area of a sphere is

$$A = 4\pi r^2$$

(a) Find the surface area of a sphere whose radius is 35.7 cm. (use $\pi = 3.1416$)

(b) If the radius of the sphere is doubled, by what factor is the area increased?

(c) If the radius of the earth is 6379 km, find the surface area of the earth.

(d) If 70.6% of the surface area of the earth is water, how much km^2 of the surface is land?

5. The formula to calculate the volume of a sphere is

$$V = \frac{4}{3}\pi r^3.$$

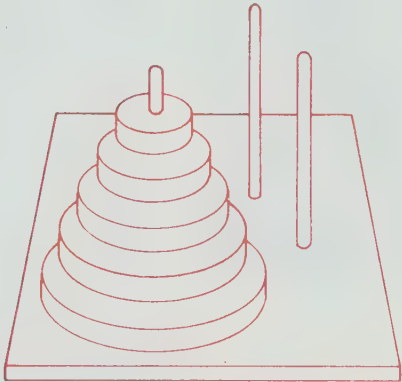
- (a) Find the volume of a sphere whose radius is 47.4 cm. (use $\pi = 3.1416$)
- (b) If the radius of the sphere is doubled, by what factor is the volume increased?
- (c) If the radius of the earth is 6379 km, find the volume.

6. The braking distance of a car equipped with steel belted radial tires is given by the formula

$$d = v(0.0075v - 0.63).$$

Find the braking distance for such a car at

- (a) 50 km/h
 - (b) 85 km/h
7. If the radius of the earth is 6379 km, how far will a satellite 295 km above the surface of the earth travel in one rotation? (Assume a circular orbit).
8. The object of the puzzle is to move all of the disks to another peg one at a time. More than one disk may be placed on any peg, but they must always be arranged in order with the largest on the bottom. The idea is to complete the task in the least number of moves.



Formula for least number of moves:

$$M = 2^n - 1$$

Compute the value of M for each value of n .

Number of disks n	Number of moves M
3	
4	
5	
6	
7	
8	
9	
10	

Probability and Statistics

Probability is the very guide of life.

Cicero



GIROLAMO CARDANO (1501–1576)

Cardano, the foremost pioneer of the theory of probability, was a man of many talents, active in a dozen arts and sciences. Aside from being a mathematician, he was one of the two most famous doctors of his day, the most widely read popularizer of science in the sixteenth century, a gambler, an astrologer, a dabbler in spiritualism, and an alchemist. While he was a student at the University of Padua, gambling was both his favourite activity and his main source of income. He applied his mathematical genius to gambling and discovered some rules of probability which were published in his "Book on Games of Chance". Another of his books, "The Great Art", established him as the major expert of his day in the field of algebra.

Cardano led a very stormy life. He was constantly involved in disputes of all sorts and was finally put in jail for heresy. Some men praised him for his genius, others believed him to be an evil spirit. The contradictions in his character are summed up by the following single sentence which Cardano wrote about himself: "Nature has made me capable in all manual work, it has given me the spirit of a philosopher and ability in the sciences, taste and good manners, voluptuousness, gaiety, it has made me pious, faithful, fond of wisdom, meditative, inventive, courageous, fond of learning and teaching, eager to equal the best, to discover new things and make independent progress, of modest character, a student of medicine, interested in curiosities and discoveries, cunning, crafty, sarcastic, an initiate in the mysterious lore, industrious, diligent, ingenious, living only from day to day, impertinent, contemptuous of religion, grudging, envious, sad, treacherous, magician and sorcerer, miserable, hateful, lascivious, solitary, disagreeable, rude, divinator, envious, obscene, lying, obsequious, fond of the prattle of old men, changeable, irresolute, indecent, fond of women, quarrelsome, and because of the conflicts between my nature and soul I am not understood even by those with whom I associate most frequently."

Everyone has some intuitive ideas about probability. It is common to hear such statements as "The Argonauts will probably win tomorrow's game," "There is a good chance of rain today," and "It is very unlikely that all four children in a family will be girls."

In the following sections we shall make these intuitive ideas precise by giving a mathematical definition of probability.

13.1 MATHEMATICAL PROBABILITY

Tossing a coin is an example of an experiment. Other examples of experiments are "rolling a die," "dealing 5 cards from a deck," and "tossing a coin, then rolling a die." We call the set of all possible outcomes the *sample space* of the experiment and we can list these possible outcomes with the aid of a figure known as a *tree diagram*. Figure 13-1 shows the 2 possible outcomes when a coin is tossed. Figure 13-2 pictures the 6 possible outcomes when a die is rolled.

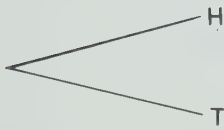


Figure 13-1

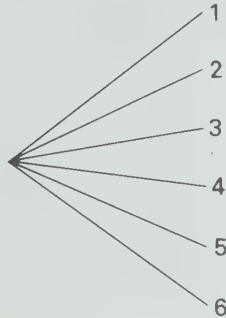


Figure 13-2

In the coin-tossing experiment the sample space is $\{H, T\}$. Of the 2 equally likely outcomes only 1 is heads, and so we say the probability of tossing heads is $\frac{1}{2}$. In symbols, we write

$$P(\text{heads}) = \frac{1}{2}.$$

In general if the sample space of an experiment consists of N equally likely outcomes and if S of those outcomes are considered successful (or favourable) for an event E , then we define probability as follows.

$$\begin{aligned} \text{probability of} &= \frac{\text{number of successful outcomes}}{\text{total number of possible outcomes}} \\ \text{an event } E & \\ P(E) &= \frac{S}{N} \end{aligned}$$

When we say that the probability of obtaining a head in a coin toss is $\frac{1}{2}$, we do not mean that heads will occur exactly half the time. However, after a large number of tosses we expect to obtain heads about half the time. Probability theory tells us what to expect in the long run, but not in a specific event. Even if heads have come up 20 times in a row, it would not be wise to bet a fortune that the next toss will be tails, because the probability of heads on the next toss is still $\frac{1}{2}$.

EXAMPLE 1. Two coins are tossed. What is the probability of obtaining (a) at least one head? (b) no head?

Solution

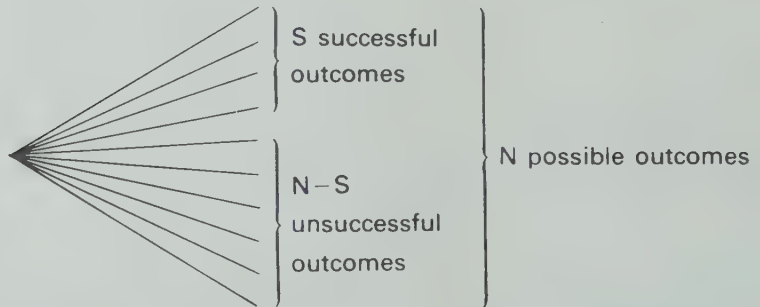
There are 4 equally likely outcomes which we list in a tree diagram.



Three of the four outcomes contain a head and so

$$(a) P(\text{at least one head}) = \frac{3}{4} \quad (b) P(\text{no head}) = \frac{1}{4}$$

Notice that the sum of the probabilities in Example 1 is 1. This always happens when we calculate the probabilities of the occurrence and non-occurrence of an event.



The probability of a successful outcome is $p = \frac{S}{N}$ and the probability of an unsuccessful outcome is $q = \frac{N-S}{N}$. Adding we get

$$p + q = \frac{S}{N} + \frac{N-S}{N} = \frac{S + (N-S)}{N} = \frac{N}{N} = 1.$$

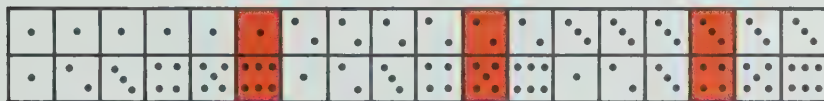
Thus

$$p + q = 1.$$

If p is the probability that an event will occur, then $0 \leq p \leq 1$. If $p = 1$, the event is a certainty. If $p = 0$, it is impossible. The closer p is to 1, the more likely the event. The closer p is to 0, the less likely the event. If q is the probability that the event will not occur, then $q = 1 - p$.

EXAMPLE 2. (a) What is the probability of rolling a 7 with two dice?
 (b) What is the probability of not rolling a 7?

Solution



(a) Total number of possible outcomes: $N = 36$ (b) $P(\text{not a } 7) = 1 - P(7)$
 Number of successful outcomes: $S = 6$

$$\therefore P(7) = \frac{6}{36} = \frac{1}{6}$$

$$= 1 - \frac{1}{6} = \frac{5}{6}$$

EXERCISE 13.1

- A** 1. A single die is rolled.
 (a) How many possible outcomes are there?
 (b) If a success is rolling a number greater than 2, how many successes are there?
 (c) What is the probability of rolling a number greater than 2?
 (d) What is the probability of rolling a number less than 3?
2. A letter is selected at random from the letters of the word BRAVE. What is the probability that the letter selected is
 (a) a vowel? (b) a consonant?
3. A student, trying to find the probability of rolling a 7 with two dice argued as follows: "There are 11 possible outcomes when 2 dice are rolled: 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12. Since only one of these is 7, the probability of rolling 7 is

$$P(7) = \frac{\text{number of successful outcomes}}{\text{total number of possible outcomes}} = \frac{1}{11}."$$

What is wrong with the student's argument?

- B** 4. A digit is chosen at random from the number 6 893 512.
 (a) List the sample space and draw a tree diagram.
 (b) List the event that the digit is an odd number, i.e., list the set of successful outcomes.
 (c) What is the probability that the chosen digit is an odd number?
 (d) What is the probability that the chosen digit is an even number?
5. A letter is chosen at random from the word STATISTICS.
 (a) List the sample space.
 (b) Find the probability that the letter is a vowel.
 (c) Find the probability that the letter is a consonant.

WORD LADDER

Start with the word "mine" and change one letter at a time to form a new word until you reach "coal". The best solution has the fewest steps.

m i n e
 _ _ _ _
 _ _ _ _
 _ _ _ _
 _ _ _ _
 _ _ _ _
 c o a l

6. A box contains 8 green marbles, 12 white marbles, and 4 blue marbles. You are to pick one marble at random. Find the probability of picking a marble which is
- (a) green (b) white (c) blue
 (d) not green (e) not white (f) not blue
7. A coin is tossed 3 times.
- (a) List the sample space and draw a tree diagram.
 (b) Find the probability of obtaining
- (i) 3 heads (ii) exactly 2 tails (iii) at least 2 tails.
8. Find the probability of rolling a 5 with two dice.
9. A deck of playing cards is shuffled and a card is drawn. Find the probability that the card is
- (a) a black card (b) a club (c) not a club (d) a king
 (e) the king of clubs (f) a red jack (g) not a jack (h) a face card
10. An integer from 1 to 40, inclusive, is chosen at random. What is the probability that the integer
- (a) is odd? (b) is even? (c) is less than 9?
 (d) is divisible by 4? (e) ends in a 6? (f) is prime?
11. 25 000 lottery tickets have been sold and Jean has bought one of them. There are 10 prizes. What is the probability that Jean will win
- (a) a prize? (b) first prize? (c) first or second prize?

13.2 EMPIRICAL PROBABILITIES

In the preceding section we were able to calculate probabilities by determining theoretically the number of successful outcomes of an experiment. But in many useful applications of probability this is not possible. In empirical probability we make predictions based on past performances. When the weather forecast says that the probability of rain is 70%, what is meant is that, in the past, rain has occurred on 70% of the days with similar climatic conditions. When an actuary says that the probability of a twenty year old man living to age 70 is $\frac{3}{5}$, he means that in the recent past 60% of the Canadian men who have reached the age of 20 have lived to age 70.

Ever since the eighteenth century, when the first life insurance companies were formed, actuaries have calculated life insurance premiums by using mortality tables to find the probability of a person living to a certain age. For example, Table 13-1 is a modern Canadian mortality table which gives the number living at a given age per 100 000 live births.

EXAMPLE 1. Use Table 13-1 to calculate the probability that a man of age 20 will live to (a) age 40 (b) age 70.

Solution

(a) The table shows that out of the original sample of males, 95 915 were alive at age 20 and 92 508 of those men were alive at age 40.

$$N = 95\,915 \quad S = 92\,508$$

If $AD = 3$ and $AE = 4$, find the length of arc BC .

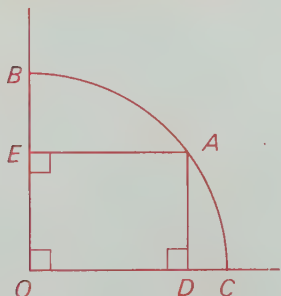


Table 13-1 Canadian Life Table

Age	Male			Female		
	Number living at each age	Number dying between each age and the next	Expectation of life (a)	Number living at each age	Number dying between each age and the next	Expectation of life (a)
At birth	100 000	2 525	68.75	100 000	2 008	75.18
1 a	97 475	156	69.53	97 992	130	75.71
2 a	97 319	102	68.64	97 862	86	74.81
3 a	97 217	88	67.71	97 776	68	73.88
4 a	97 129	74	66.77	97 708	61	72.93
5 a	97 055	268	65.82	97 647	202	71.79
10 a	96 787	271	61.00	97 445	154	67.12
15 a	96 516	601	56.16	97 291	235	62.22
20 a	95 915	855	51.50	97 056	271	57.37
25 a	95 060	753	46.94	96 785	311	52.52
30 a	94 307	767	42.29	96 474	428	47.68
35 a	93 540	1 032	37.62	96 046	630	42.88
40 a	92 508	1 615	33.01	95 416	979	38.15
45 a	90 893	2 594	28.55	94 437	1 536	33.51
50 a	88 299	4 180	24.31	92 901	2 333	29.02
55 a	84 119	6 258	20.38	90 568	3 467	24.70
60 a	77 861	8 877	16.81	87 101	5 160	20.58
65 a	68 984	11 436	13.63	81 941	7 568	16.71
70 a	57 548	13 544	10.83	74 373	10 892	13.14
75 a	44 004	14 859	8.37	63 481	15 273	9.94
80 a	29 145	13 552	6.36	48 208	18 321	7.26
85 a	15 593	9 401	4.79	29 887	16 534	5.16
90 a	6 192	4 551	3.60	13 353	9 767	3.60
95 a	1 641	1 389	2.71	3 586	3 148	2.48
100 a	252		2.04	438		1.69

From: *Canadian Life Table, 1966, Canada Year Book, 1973*

The probability of a twenty year old man living to age 40 is

$$p = \frac{92\,508}{95\,915} \doteq 0.964$$

(b) Similarly the probability that a twenty year old man will live to age 70 is

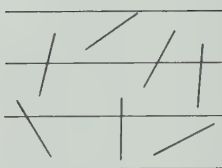
$$p = \frac{57\,548}{95\,915} \doteq 0.600$$

The answer to Example 1(b) does not mean that $\frac{3}{5}$ of every group of twenty year old men will live to age 70. We cannot predict the life span of a single person or members of a small group. Empirical probabilities only apply to large populations and can depend on time and local conditions.

EXERCISE 13-2

In questions 1-3 use Table 13-1 and calculate the answers to 3 decimal places.

- Find the probability that a 20 year old woman will live to
(a) age 40 (b) age 70.
- Find the probability that a man of the given age will live another 10 a.
(a) 30 a (b) 40 a (c) 50 a (d) 60 a
- Find the probability that a fifteen year old person will live to be the following ages if the person is
(a) a boy (b) a girl.
(i) 50 (ii) 70 (iii) 90



- For this question you need a toothpick, needle, or matchstick. On a blank piece of paper construct several parallel lines so that the distance between the lines is equal to the length of the object that you are using. If the object is dropped at random on the paper within the set of parallel lines, determine the empirical probability that it will land touching one of the lines. Do this by throwing the object 100 times and calculating the cumulative value of $\frac{S}{N}$ after every 5 throws.

[In 1777 a French mathematician, Buffon, found the probability to be $\frac{2}{\pi}$. Does your value agree with his?]

13.3 MUTUALLY EXCLUSIVE EVENTS

Two events are *mutually exclusive* if they cannot both occur at once. If an event E can occur as either of two mutually exclusive outcomes then we can add their probabilities to obtain the probability of E .

EXAMPLE 1. A game consists of rolling a single die. You win if the die shows a 3 or a 5. What is the probability of winning?

Solution

(i) Total number of possible outcomes: $N = 6$

Number of successful outcomes: $S = 2$

$$P(\text{winning}) = \frac{2}{6} = \frac{1}{3}$$

(ii) Rolling a 3 and rolling a 5 are mutually exclusive events (the die cannot turn up 2 and 5 simultaneously), and so we add probabilities:

$$\begin{aligned} P(\text{winning}) &= P(\text{rolling 2 or 5}) \\ &= P(2) + P(5) \\ &= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} \\ &= \frac{1}{3} \end{aligned}$$

If an event can occur in two mutually exclusive ways which have probabilities p_1 and p_2 , then the probability that the event will occur is the sum:

$$p = p_1 + p_2$$

When applying this rule, we must be certain that the two events are mutually exclusive. The following example shows that the addition rule cannot be used when events are not mutually exclusive.

EXAMPLE 2. An integer from 1 to 10 is chosen at random. What is the probability of choosing an even number or a number less than 5?

Solution

The sample space is $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. $\therefore N = 10$

The event is $\{1, 2, 3, 4, 6, 8, 10\}$. $\therefore S = 7$

The probability is $p = \frac{7}{10}$.

Another Solution

Let A be the event of choosing an even number.

Let B be the event of choosing a number less than 5.

$$p_1 = P(A) = \frac{5}{10} = \frac{1}{2}.$$

$$p_2 = P(B) = \frac{4}{10} = \frac{2}{5}.$$

$$p_1 + p_2 = \frac{1}{2} + \frac{2}{5} = \frac{9}{10}$$

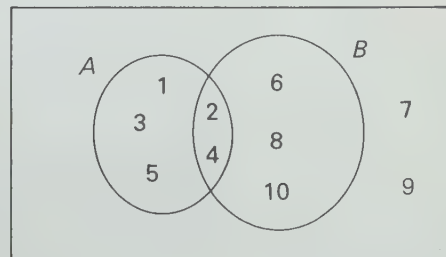
Note that $p \neq p_1 + p_2$ because A and B are not mutually exclusive events. It is possible to choose a number which is simultaneously even and less than 5 (namely 2 or 4).

Since A and B are not mutually exclusive we must use

$$P(E) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{5}{10} + \frac{4}{10} - \frac{2}{10}$$

$$= \frac{7}{10}$$



We subtract $P(A \cap B)$ since the numbers in $A \cap B = \{2, 4\}$ have been counted twice.

EXERCISE 13-3

- A** 1. Which of the following pairs of events are mutually exclusive?
- (a) Picking a green marble and picking a white marble.
 - (b) Having blue eyes and black hair.
 - (c) Dealing a spade and dealing a jack.
 - (d) Dealing a spade and dealing a club.
 - (e) Living in Ottawa and living in Nova Scotia.
 - (f) Living in Vancouver and living in British Columbia.
 - (g) Choosing an even number and choosing a prime number.
- B** 2. A box contains 6 green marbles, 4 white marbles, 9 red marbles, and 5 blue marbles. You are to pick one marble at random. Find the probability of picking
- (a) a green marble
 - (b) a green or a white marble
 - (c) a red or a blue marble
 - (d) a red or a green marble
 - (e) a white or a red or a blue marble.
3. A deck of playing cards is shuffled and a card is drawn. Find the probability that the card is
- (a) a red ace or a black face card
 - (b) a diamond or a black queen.
4. In the game of craps, the player wins on the first roll if he rolls a 7 or 11 with a pair of dice. He loses if he rolls a 2, 3, or 12 on the first roll.
- (a) What is the probability of winning on the first roll?
 - (b) What is the probability of losing on the first roll?
5. Two dice are rolled. What is the probability that the result is
- (a) greater than 8?
 - (b) less than 8?
 - (c) a prime number?
6. Three dice are rolled. Find the probability that the total is
- (a) 2
 - (b) 3
 - (c) 4
 - (d) 5
 - (e) less than 6.
7. Four coins are tossed. What is the probability of getting
- (a) 2 tails and 2 heads?
 - (b) 3 tails and a head?
 - (c) at least 2 tails?

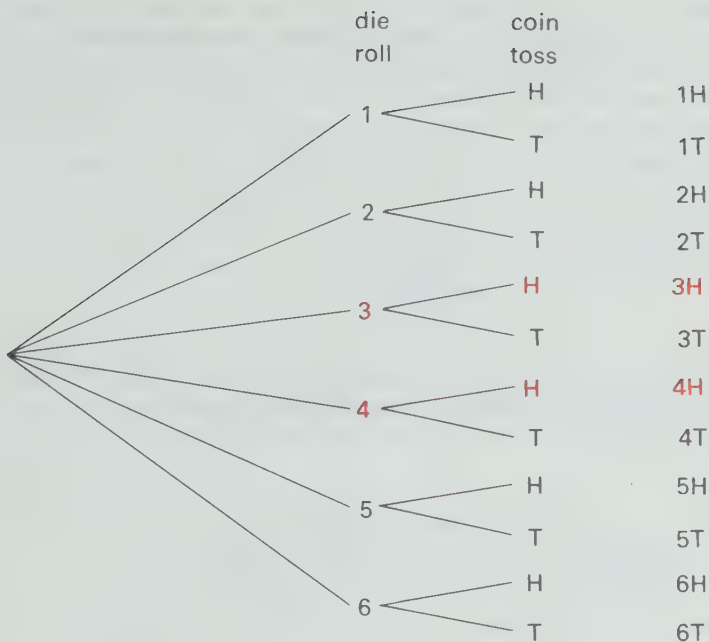
13.4 INDEPENDENT EVENTS

Two events are called *independent* if neither has an influence on the other. For instance if a die is rolled and a coin is tossed, the events of the die coming up 3 or 4 and the coin coming up heads are independent.

EXAMPLE 1. A game is played by rolling a die and then tossing a coin. You win if the die shows a 3 or a 4 and the coin shows heads. What is your probability of winning?

Solution

The tree diagram involves two events: first the roll of a die and then the toss of a coin.



The sample space $\{1H, 1T, 2H, 2T, 3H, 3T, 4H, 4T, 5H, 5T, 6H, 6T\}$ contains 12 equally likely outcomes. (For example $5T$ means the die shows 5 and the coin shows tails.) Only 2 of the outcomes are favourable: $3H$ and $4H$. Thus $N = 12$ and $S = 2$.

$$P(\text{winning}) = \frac{2}{12} = \frac{1}{6}$$

Note that this probability is the product of the individual probabilities of the two events.

$$P(3 \text{ or } 4) = \frac{2}{6} = \frac{1}{3} \quad P(\text{heads}) = \frac{1}{2}$$

The probability of both events happening is

$$P(\text{winning}) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$$

The reason for multiplying the probabilities can be seen in the tree diagram. There are 2 small branches on each of 6 main branches and so there are 12 possible outcomes which in turn means a probability of $\frac{1}{12}$ for each outcome.

If the probabilities of 2 independent events are p_1 and p_2 , then the probability that both events will occur is the product

$$p = p_1 p_2$$

Put the numbers from 1 to 9 in the spaces to make the statements true.

$$\square - \square + \square = 4$$

$$\square + \square \div \square = 4$$

$$\square \times \square \div \square = 4$$

$P(A \cap B) = P(A) \cdot P(B)$ for independent events.

EXAMPLE 2. (a) Suppose a marble is drawn at random from a bag containing 4 red and 3 blue marbles. Then the marble is replaced in the bag and another is drawn. What is the probability that both marbles are red?

(b) If the first marble is not replaced, what is the probability that both marbles are red?

Solution

(a) On the first draw $N=7$ and $S=4$.

$$P(\text{red}) = p_1 = \frac{4}{7}$$

Since the marble is replaced and the second draw is again random it is independent of the first draw. Again the probability of getting a red marble is

$$p_2 = \frac{4}{7}.$$

Thus the probability that both marbles are red is

$$p_1 p_2 = \frac{4}{7} \cdot \frac{4}{7} = \frac{16}{49}$$

(b) As in part (a)

$$P(\text{red}) = p_1 = \frac{4}{7}.$$

If the first marble is red, there are 3 red marbles left in the bag.
 $\therefore N=6, S=3$.

The probability that the second marble is red is

$$p_2 = \frac{3}{6} = \frac{1}{2}.$$

The probability that both marbles are red is

$$p_1 p_2 = \frac{4}{7} \cdot \frac{1}{2} = \frac{2}{7}$$

In Example 2(b) note that if the first event has occurred before the second and we know its outcome, then we can use the outcome of the first event as one of the conditions in calculating the probability of the second event.

EXERCISE 13-4

1. What is the probability of tossing 3 heads in a row with a coin?
2. A game is played by first tossing a coin and then rolling a die. You win if the coin shows tails and the die shows an even number.
 - (a) Draw a tree diagram.
 - (b) Find your probability of winning.

3. A player rolls a pair of dice, one red and the other green.
 - (a) Find the probability of rolling a 2 on the red die and 3 on the green die.
 - (b) Find the probability of rolling a 2 on one die and a 3 on the other die.
4. A player rolls a pair of dice and picks a card from a deck of playing cards. What is the probability of rolling a 9 and picking a spade?
5. Two letters are chosen at random from the English alphabet. If y is considered to be a consonant, find the probability that
 - (a) both are vowels (b) both are consonants.
6. A pair of dice is rolled 4 times in a row. Find the probability of rolling less than 5 on every roll.
7. A bag contains 4 white, 3 blue, and 6 red marbles. A marble is drawn from the bag, replaced, and another marble is drawn. Find the probability that
 - (a) both marbles are red (b) both marbles are blue
 - (c) the first marble is red and the second is blue
 - (d) one marble is red and the other is blue (e) neither is red.
8. Do question 7 assuming that the first marble is not replaced.
9. A card is dealt from a deck of playing cards and is replaced. The deck is shuffled and another card is dealt. What are the probabilities of the following events?
 - (a) Both cards are diamonds (b) The first card is a diamond and the second is a club (c) One card is a diamond and the other is a club
 - (d) Both cards are aces (e) A king and a jack are dealt (f) The ace of spades is dealt twice (g) The same card is dealt twice.
10. Do question 9 assuming that the first card is not replaced.
11. 5000 lottery tickets have been sold and Tom has bought 4 of them. There are 10 prizes. What is the probability that Tom will win
 - (a) first prize? (b) first and second prizes?
 - (c) first, second, and third prizes?
- C 12. Three identical bags A , B , and C contain black and white marbles as follows: A , 4 black and 1 white; B , 2 black and 3 white; and C , 3 black and 7 white. A bag is chosen at random and a marble is chosen from that bag. Find the probability that the marble is white.

13.5 STATISTICS

Statistical thinking will one day be as necessary for efficient citizenship as the ability to read and write.

H. G. Wells

Statistics is concerned with the collection of numerical facts, called *data*, the organization and presentation of these facts, and finally the interpretation of the data.

WORD LADDER

Start with the word "four" and change one letter at a time to form a new word until you reach "five". The best solution has the fewest steps.

f o u r

f i v e

The subject of statistics has been applied to many different areas, including insurance, medicine, advertising, electronics, television audiences, population growth, energy consumption, student enrollment, etc.

The following are some examples of the familiar use of statistics.

- (a) The Gallup Poll uses statistics to predict the outcome of elections.
- (b) Based on statistics, drivers under 25 pay more for car insurance.
- (c) The Nielsen television ratings help determine which TV programs should be discontinued.
- (d) Statistics show what brand of toothpaste is most effective in reducing tooth decay.
- (e) Schools are built or closed according to statistics.
- (f) Betting odds on sports events are calculated using statistics.

You rarely live through a day without having statistics forced on you. Everyone is bombarded with statistics by advertisers, commentators, politicians, pollsters, and salesmen. Knowledge of statistics is fast becoming an important tool for everyone.

SAMPLING

The Nielsen television ratings are very important to producers of television programs. In order to determine the popularity of a show, the pollsters cannot ask every individual who watches television for a reaction to the show. What they do is take a *sample*. A relatively small group of TV viewers is asked for its reactions. From their comments, the polling company will then generalize for all television viewers.

A *sample* is a small group of individuals (or objects) selected to stand for a larger group called the *population*.

There are a number of problems that can arise in taking a sample. The first problem is that of sample size. If the sample is too small the individuals (or objects) may not be representative of the *population*. On the other hand a large sample is usually very costly.

A good sample should be large enough to contain sufficient information about the population.

The most important requirement of a sample is that it be a *random* one. This means that each individual in the population should have an equally likely chance (or probability) of being selected. Also the "target" population must be kept in mind. For example, it is unlikely that you would get an unbiased reaction on whether Sunday sports should be permitted if you sampled a crowd at a ball game.

EXERCISE 13-5

- A 1. Selecting names for a survey from a telephone book does not produce a random sample. Why not?
- 2. (a) Conducting an election survey at a football game would omit a large section of the population. Who?
- (b) What type of survey would you conduct at a football game?

3. Surveying every 10th person that passes through a turnstile at a subway station does not give a random sample of a city's population. Why not?
4. In order to predict the results of an election a national magazine sent a questionnaire to all of its subscribers. What is wrong with this method of surveying?
5. The Ace Fireworks Company tests (samples) one out of every 500 firecrackers it makes.
 - (a) Why is it not feasible to sample the total population?
 - (b) What other industries would test a small percentage of the product?
 - (c) What industries test 100% of their products?

13.6 FREQUENCY DISTRIBUTIONS

Following is the "raw data" resulting from fifty rolls of a single die.

14231 64536 11336 33265 46613
55522 42544 61243 54345 63253

We can tabulate the data in a frequency distribution table where we determine the number of times each value occurs.

Value	Tally mark	Frequency
1	++++ /	6
2	++++ //	7
3	+++++ + /	11
4	++++ + /	9
5	++++ + /	9
6	++++ + /	8

Formulating the data in a frequency table hides the order in which the values occurred. From the table, we see the frequency of each value. This information can be presented in a graph called a histogram as in Figure 13-3, where the heights of the rectangles represent the frequencies. In this histogram, each bar occupies a space 0.5 below the observed value to 0.5 above.

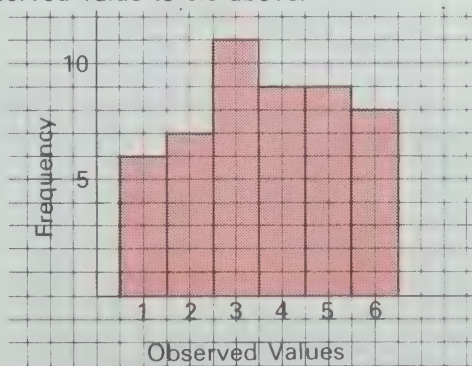
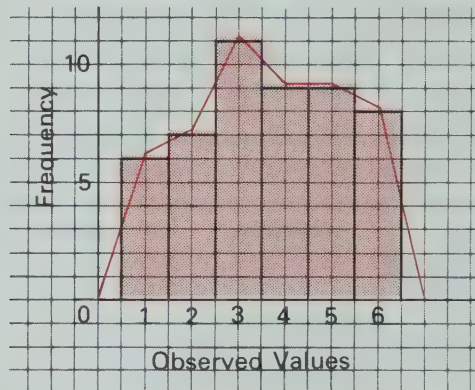


Figure 13-3

We can now read the frequencies from the vertical axis. When displaying a frequency distribution in a histogram, we are not really interested in the order, or in the individual values, but rather in general characteristics such as the shape of the distribution, most frequent scores, and range.

If we join the midpoints of the tops of the rectangles of the histograms in Figure 13.3, the graph so formed is called the frequency polygon which is shown in Figure 13.4. The distribution is extended one unit beyond the highest and lowest observed values so that the polygon is completed by drawing lines to these values on the baseline. Examination of Figure 13.4 shows that the histogram and the frequency polygon showing the same data and drawn with the same scale are equal in area. This is significant because the area is proportional to the frequency of the values. Either a histogram or a frequency polygon can be used when we wish to compare the distributions of two or more groups of data.

Figure 13-4

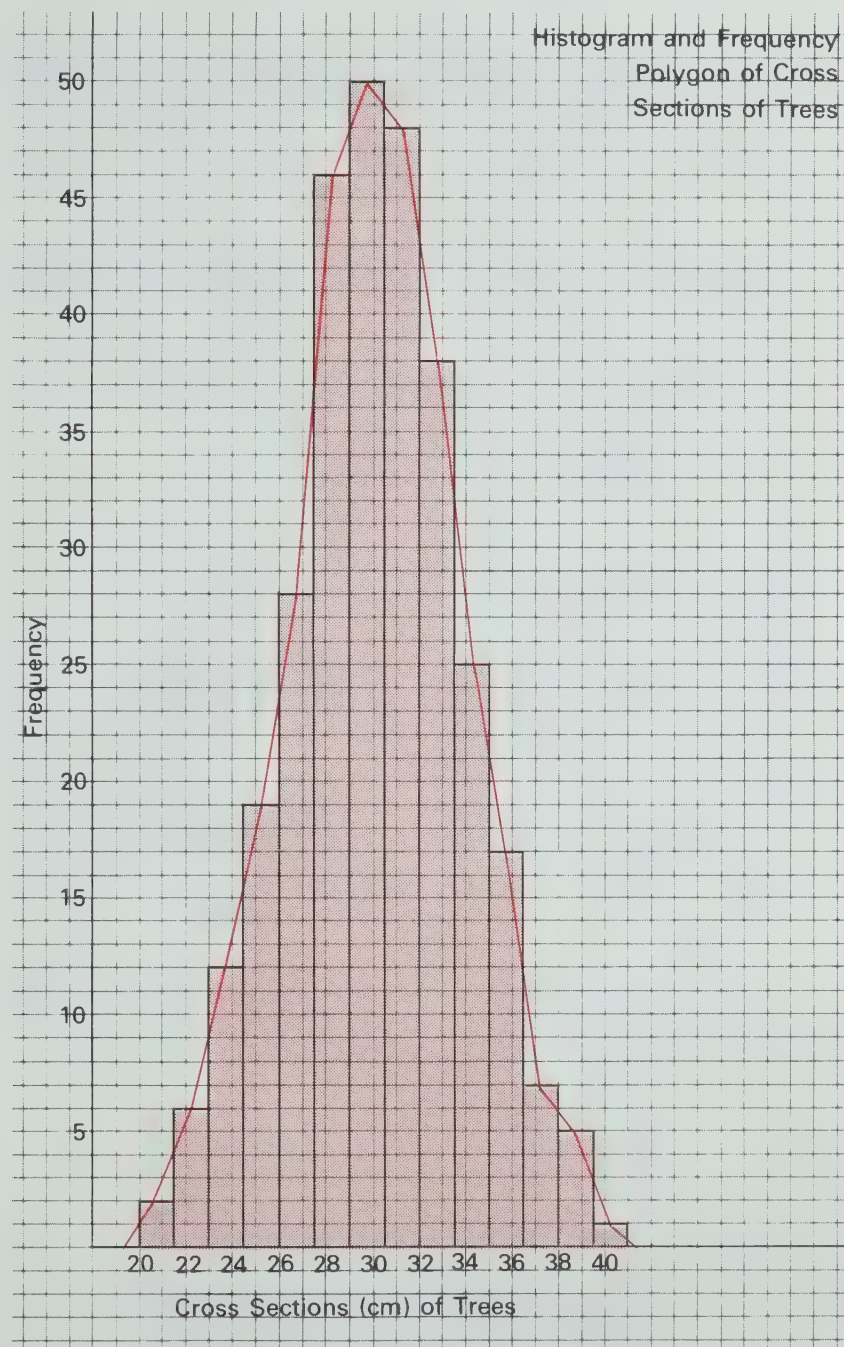


EXAMPLE 1. In a survey of cross sections of trees in a forest, data were recorded and the following frequency distribution was prepared. Prepare a histogram and frequency polygon.

Diameter in cm	Frequency
20.0–21.5	2
21.5–23.0	6
23.0–24.5	12
24.5–26.0	19
26.0–27.5	28
27.5–29.0	46
29.0–30.5	50
30.5–32.0	48
32.0–33.5	38
33.5–35.0	25
35.0–36.5	17
36.5–38.0	7
38.0–39.5	5
39.5–41.0	1

Solution

Histogram and frequency polygon of cross sections of trees.



EXERCISE 13-6

- B** 1. Tabulate the following data to find the frequency, then construct a corresponding histogram and frequency polygon.

4, 2, 4, 6, 1, 4, 2, 2, 3, 6, 1, 4, 4, 5, 2, 3, 4, 4, 5, 5, 1, 1, 6, 3, 4, 6, 3, 2, 3, 2, 4, 3, 3, 5, 2, 5, 6, 5, 2, 4, 6, 4, 6, 1, 2, 2, 5, 1, 3, 3.

2. Fifty applicants for employment at Acme Graphics were required to answer an aptitude test consisting of 50 questions. The raw scores were

25, 25, 26, 27, 28, 28, 29, 30, 30, 30, 30, 31, 31, 31, 31, 32, 32, 32, 33, 33, 33, 34, 34, 34, 34, 35, 35, 35, 35, 35, 35, 35, 36, 36, 36, 36, 37, 37, 37, 37, 38, 38, 38, 41, 41, 42, 43.

- (a) Prepare a frequency distribution table and construct the corresponding histogram and frequency polygon.

- (b) Prepare a frequency distribution table using class intervals of 2. (25–26, 27–28, 29–30, . . . , 43–44).

3. Given the data for masses of 50 students in kilograms:

55, 56, 56, 57, 58, 59, 59, 60, 60, 62, 62, 63, 64, 64, 65, 66, 66, 67, 68, 69, 70, 70, 70, 71, 72, 72, 72, 72, 72, 73, 73, 74, 75, 75, 77, 77, 78, 78, 79, 79, 80, 80, 81, 83, 84, 86, 88, 89, 91, 98.

- (a) Tabulate the data using two class intervals 50–75 and 76–100 then construct the corresponding histogram and frequency polygon.

- (b) Tabulate the data using five equal class intervals, then construct the corresponding histogram and frequency polygon.

4. In order to determine whether or not to request a radar trap in front of a school where the speed limit was 45 km/h, a class of students recorded the following car speeds in km/h during 1 h.

39	57	42	40	52	43	48	51	41	42
53	24	45	49	41	53	55	48	51	51
44	50	57	48	50	47	51	48	57	43
36	54	50	45	29	49	57	47	42	49
43	51	50	53	51	48	47	51	50	45
54	42	44	52	42	58	45	53	51	38
61	45	51	50	57	49	47	47	42	48
44	57	41	43	35	59	54	51	50	50

- (a) Prepare a histogram and frequency polygon.

- (b) Should a speed trap be installed?

13.7 MEASURES OF CENTRAL TENDENCY

In the previous section we used graphs to provide a “picture” of the data in order to help analyze the data. Statisticians also describe data by indicating a centre of the distribution called a *measure of central tendency*. Three common measures of central tendency are the *mean*, *median*, and *mode*.

(i) The MEAN:

The mean or average of a set of numbers is found by adding them together and dividing the total by the number of numbers added.

For a set of values $x_1, x_2, x_3, \dots, x_n$ the mean is

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \dots + x_n}{n}$$

(ii) The MEDIAN:

When a set of numbers is arranged in order (smallest to largest), then the middle number is the median. If there is an even number of numbers, then the median is the average of the middle two numbers. Knowing the median enables one to tell whether any number is in the top half or bottom half of the group.

(iii) The MODE:

The mode of a set of numbers is the number that occurs most often. If every number occurs only once, then we say there is no mode. It is also possible, however, for a set of numbers to have several modes.

EXAMPLE 1. A baseball team pays the following annual salaries to its players.

Number of players	Salary
1	\$250 000
1	\$200 000
2	\$190 000
3	\$170 000
5	\$100 000
4	\$90 000
3	\$80 000
3	\$70 000
2	\$60 000
1	\$50 000

Determine the (a) mean, (b) median, and (c) mode of these salaries.

Put the numbers from 1 to 9 in the spaces to make the statements true.

$$\square + \square \div \square = 3$$

$$\square - \square \div \square = 3$$

$$\square - \square - \square = 3$$

Solution

(a) For data of this type we can use the following method to determine the mean.

x_i	f	$f \cdot x_i$
250 000	1	250 000
200 000	1	200 000
190 000	2	380 000
170 000	3	510 000
100 000	5	500 000
90 000	4	360 000
80 000	3	240 000
70 000	3	210 000
60 000	2	120 000
50 000	1	100 000
TOTAL	25	2 870 000

$$\begin{aligned}\bar{x} &= \frac{2\,870\,000}{25} \\ &= 114\,800\end{aligned}$$

The mean salary is \$114 800.

(b) There are 25 people on the payroll so the median is in the 13th position. The median salary is \$90 000.

(c) The mode is \$100 000.

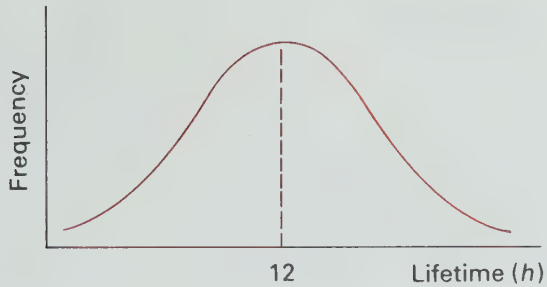
EXAMPLE 2. A purchasing agent for a calculator manufacturer is considering three different brands of batteries to be used in calculators. The following data were received from a testing laboratory.

	Brand of Battery		
	Alpha	Beta	Delta
Mean lifetime (h)	12	12	12
Median lifetime (h)	12	8	15

How should the agent interpret these data?

Solution

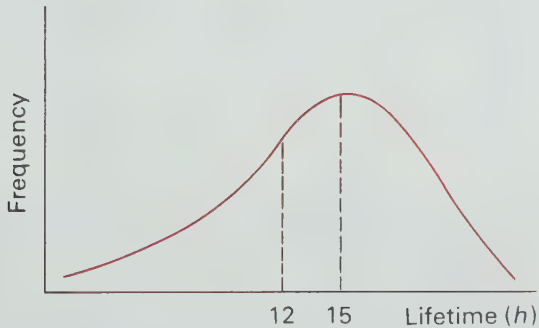
The agent should see that the graph of the lifetime of the Alpha battery is symmetric.



The data on the Beta battery indicates that most of these batteries wear out very quickly but a few have a high life expectancy. When the median is less than the mean the numbers probably “tail off” to the right.



The data on the Delta brand suggest that a few batteries wear out very quickly but a lot of them last a long time. When the median is higher than the mean the numbers probably “tail off” to the left.



The purchasing agent may then assess the position as follows:

- (a) If Alpha batteries are used there will be a balance between the batteries that last a short time and those that last a long time.
- (b) If Beta batteries are used there will be many customers whose batteries won't last very long.
- (c) If Delta batteries are used, there will be a few customers whose batteries will fail almost instantly. On the other hand, however, many customers will have batteries that last a long time.

Which one should he choose?

EXERCISE 13-7

- A** 1. State the measure of central tendency that best describes the following:
- (a) the number of L/100 km you can expect from a new car
 - (b) the hat sizes to be kept in stock
 - (c) a set of class marks
 - (d) the amount of rainfall for a country
 - (e) the helmet size of a football team.
- B** 2. Determine the mean, median and mode(s) of the following sets of data.
- (a) 5, 10, 16, 17, 7, 12, 15, 9, 12
 - (b) 15, 12, 10, 13, 10, 11, 13, 14
 - (c) 15, 18, 9, 13, 17, 10, 22, 16
 - (d) 5, 2, 3, 4, 2, 6, 9, 8, 7, 5, 2
3. The lifetime of a certain brand of tires has a mean of 50 000 km and a median of 40 000 km. Describe the performance of these tires.
4. The lifetime of a certain brand of TV picture tube has a mean of 18 000 h and a median of 2 700 h. Describe the performance of these picture tubes.
5. A firm pays the following annual salaries:
- | | |
|-------------------------------|---------------------------|
| 1 President at \$130 000 | 3 Accountants at \$50 000 |
| 2 Vice-presidents at \$70 000 | 8 Sales staff at \$35 000 |
| 1 Treasurer at \$60 000 | 9 Consultants at \$30 000 |
- (a) Determine the mean, median and mode.
 - (b) Which statistic best indicates the level of salary paid by the company? Why?
6. The sales staff for an automobile dealership have the following new car sales for the month of June.
- | | | | |
|-------|----|--------|----|
| Carol | 15 | Pete | 12 |
| Al | 10 | Sharon | 14 |
| Terry | 9 | Bill | 9 |
| Marty | 7 | Sam | 11 |
- (a) Find the mean, median and mode of the sales.
 - (b) Suppose that the salesmanager is hiring a new salesman. What would be the most realistic estimate of expected sales he could give the new salesman?
7. A company vice-president gathered the following data on the length of time spent on coffee breaks by employees: mean 13 min; median 10 min. Describe the coffee break habits of the employees.

13.8 MEASURES OF DISPERSION

The measures of central tendency discussed in the previous section may not tell everything we want to know about a set of data. Two sets of data could have the same mean or median and yet differ greatly in their dispersion or spread.

One measure of dispersion is the range.

The range of a set of numbers is found by subtracting the smallest number from the largest.

The range is very sensitive to extreme values but does not tell us anything about how the numbers vary. For this reason we need another measure of the dispersion called the **standard deviation** (σ).

The standard deviation of a set of numbers is determined by finding (in order)

- (a) the mean of the numbers
- (b) the difference between each number and the mean
- (c) the squares of each of these differences
- (d) the mean of the squares
- (e) the square root of this mean.

The five steps may be abbreviated in the formula

$$\sigma = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \dots + (x_n - \bar{x})^2}{n}}$$

EXAMPLE 1. A basketball coach measured the heights of his players and found them to be 173 cm, 177 cm, 177 cm, 179 cm, 181 cm, 181 cm, 182 cm, 182 cm, 183 cm and 185 cm. Find the standard deviation of this set of data.

Solution

Heights	Differences	Squares
x_i	$(x_i - \bar{x})$	$(x_i - \bar{x})^2$
173	-7	49
177	-3	9
177	-3	9
179	-1	1
181	1	1
181	1	1
182	2	4
182	2	4
183	3	9
185	5	25

$$\text{sum} = 1800$$

$$\text{sum} = 112$$

$$\bar{x} = \frac{1800}{10}$$

$$\text{mean} = \frac{112}{10}$$

$$= 180$$

$$= 11.2$$

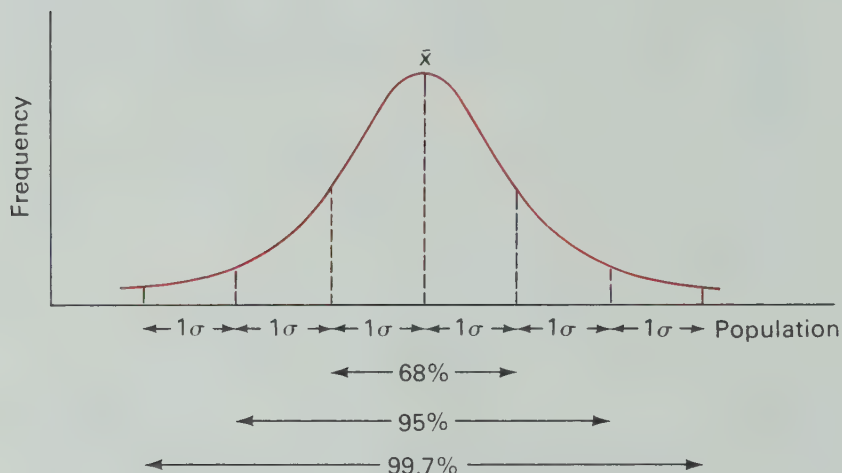
$$\sigma = \sqrt{11.2}$$

$$\doteq 3.3$$

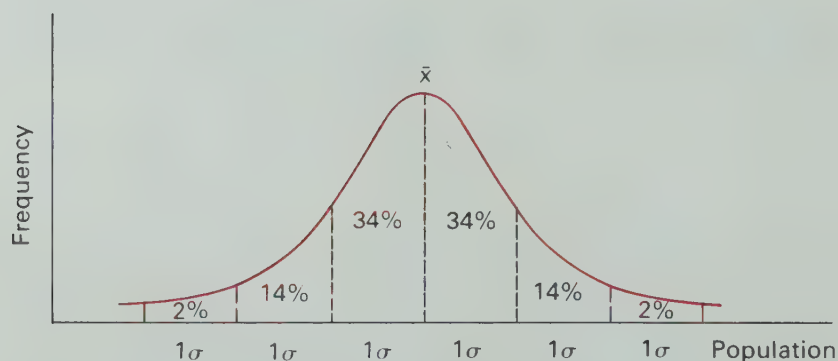
the standard deviation $\sigma \doteq 3.3$

NORMAL DISTRIBUTION

When data are distributed normally, 68% of the population will lie within one standard deviation of the mean and 95% of the population will lie within two standard deviations of the mean. Almost all (99.7%) of the population will lie within three standard deviations of the mean.



In order to facilitate calculations the distribution is simplified as follows.

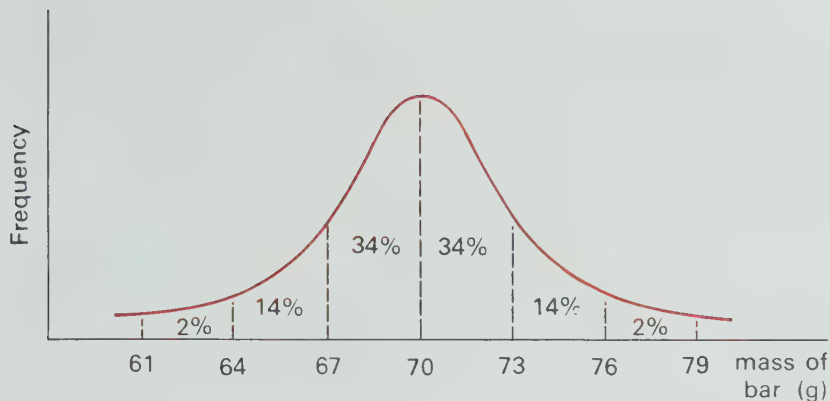


EXAMPLE 2. The quality controller at a candy factory determined that the mean mass of Krispy Nut chocolate bars is 70 g with a standard deviation of 3 g. Assuming a normal distribution,

- what percent of the bars have masses between 70 g and 76 g?
- if the company makes 8000 bars/day, how many of them will have masses less than 67 g?

Solution

Draw a normal curve.



The standard deviation is 3 g, so we can mark the horizontal axis accordingly.

- (a) 48% of the bars have masses between 70 g and 76 g.
- (b) 16% of the bars have masses less than 67 g.

$$0.16 \times 8000 = 1280$$

1280 bars will have masses less than 67 g.

EXERCISE 13-8

A 1. State the range of the following

- (a) 3, 7, 15, 19, 24
- (b) 13, 5, 36, 48, 9
- (c) 54, 13, 25, 61, 47, 8

B 2. Determine the standard deviation for the following sets of data.

- (a) 4, 5, 8, 9, 9, 9, 10, 10, 12, 14
- (b) 20, 24, 17, 15, 28, 25, 22, 20, 17, 12, 20

3. A hospital determines that the average stay in the hospital for pneumonia cases is 7 d with a standard deviation of 2 d. Assuming a normal distribution.

- (a) what percent of pneumonia cases stay in longer than 9 d?
- (b) what percent of cases stay in between 3 d and 7 d?

4. The amount of detergent used daily by a hospital is normally distributed with a mean of 150 kg and a standard deviation of 10 kg.

- (a) What percent of the days will they use less than 130 kg?
- (b) What percent of the days will they use more than 160 kg?

5. The Long Life Light Company advertises that its light bulbs have a mean life of 900 h with a standard deviation of 50 h. Assuming normal distribution,

- (a) what percent of the bulbs will last between 900 h and 1000 h?
- (b) what percent of the bulbs will last longer than 1000 h?
- (c) if a business purchases 3000 bulbs, how many can be expected to last less than 850 h?

6. A tire manufacturer advertises that its Mac III truck tire has a mean life of 60 000 km with a standard deviation of 4000 km. The local streets and sanitation department buys 600 tires. Assuming a normal distribution

- what percent of the tires will last longer than 56 000 km?
- what percent of the tires will last less than 52 000 km?
- how many tires should last longer than 68 000 km?
- how many tires will last between 56 000 km and 64 000 km?

7. The mean number of fielding errors committed by major league second basemen in one season is 7 with a standard deviation of 1. For third basemen, the mean is 9 with a standard deviation of 2.

- What would account for the differences in the mean?
- If the second baseman for the Yankees commits 9 errors in a season and the third baseman for the Blue Jays commits 11 errors, who is the better fielder?

8. The following data show the times (in minutes) that it takes firemen to reach fires in their area.

Station 1	Station 2
5.5	7
5	8
6.5	7.5
5.5	7
4.5	6.5
7	6
3	6.5
6	7
4	7.5
3	7

- Find the mean and the standard deviation of the response time for Station 1.
- Find the mean and the standard deviation of the response time for Station 2.
- What could account for the difference in the means?
- What could account for the difference in the standard deviation?
- Can we use this data to determine which station area had the better fire protection? Explain.

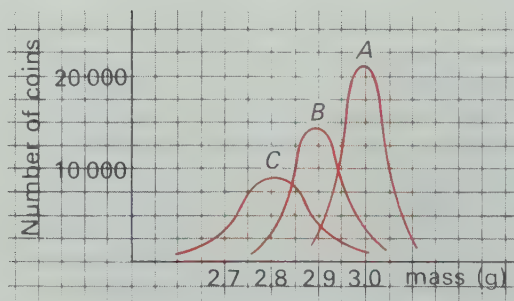


Figure 13-5

9. The graphs (Figure 13-5) show the masses of pennies that were put into circulation at the same time. Curve A shows the mass distribution when the coins were new and the other two show the mass distribution after 4 a (B) and 8 a (C) of circulation.

- As circulation time increases, what happens to the mean mass of the coins? Why?
- As circulation time increases, what happens to the standard deviation? Why?

REVIEW EXERCISE

1. A word is selected at random from the sentence *The sky is blue*. Find the probability that the number of letters in the selected word is (a) 2 (b) 3 (c) 5 (d) less than 5.

2. A ball is drawn at random from a box containing 12 red, 15 green, 5 black, and 8 white balls. Find the probability that the ball is (a) red (b) green (c) black (d) white (e) red or green (f) not white (g) neither red nor white (h) blue (i) green or white or black.

3. A number is formed by arranging the digits 1, 2, 3 in random order.

(a) List the sample space.

(b) What is the probability that the number is

(i) greater than 100?

(ii) greater than 200?

(iii) greater than 220?

(iv) greater than 330?

4. Find the probability of the following events.

(a) Drawing a club or a diamond from a deck of cards.

(b) Rolling a number less than 10 with a pair of dice.

(c) Obtaining at least one head when a coin is tossed 3 times.

(d) Not drawing an ace from a deck of cards.

(e) Arriving at a traffic light when it is green if it stays green for forty seconds, red for thirty seconds, and yellow for two seconds.

(f) Drawing a black card with an even number on it from a deck of cards.

5. Lynn's laundry contains 2 pairs of green socks, 3 pairs of blue socks, and a pair of black socks. When the dryer stops a green sock is pulled out. If a second sock is then pulled out, what is the probability that it will match the first sock?

6. A test consists of multiple choice questions. Each question has five choices, one of which is correct. An unprepared student makes random guesses for each answer. What is the probability that the student answers

(a) the first question correctly?

(b) the first 3 questions correctly?

(c) the first 3 questions incorrectly?

7. Find the probability of tossing a coin 10 times and getting tails every time.

8. Each letter of the alphabet is printed on a separate slip of paper. The 26 slips are then mixed in a bowl. Find the probability of each of the following, assuming that each slip is replaced before the next slip is drawn.

(a) Choosing 2 slips and getting NO.

(b) Choosing 3 slips and getting YES.

(c) Choosing 4 slips and getting MATH.

9. Do question 8 assuming that the slips are not replaced.

Cut this figure into 4 identical pieces that have the same shape as the original figure.



10. Determine the mean, median and mode(s) of the following sets of data.

- (a) 10, 7, 5, 9, 13, 12, 17, 5, 15
- (b) 11, 12, 15, 20, 22, 25, 24, 21, 12, 11, 10, 12, 15
- (c) 10, 16, 13, 9, 10, 12, 11, 14, 9, 10, 15, 9

11. Determine the range and standard deviation for the following sets of data.

- (a) 13, 9, 15, 12, 11
- (b) 31, 28, 21, 25, 24, 30, 33, 27, 36, 35

12. The lifetime of a certain brand of calculator battery has a mean of 60 h and a median of 50 h. Describe the performance of these batteries.

13. After one season the manager of a baseball team determined that the team had a mean batting average of 0.242. The median batting average was 0.263. Describe the batting performance of the team.

14. The members of a police department have a mean height of 180 cm with a standard deviation of 4 cm.

- (a) What percent of the department are taller than 172 cm?
- (b) What percent of the department are shorter than 184 cm?
- (c) If the department has 450 employees, how many are taller than 176 cm?
- (d) How many have heights between 172 cm and 180 cm?

15. A company uses an automated packaging device to produce 50 g bags of Karmel Korn. The machine needs frequent checking to see if it is actually putting 50 g in each bag. The following are masses of thirty bags of Karmel Korn.

54 50 47 50 51 50
53 50 47 51 50 51
52 49 46 52 50 49
52 48 48 53 49 49
51 48 49 52 49 50

- (a) Find the mean and standard deviation of this data.
- (b) What problems will be encountered if the standard deviation gets too high?
- (c) Can any problems result if the median goes below 50 g?

ANSWERS

REVIEW AND PREVIEW TO CHAPTER 1

Exercise 1.

- | | | |
|------------------------|-------------------------|-----------------------|
| 1. (a) $7w-7x+15y$ | (b) $15x-7y+12z$ | (c) $16xy+4yz-3xz$ |
| (d) $9r+s+7t$ | (e) $-x^2-2x+20$ | (f) $7m^2+4m-11$ |
| (g) $22xy+14x-9yz$ | (h) $3m-2n+4x-7y+8$ | (i) x^2+5y^2 |
| (j) $7.3u-11.4v+13.2w$ | (k) $3.4xy-4.1yz-0.2xz$ | |
| 2. (a) $2x+3y+4z$ | (b) $2x+8y+4z$ | (c) $2u-2v-w$ |
| (d) $x-3y+4z$ | (e) $7n-6mn-4m$ | (f) $-5x^2+2x+10$ |
| (g) $2x-y-4m+3n$ | (h) $-0.5x+8.3y$ | (i) $-4.8x-2.8y-6.1z$ |
| (j) $-1.8x^2+10.2x-13$ | | |
| 3. $-3u+w+4v$ | 4. x^2-6x+8 | 5. $x^2+9xy-4y^2$ |
| 6. $xy-xz+7yz$ | 7. $-2x^2+4x+8$ | 8. $-3m-4mn+5n$ |

Exercise 2.

- | | | | | | |
|-----------|---------|---------|---------|---------|----------|
| 1. (a) 24 | (b) 168 | (c) 60 | (d) 18 | (e) 19 | (f) 29 |
| (g) 3 | (h) -5 | (i) 11 | (j) 60 | (k) -38 | (l) 19 |
| (m) 20 | (n) 0 | (o) -24 | (p) -96 | (q) -10 | (r) -144 |
| 2. (a) 0 | (b) 24 | (c) -8 | (d) -5 | (e) 5 | (f) 9 |
| (g) 10 | (h) 1 | (i) -13 | (j) -1 | (k) 5 | (l) -16 |
| (m) -40 | (n) 2 | (o) -3 | (p) 21 | (q) -68 | (r) -8 |

Exercise 3.

- | | | | | | | | |
|-----------|---------|--------|---------------------|---------|---------|----------|----------|
| 1. (a) 5 | (b) 6 | (c) 12 | (d) 24 | (e) 4 | (f) 0 | (g) -6 | (h) -196 |
| 2. (a) 4 | (b) 19 | (c) -2 | (d) -17 | (e) 40 | (f) -47 | (g) -302 | (h) 157 |
| 3. (a) -8 | (b) -33 | (c) 16 | (d) -24 | (e) -24 | (f) 136 | (g) 256 | (h) 291 |
| 4. (a) 0 | (b) -4 | (c) 6 | (d) a^3-2a^2-5a-6 | (e) 0 | (f) 0 | (g) 0 | (h) 0 |
| (f) -24 | (g) 56 | | | | | | |

Exercise 4.

- | | | | | |
|------------------------|----------------------------|-------------------------------|----------------------|-------------------|
| 1. (a) $2x+12$ | (b) $3x+12$ | (c) $4x-20$ | (d) $35-7m$ | (e) $16-12x$ |
| (f) $5x-5y$ | (g) $-4x+14$ | (h) $-21m+28n$ | (i) $-3m+7$ | (j) $6x^2-12x+15$ |
| (k) $-6t^2+15t-12$ | (l) $-3x^2+4xy-y^2$ | (m) $2x^2-14x$ | (n) $5x^3-15x^2-20x$ | (o) m^3+3m^2-m |
| 2. (a) $6x^2y-8xy^2$ | (b) $-2x^3y+2x^2y^2-2xy^3$ | (c) $3t^5-6t^4-12t^3$ | | |
| (d) $2\pi rw-2\pi rx$ | (e) $-3\pi m+2\pi n$ | (f) $1.2x-2y$ | | |
| (g) $-0.6x^2+0.8x$ | (h) $-14x+1.4y$ | (i) $0.3mx-0.7nx$ | | |
| (j) $3t^2-7t$ | (k) $-48x^3+73x^2y$ | (l) $0.004m-0.005$ | | |
| (m) $-0.03m^3+0.06m^2$ | (n) $30x^4y^2-7.5x^3y^3$ | (o) $12m^3nt-16mn^3t+20mnt^3$ | | |
| 3. (a) $5x+32$ | (b) $26x+53$ | (c) $46w-33$ | (d) $20m-22$ | |
| (e) $2x-3$ | (f) $12t-28$ | (g) $7x^2-3x+9$ | (h) x^2+9x-2 | |
| (i) $-2x-10y$ | (j) $8x^2-20xy-4y^2$ | (k) $5x^2-46x+1$ | (l) $-7-8t+10t^2$ | |
| (m) $8m+3n-6$ | (n) $-32m^2+4mn-n^2$ | (o) $-6x^2+17x$ | (p) $-11w^2$ | |
| (q) $11x^2-12x$ | (r) x^3-5x^2+8x | (s) $-4m^3n+5mn^3$ | (t) $28x^3-19x^2-2x$ | |
| (u) $-4x^3-7x^2+45x$ | (v) $-3x^2y+1.1xy^2$ | (w) $2m^3n^2-6m^2n^3$ | (x) $-2x^2+2y^2$ | |
| 4. (a) -6 | (b) 9 | (c) -11 | (d) -12 | (e) 11 |
| (g) 3 | (h) 1 | (i) 19 | | (f) 24 |

CHAPTER ONE

Exercise 1-1

- | | | | |
|----------------------|--------------------|--------------------|--------------------|
| 1. (a) monomial; 4th | (b) monomial; 8th | (c) monomial; 3rd. | (d) binomial; 2nd |
| (e) trinomial; 2nd | (f) binomial; 13th | (g) binomial; 7th | (h) trinomial; 7th |
| (i) monomial; 6th | (j) binomial; 4th | (k) binomial; 9th | (l) trinomial; 6th |
| (m) binomial; 3rd | (n) monomial; 21st | (o) trinomial; 8th | |

2. (a) $-5x^5 + 16x^3 + 4x^2 - 3x$ (b) $15x^4 - 3x^3 + 17x^2 + 5x$ (c) $-4x^3y^3 + 5x^2y^3 + 5x^3y$
 (d) $-3x^5y^2 - 4x^4y^2 + 8x^3y + 2m^2n$ (e) $-4mn^3 + 5x^3 + 3xy + 5$ (f) $11x^5y^2 - 5m^6 - x^5 + 4x^3y$
 (g) $2m^4n^3 - 5m^6 + 2x^4$ (h) $y^4 - 5x^3 + 3xy - 3x + 2$
3. (a) rectangle; 1st (b) triangle; 1st (c) square; 1st (d) circle; 1st
 (e) quadrilateral; 1st (f) circle; 1st
4. (a) rectangle; 2nd (b) circle 2nd (c) triangle; 2nd (d) trapezoid; 2nd (e) annulus; 2nd
5. (a) rectangular solid; 3rd (b) cylinder; 3rd (c) sphere; 3rd
6. (a) $x^2 + 6x + 9$ (b) $x^2 - 4$ (c) $t^2 - 4t + 4$ (d) $m^2 - 2mx + x^2$
 (e) $r^2 - 49$ (f) $y^2 + 8y + 16$ (g) $x^2 - y^2$ (h) $4m^2 + 4m + 1$
 (i) $4x^2 - 12x + 9$ (j) $4x^2 - 9y^2$ (k) $1 - 2x + x^2$ (l) $4x^2 + 20x + 25$
 (m) $9x^2 - 24xy + 16y^2$ (n) $25s^2 + 30st + 9t^2$ (o) $x^4 - 1$ (p) $4 - 12st + 9s^2t^2$
 (q) $9x^4 - 4y^2$ (r) $9 - 42r + 49r^2$
7. (a) $x^2 + 5x + 6$ (b) $y^2 + 9y + 20$ (c) $m^2 - 5m + 6$ (d) $y^2 - 10y + 21$
 (e) $t^2 - t - 12$ (f) $r^2 + 3r - 40$ (g) $x^2 - 4x - 21$ (h) $m^2 - 12m + 35$
 (i) $x^2 + 2x - 24$ (j) $t^2 + 16t + 55$ (k) $x^2 + 21x + 110$ (l) $y^2 + 2y - 48$
 (m) $x^4 + 3x^2 - 18$ (n) $x^4 - x^2 - 2$ (o) $m^4 - 8m^2 + 15$ (p) $x^6 - 2x^3 - 48$
 (q) $5 - 4x - x^2$ (r) $18 - 9m + m^2$ (s) $80 - 18x + x^2$ (t) $56 - t - t^2$
 (u) $8 + 7y - y^2$
8. (a) $-x - 14$ (b) $2x^2 - 17x + 22$ (c) $13x - 15y + 7$ (d) $-9r - 12s + 15t$
 (e) $4x - 11$ (f) $15x - 20y - 19$ (g) $4x^2 - 18x - 18$ (h) $7x^2 - 18x - 28$
 (i) $-x^2 - 2x + 1$ (j) $-8m^2 + 6m$ (k) $3x_1 - 8x_2 + 11x_3$ (l) $6x^2 - 12xy + 22y^2$
9. (a) $3x^2 + 19x + 20$ (b) $6t^2 + 17t + 7$ (c) $4s^2 + 33s + 35$
 (d) $6x^2 - 11x + 4$ (e) $6m^2 - 25m + 24$ (f) $42t^2 - 83t + 40$
 (g) $20x^2 - x - 12$ (h) $6r^2 + 19r - 7$ (i) $2 - x - 15x^2$
 (j) $3 - 23y + 30y^2$ (k) $-6m^2 - 13m + 5$ (l) $-15x^2 + 56x - 20$
 (m) $6x^2 - 7xy - 3y^2$ (n) $12x^2 - 55xy + 50y^2$ (o) $30m^2 + 47mn + 7n^2$
 (p) $6w^2 + 7wx - 33x^2$ (q) $56x^2 - 33xy - 14y^2$ (r) $15r^2s^2 + rst - 28t^2$
 (s) $15x^4 - 2x^3 - 8x^2$ (t) $-3m^4 - 4m^3 + 4m^2$ (u) $35x^4y^2 + 13tx^2y - 4t^2$
10. (a) $x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$ (b) $w^2 + x^2 + y^2 - 2wx - 2wy + 2xy$
 (c) $r^2 + s^2 + t^2 - 2rs + 2rt - 2st$ (d) $4x^2 + y^2 + z^2 + 4xy + 4xz + 2yz$
 (e) $4w^2 + 9x^2 + y^2 - 12wx + 4wy - 6xy$ (f) $x^4 - 4x^3 + 10x^2 - 12x + 9$
 (g) $16x^4 + 24x^3 + x^2 - 6x + 1$ (h) $25m^2 + 9n^2 + 16 - 30mn + 40m - 24n$
 (i) $25x^8 - 20x^6 - 6x^4 + 4x^2 + 1$
11. (a) $2x^3 + 7x^2 + 8x + 3$ (b) $6w^3 - 11w^2 - 2w + 3$
 (c) $8m^4 + 8m^3 - 4m^2 + 11m - 3$ (d) $8w^2 + 3x^2 + 8y^2 - 14wx + 16wy - 14xy$
 (e) $5x^4 + 11x^3 - 19x^2 - 2x + 2$ (f) $3x^2 - 12y^2 - 2z^2 + 5xy - xz + 10yz$
 (g) $x^5 - 2x^4 - x^3 + x^2 - 2x + 3$ (h) $x^6 + x^5 - x^4 - 9x^3 - 13x^2 - 11x - 3$
 (i) $2m^4 - 9m^3 + 4m^2 + 13m + 5$ (j) $3x^4 - 10x^3 + 23x^2 - 32x + 16$
 (b) $-9t^2 - 90t + 96$
12. (a) $12x^2 + 53x - 54$ (d) $9m^2 + 60m - 18$
 (c) $-9m^2 - 44m - 49$ (f) $-21x^2 - 27$
 (e) $12x^2 - 8x - 102$ (h) $10x^2 - 8x + 9$
 (g) $x^2 + 27x - 49$ (j) $-2x^2 + 40xy - 35y^2$
 (i) $-11x^2 + 17x - 34$ (l) $x^2 - 14xy + y^2 + 5$
 (k) $3w^2 + 17wx - 24x^2$ (n) $4r^2 + 9s^2 - 16t^2 - 17rs - 12rt + 3st - r + s + 3t$
 (m) $-2x^3 - 8x^2 + 27x + 8$
13. $2x^2 - 15x + 25$ 14. $2\pi(2x + 5), \pi(4x^2 + 20x + 25)$
15. $\frac{1}{2}(6x^2 + 13xy - 5y^2)$ 16. $\frac{3}{2}x^2 + \frac{13}{2}xy + 7y^2$
17. (a) $6x^3 + 11x^2 - 47x + 20$ (b) $2x^3 - 3x^2y - 17xy^2 + 30y^3$
 (c) $w^2 + x^2 + y^2 + z^2 + 2wx + 2wy + 2wz + 2xy + 2xz + 2yz$
 (d) $x^2 - \frac{1}{x^2}$ (e) $m^2 + 1 - \frac{6}{m^2}$ (f) $-x^2 - x + 4 + \frac{1}{x} - \frac{1}{x^2}$

Exercise 1-2

1. (a) $5(x + 2)$ (b) $2(x - 3)$ (c) $2(x - 2y)$ (e) $x(3y - 4)$ (f) $5(2m - 1)$
 (g) $x(3y + 4w)$ (h) $2(t^2 + 2t - 3)$ (i) $mn(5 - 6t)$ (j) $4x(2x - 1)$ (k) $7r(1 - 2r)$
 (l) $2s(w + 2x + 3y)$ (m) $(x - 1)(r + t)$ (n) $(w - z)(7 - 5x)$
2. (a) $5x^2(1 + 5x - 6x^2)$ (b) $x^2(x^3 - x^2 + x - 1)$ (c) $4x^2(7x + 6)$
 (e) $2m(3m^2 - 5mt + 3t^2)$ (f) $7rst(2 - r + 3s - 4t)$ (g) $xy(x^2y^2 - x + y)$
 (h) $13m^2n(1 - 2n)$ (i) $4p^3q^3(4p - 2q - 1)$ (j) $3xy(6xy + 3 - x^2y^3)$
 (l) $x^2(x^4 - x^3 + x^2 - x - 1)$ (m) $4p^2x^4(9 - 4px + 6p^3)$ (n) $m^9n^{11}(m^6 - 3mn - 4n^4)$
 (o) $12x^9y^7(2 - 4x^2y^6 + 3xy^3)$ (p) $2x^8y^4(1 - 3x^9y^{14} + 2y^5 - 5x^6y^{17})$

3. (a) $(p+q)(2x+3)$ (b) $(x-1)(2m+5)$ (c) $(x-y)(9m-14n)$ (d) $(m+4)(8-9x)$
 (e) $(m+n)(5ty-6x)$ (f) $(x-3)(1+y)$ (g) $(m+7)(x-1)$ (h) $(x^2-2x-1)(2-y)$
 (i) $7m(x-y)(2m-1)$ (j) $(x+y)(x-y+3)$ (k) $(x-4)(2x-3)$ (l) $(2m+1)(5m-4)$
 (m) $(x-5)(x-5-m)$ (n) $(x-y)(x-y-1)$ (o) $(y+2)(x-7)$ (p) $(m-n)^2(m-n-2)$
 4. (a) $(x+y)(m+n)$ (b) $(x+3)(t+w)$ (c) $(x+y)(m-3)$ (d) $(x+1)(x^2+1)$
 (e) $(x-y)(x^2+y^2)$ (f) $(a+3)(m+n)$ (g) $(1-x)(1+x^2)$ (h) $(x-y)(5-x)$
 (i) $(x+y)(3x+1)$ (j) $(4x-y)(m-n)$ (k) $(x-y)(2m-3n)$ (l) $(x-y)(2m+3n)$
 5. $2\pi r(r+h)$
 6. (a) $(x+y)(2m-3)$ (b) $2(m+n)(2x-1)$ (c) $(x-y)(6n-5)$ (d) $(x-4)(7m+3)$
 (e) $(3x-y)(5m+2)$ (f) $(m-n)(3x+1)$

Exercise 1-3

1. (a) $(x+2)$ (b) $(x+2)$ (c) $(x-3)$ (d) $(m-6)$ (e) $(t-11)$ (j) $(n-2)$
 (g) $(y+8)$ (h) $(x-3)$ (i) (m^2-10) (j) (x^3-18) (k) $(x-8)$ (l) $(x+4)$
 (m) $(x+1)$ (n) $(s+1)$ (o) $(x+3)$ (p) $(8-t)$ (q) $(7-m)$ (r) $(9-x)$
 2. (a) 3,4 (b) -3, -5 (c) (d) 4,1 (e) -6,3 (f) 9, -2
 (g) (h) -4,3 (i) 8, -5 (j) (k) -4, -4 (l) 7, -4
 (m) 15,5 (n) -9,2 (o) -9,3 (p) 9, -3 (q) -1, -4 (r)
 3. (a) $(x+2)(x+5)$ (b) $(y-5)(y-3)$ (c) $(m+7)(m+3)$ (d) $(w-8)(w+7)$
 (e) (f) $(s-12)(s+8)$ (g) $(t+16)(t+20)$ (h)
 (i) $(x+11)(x+15)$ (j) $(w-7)(w-6)$ (k) $(z+8)(z-5)$ (l) $(m-9)(m+2)$
 (m) $(x-4)(x-4)$ (n) (o) $(n-9)(n-4)$ (p) $(x-4)(x-5)$
 (q) $(7+x)(6-x)$ (r) $(8+y)(11-y)$
 4. (a) $2(x+1)(x+1)$ (b) $3(q-5)(q+3)$ (c) $(m+7)(m+4)$ (d) $5(p-7)(p-2)$
 (e) $2(x^2-2x-10)$ (f) $(x-13)(x-9)$ (g) $(12-m)(5+m)$ (h) $10(y-8)(y+5)$
 (i) (j) $(x+6)(x+8)$ (k) $2(12-x)(5+x)$ (l) $(s-3)(s-3)$
 (m) $3(x^2-12x+28)$ (n) $(m-6)(m-6)$ (o) $(w+11)(w-7)$
 5. (a) $(x+16y)(x-10y)$ (b) $(s+2t)(s+7t)$ (c) $(x+4y)(x-3y)$ (d) $(t-7)(t-8)$
 (e) $(x-11y)(x+5y)$ (f) $(m-17)(m-24)$ (g) (h) $(x+13y)(x-7y)$
 (i) $(x-7)(x-6)$ (j) $(x^2-10)(x^2+9)$ (k) $(a^3+7)(a^3-5)$ (l) $(s-5t)(s-4t)$
 (m) $2(y-12)(y+2)$ (n) $(z+14)(z+8)$ (o) $(m-31)(m+24)$ (p) $(m+12n)(m-20n)$
 (q) $(x+22y)(x-30y)$ (r) $(x^2-12)(x^2+4)$
 6. (a) $(2x-5)(3x+4)$ (b) $(3x+5)(4x+1)$ (c) $(11t+1)(2t+1)$ (d)
 (e) $(3m+1)(2m-3)$ (f) $(2y+7)(y-4)$ (g) $(4q+3)(3q+5)$ (h) $(5x+3)(4x-3)$
 (i) $(6x+1)(2x-7)$ (j) (k) $(2t-5)(5t-3)$ (l)
 (m) $(2q-7)(3q-1)$ (n) $(m+7)(4m-5)$ (o) $(3y-2)(y+8)$ (p) $(x+6)(3x+7)$
 (q) $(s+9)(4s-5)$ (r) $(5x+4)(4x-1)$ (s) $(10x-7)(2x-5)$ (t) $(9m+5)(4m-3)$
 (u) $(8t+3)(2t-3)$ (v) (w) $(4s+9)(s+3)$ (x) $(4x-3)(2x+1)$
 7. (a) $2(4x+1)(3x-1)$ (b) $(11y+1)(2y+1)$ (c) $3(2x+1)(x-5)$ (d) $(2x+5y)(3x+y)$
 (e) $(3x+7y)(4x-5y)$ (f) $10(6m+n)(m+6n)$ (g) $(8x-5y)(3x-4y)$ (h) $(5t+4s)(3t+2s)$
 (i) $2(1+7x)(1-6x)$ (j) $y(x+4)(5x-2)$ (k) $(x^2+7)(4x^2+7)$ (l) $2(x^2+xy+y^2)$
 (m) (n) $(3-10x)(5+2x)$ (o) $2(5-3x)(2-3x)$
 8. (a) $(2x+5)(4x+9)$ (b) $(10y-3)(2y+5)$ (c) $(8t-3)(5t-4)$ (d) $(6x+7)(8x+3)$
 (e) $(7m-5)(6m-3)$ (f) $(10x+7)(4x+1)$ (g) $(4m+9)(2m+7)$ (h) $(8y-3)(6y-1)$
 (i) $(5s-11)(4s+3)$ (j) $(3-7x)(5-6x)$ (k) $(7-8t)(6+7t)$ (l) $(7m+11)(4m+9)$
 9. (a) $(x+6)(x+7)$ (b) $(x+y-6)(x+y-3)$ (c) $(x-y-5)(x-y+3)$
 (d) $(m+n-7)(m+n-3)$ (e) $(x^2-x-7)(x^2-x-4)$ (f) $(xy-5)(xy-4)$
 (g) $(x^2y+5)(x^2y+3)$ (h) $(2xy-3z)(3xy+4z)$ (i) $(r+s-4x-4y)(r+s+2x+2y)$

Exercise 1-4

1. (a) x^2-9 (b) x^2-16 (c) t^2-81 (d) m^2-16
 (e) $x^2+8x+16$ (f) x^2-6x+9 (g) $4x^2-9$ (h) $16y^2-1$
 (i) $w^2+14w+49$ (j) $x^2-10x+25$ (k) $4x^2-9y^2$ (l) $16m^2-25n^2$
 (m) $4m^2+12m+9$ (n) $9x^2-24x+16$ (o) $4x^2-25y^2$ (p) $49x^2-1$
 (q) $1-25t^2$ (r) m^2-n^2
 2. (a) $(x-4)(x+4)$ (b) $(m-5)(m+5)$ (c) $(x-9)(x+9)$
 (d) $(y-10)(y+10)$ (e) $(x+3)^2$ (f) $(t-2)^2$
 (g) (h) $(v+5)^2$ (i) $(r-10)^2$
 (j) $(5x-4)(5x+4)$ (k) (l) $(4x-3y)(4x+3y)$
 (m) $(x-11)^2$ (n) $(s+8)^2$ (o) $(6x-7y)(6x+7y)$
 (p) $(8m-n)(8m+n)$ (q) $(n-6)^2$ (r) $(10x^2-3y)(10x^2+3y)$

3. (a) $(2x+3y)^2$ (b) $(4x-3y)^2$ (c) $(5x+6y)^2$ (d) $(3r-2s)^2$ (e) $(7x-4y)^2$
 (f) $(6m-n)^2$ (g) $(9m+2n)^2$ (h) $(8r-3s)^2$
4. (a) $(x+y-5)(x+y+5)$ (b) $(x+y+1)(x+y-1)$
 (c) $(m-3-n)(m-3+n)$ (d) $(s+5-t)(s+5+t)$
 (e) $(x-m-n)(x+m+n)$ (f) $(m-x+3)(m+x-3)$
 (g) $(4m-n+2)(4m+n-2)$ (h) $(3x-2y+4)(3x+2y-4)$
 (i) $(5y-3x-3)(5y+3x+3)$ (j) $(x-y+3)(x+y+1)$
 (k) $(m-n+7)(m+n-1)$ (l) $(2x-y-3)(2x+y+11)$
 (m) $(4x-4y-7w+21)(4x-4y+7w-21)$ (n) $(5m-n+24)(5m+n+26)$
 (o) $(t^2-4)(t^2+4)$ (p) $(x^5-5y^3)(x^5+5y^3)$
 (q) $(4m^4-3n^5)(4m^4+3n^5)$ (r) $(6x^7-y)(6x^7+y)$
5. (a) $(x+1-y)(x+1+y)$ (b) $(x-3-y)(x-3+y)$ (c) $(m-2-n)(m-2+n)$
 (d) $(r-5-s)(r-5+s)$ (e) $(r+7-5s)(r+7+5s)$ (f) $(x-6-3a)(x-6+3a)$
 (g) $(1-x-y)(1+x+y)$ (h) $(3-x+2y)(3+x-2y)$ (i) $(4x-y+3)(4x+y-3)$
 (j) $(5m-n-4)(5m+n+4)$ (k) $(x-3-6y)(x-3+6y)$ (l) $(4y-2x-1)(4y+2x+1)$
 (m) $(5y-3x+2)(5y+3x-2)$ (n) $(3s-4r+3)(3s+4r-3)$
6. (a) $(x+y-m-1)(x+y+m+1)$ (b) $(m+3n-x+2)(m+3n+x-2)$
 (c) $(m-3n-x+4)(m-3n+x-4)$ (d) $(x+y-s-r)(x+y+s+r)$
 (e) $(2x+3y-2p+3q)(2x+3y+2p-3q)$ (f) $(5-2x-2m+n)(5-2x+2m-n)$
7. (a) 32π (b) 64π (c) 341π

Exercise 1-5

1. (a) $8x$ (b) $2t^3$ (c) 5 (d) $9x$ (e) $-12rs$ (f) $-9mn^2$
 (g) $5mn$ (h) $-4m^2$ (i) $-8xyz^5$ (j) $-10m^3n^7$ (k) 3 (l) m^4n^6
 (m) $5r^5s^6t$ (n) $-6x^5y^7$ (o) $-5z^2$ (p) $-8rs^5$
2. (a) $\frac{1}{x}$ (b) $\frac{1}{y}$ (c) $\frac{1}{2t^2}$ (d) $\frac{x}{4}$ (e) $\frac{5}{y^5}$ (f) $\frac{1}{4xy}$ (g) $\frac{n}{2}$ (h) $\frac{3y}{x^2}$
3. (a) $m+n$ (b) $2x-5y$ (c) $x+y$ (d) $3x-2$ (e) $5s+1$
 (f) $6m^2-2m$ (g) $-x+2$ (h) m^2-2m-4 (i) $2x^4+3x^2-1$
4. (a) $x+2y$ (b) $3x^2y-9x-3$ (c) $4m^3n-3mn^2-1$ (d) $2r^2st^3-s-3r^3$
 (e) $x^2m^4-3xm^2-6m$ (f) $5m^4n^4-2m^2n+3n$ (g) $3xy^2-y^4+2x^2y^2$ (h) $-1-2xy^2-3x^3y^4$
5. (a) $-x+2y$ (b) $-t+2t^2$ (c) $4rs-3$ (d) $-7s+5r+2t$
 (e) $-4mn^3+1+3n^4$ (f) $-15x^2yz^2+3x^3$ (g) $-3xy^2+2z+1$
6. (a) $4x+2+\frac{1}{x}$ (b) $6x^2-4x+2+\frac{1}{x^2}$
 (c) $\frac{m^2}{3}-2m+3$ (d) $3t^2-2t-1-\frac{1}{7t}$
 (e) $xy^2-\frac{y^3}{2}+\frac{3}{2xy}$ (f) $3w^2-2w-\frac{1}{w}+\frac{3}{17w^2}$
 (g) $-10m^2n+4mn^2+\frac{2}{mn}-\frac{5}{2m^2n^2}$ (h) $-21rs^3+27r^2s^4-6rs+\frac{3}{s}$
 (i) $-\frac{x^2}{8y^3}-3xy^2+y^3+\frac{1}{8x^3}$

Exercise 1-6

1. (a) $3x^5+x^3-4x^2+7$ (b) $-x^4+2x^2-5x+1$ (c) $2m^5-4m^3+3m^2+5m$
 (d) $r^4+2r^2s^2+rs^3-s^4$ (e) $2x^5-5x^4+4x^2+3x$ (f) $4x^4-3x^3y+2x^2y^2+xy^3$
 (g) $-t^5+t^4+4t^3-5t^2+1$ (h) $5m^5-3m^3+2m^2+m$
2. (a) $(6 \times 3)+1=19$ (b) $(3 \times 2)+1=7$ (c) $(1 \times 7)+4=11$ (d) $(2 \times 20)+6=46$
 (e) $(2 \times 9)+3=21$ (f) $(10 \times 7)+2=72$ (g) $(5 \times 12)+5=65$ (h) $(10 \times 9)+8=98$
3. (a) $2x+5, x \neq -1$ (b) $x^2+2x-3; R=1, x \neq 4$
 (c) $2y^2-6y+7; R=-2, y \neq 4$ (d) $3w-4; R=-10, w \neq \frac{7}{2}$
 (e) $3m-5, m \neq -\frac{4}{3}$ (f) $t^2-3t+5; R=-5, t \neq -\frac{5}{2}$
 (g) $2m^2-3m-4; R=9, m \neq -\frac{1}{2}$ (h) $3x^2-2x+7, x \neq \frac{5}{4}$
 (i) $4y^2-5; R=16, y \neq -\frac{3}{2}$
4. (a) $8x^2-1; R=-6$ (b) $x+4$ (c) $m^2-2m+1; R=4$
 (d) $2x+7; R=-4$ (e) $t^2+2t-1; R=-6$ (f) $4r^2+2r+1; R=8$
 (g) $5s^3-2$ (h) $x^3+2x^2+4x+8; R=-3$ (i) $x^4+x^3+x^2+x+1; R=-1$

5. (a) $x+y$; $R=y^2$ (b) $x-y$; $R=3y^2$ (c) $x-3y$; $R=-2y^2$ (d) x^2+xy+y^2
 (e) x^2+xy+y^2 (f) $x^2-xy+2y^2$ (g) $x^2-2xy-y^2$; $R=2y^4$ (h) $2m^2-3mn+n^2$
 6. $3x+4$
 7. $x^6+2x^5+4x^4+8x^3+16x^2+32x+64$

Exercise 1-7

1. (a) 18 (b) 7 (c) 4 (d) -2
 2. (a) 0 (b) -4 (c) -42 (d) 62
 3. (a) 35 (b) $3\frac{1}{2}$ (c) $7\frac{1}{2}$ (d) $3\frac{1}{2}$
 4. (a) 13 (b) -13 (c) 0 (d) 61 (e) -3 (f) 0 (g) -14 (h) -5
 5. (a) -1 (b) 5 (c) $-10\frac{1}{8}$ (d) 3 (e) 0

Exercise 1-8

1. (a), (c), (f), (g)
 2. Both are factors.
 3. $(x+1)$ is a factor.
 5. No
 6. (a) $(x-1)(x-2)(x-3)$ (b) $(x+1)(x+3)(x+4)$ (c) $(t-2)(t-3)(t+3)$ (d) $(m+3)(m^2+m-1)$
 (e) $(x+2)(x+4)(x-5)$ (f) $(x+1)(x-4)(x+4)$ (g) $(w-4)(w^2+2w+2)$ (h) $(n-3)(n+4)(n+5)$
 (i) $(x-3)(x^2+3x+9)$ (j) $(x-5)(x^2+5x-2)$
 7. (a) $(x-1)(x-3)(2x-1)$ (b) $(x-2)(x+2)(3x-2)$ (c) $(x-2)(2x^2+x+5)$
 (d) $(m+1)(2m+1)(2m-3)$ (e) $(t-2)(t-3)(5t+2)$ (f) $(y+3)(4y^2+y+1)$
 (g) $(x-3)(3x+5)(2x-1)$ (h) $(x-2)(2x^2+11)$ (i) $(x+1)(x+3)(x-4)(2x+1)$
 (j) $(x+2)(x-5)(x^3+3)$
 8. (b) $(x-2)(x^2+2x+4)$
 9. (b) $(m+3)(m^2-3m+9)$
 11. (a) $(x-3)(x^2+3x+9)$ (b) $(x+4)(x^2-4x+16)$
 (c) $(m+2)(m^2-2m+4)$ (d) $(y-5)(y^2+5y+25)$
 (e) $(2x-1)(4x^2+2x+1)$ (f) $(x-y)(x^2+xy+y^2)$
 (g) $(m+n)(m^2-mn+n^2)$ (h) $(4m-1)(16m^2+4m+1)$
 (i) $(3x+2y)(9x^2-6xy+4y^2)$ (j) $(x^2+y^3)(x^4-x^2y^3+y^6)$
 (k) $(s^2-3t^3)(s^4+3s^2t^3+9t^6)$ (l) $(10x^4+y^5)(100x^8-10x^4y^5+y^{10})$
 12. $k=-2$
 13. $k=9$
 14. $k=3$
 16. (a) No (b) Yes
 17. Yes

Exercise 1-9

1. (a) m (b) wy (c) 1 (d) 1 (e) $3x+3y$ (f) $7m-7n$
 (g) $x-1$ (h) $y-x$ (i) 2 (j) 2
 2. (a) $x \neq 0$ (b) $x \neq 4$ (c) $m \neq 2$ (d) $t \neq -3$ (e) $x \neq 1, -3$
 (f) $x \neq -2, -7$ (g) $r \neq -5$ (h) $m \neq 0, -8$ (i) $x \neq \frac{1}{2}$ (j) $x \neq m$
 (k) $x \neq -\frac{1}{2}, \frac{2}{3}$ (l) $x \neq 1, 3$
 3. (a) $\frac{2}{5}$ (b) $\frac{12x}{5}$ (c) $\frac{x}{y}$ (d) $-2t$ (e) $\frac{4m}{3n}$ (f) $2y$ (g) $\frac{mx}{ny}$
 (h) $\frac{5xy}{w}$ (i) $4x^2$ (j) $\frac{-7y^2z}{x^2}$ (k) $\frac{1}{x^5}$ (l) $\frac{r^4s^6t^7}{xy}$ (m) $\frac{1}{4x^3y^2}$ (n) $\frac{-m}{3n^6}$
 (o) $\frac{-5}{x^2}$ (p) $\frac{x+2}{x-2}$ (q) $\frac{m-3}{m+3}$ (r) $\frac{2}{3}$
 4. (a) $\frac{x+4}{x+3}$, $x \neq 7, -3$ (b) $\frac{x}{x+7}$, $x \neq -7$ (c) $\frac{2}{t-1}$, $t \neq 1, 2$
 (d) $\frac{2(m+3)}{(m-2)(m+2)}$, $m \neq 2, -2$ (e) $\frac{2(x+y)}{x-y}$, $x \neq y$ (f) $\frac{2}{x+7}$, $x \neq 5, -7$
 (g) $\frac{x-7}{x-1}$, $x \neq 1, -7$ (h) $\frac{(x-3)(x-2)}{(x-6)(x+1)}$, $x \neq 6, -1$ (i) $-x-2$, $x \neq 2$
 (j) -1 , $x \neq 4$ (k) $\frac{2(m-1)}{(m+1)}$, $m \neq -1$ (l) $\frac{-x-7}{x+3}$, $x \neq 4, -3$

- (m) $\frac{-3}{x+4}, x \neq 1, -4$ (n) $\frac{x+y}{x-y}, x \neq y$ (o) $\frac{m}{m+n}, m \neq -n$
5. (a) $\frac{2x-1}{x-2}, x \neq 2, -2$ (b) $\frac{2x+3}{2x-3}, x \neq \frac{3}{2}, -\frac{1}{3}$
- (c) $-x-y, x \neq y$ (d) $\frac{x}{x+4}, x \neq 5, -4$
- (e) $\frac{(2m+1)(m+3)}{(2m+3)(m+1)}, m \neq -1, -\frac{3}{2}$ (f) $\frac{-m-4}{m-2}, m \neq 2, \frac{3}{2}$
- (g) $\frac{6(m-3)}{2m+1}, m \neq -3, -\frac{1}{2}$ (h) $\frac{x^2+2x+4}{3x+5}, x \neq 2, -\frac{5}{3}$
- (i) $\frac{-m}{x+y}, x \neq -y$ (j) $\frac{3}{(x-2)(x^2+4)}, x \neq 2, -2$
- (k) $\frac{4(x+y)}{(x-y)}, x \neq y$ (l) $\frac{(2x-1)(x+4)}{(3x+5)(x-2)}, x \neq 2, -\frac{5}{3}$
6. (a) $\frac{1}{x+4}$ (b) $x+2$ (c) $\frac{7x-8}{9x-10}$ (d) $\frac{x+4}{(x+3)(x-4)}$ (e) $\frac{x-3}{2x+3}$
- (f) $\frac{x+2}{3}$ (g) $\frac{x-y}{x^2-xy+y^2}$ (h) $\frac{-1}{2x-1}$ (i) $\frac{-1}{2(x+3)}$ (j) $\frac{5x-2y}{2x-y}$

Exercise 1-10

1. (a) $\frac{x}{w}$ (b) $\frac{y}{x}$ (c) $\frac{5}{12}$ (d) $\frac{2x}{y}$ (e) $\frac{7}{3t}$ (f) xy
- (g) $\frac{3}{5}$ (h) $\frac{y}{2x}$ (i) x^2 (j) $\frac{1}{x}$ (k) $\frac{m^2}{n^2}$ (l) $\frac{1}{9}$
- (m) $\frac{1}{2}$ (n) 3 (o) $\frac{1}{7}$ (p) $\frac{y^2}{x^2}$ (q) $\frac{1}{(x+y)^2}$ (r) $\frac{x+3}{x+4}$
- (s) 1 (t) $\frac{x+2}{x-3}$ (u) $\frac{x+y}{y}$
2. (a) $\frac{10y^2}{7x}$ (b) $\frac{3x^3}{y}$ (c) $\frac{6x}{y^2}$ (d) $\frac{4y^4}{z}$ (e) $\frac{r^2s^2}{x}$ (f) $\frac{3mn}{10xy^3}$
3. (a) 2 (b) $3x$ (c) $14t^3$ (d) $-\frac{2}{3}$ (e) $\frac{-2x}{x-2}$ (f) $\frac{x+y}{4}$
4. (a) $\frac{x+4}{x+1}$ (b) $\frac{x+4}{x-6}$ (c) $\frac{1}{(x-9)(x+6)}$ (d) $\frac{t+7}{t-6}$ (e) 1
- (f) 1 (g) -5
5. (a) 1 (b) $-\frac{3x+1}{3x-1}$ (c) 1 (d) $\frac{4x-5}{x-3}$ (e) $-\frac{m+8}{m+2}$
6. (a) 2 (b) $\frac{(x+2)(x+5)}{(x+6)}$ (c) $\frac{(x-3y)}{2(2x-y)}$
7. (a) $\frac{2(x-2)}{x}$ (b) $m(m+1)$ (c) $\frac{-1}{x(x+4)}$

Exercise 1-11

1. (a) 42 (b) 24 (c) 360 (d) 300 (e) 1008 (f) 252
2. (a) x^2y^2 (b) $4x^2y^2$ (c) $6m^2n^2$ (d) $24m^3n^4$ (e) $60x^3y$ (f) $270tx^5y^4$
3. (a) $(x-3)(x+3)(x-4)$ (b) $mn(x-y)$ (c) $2(x+3)^2$
- (d) $x(x-y)(x+y)$ (e) $30(x-2)(x+2)$ (f) $5(x+3)(x+5)(x-1)$
- (g) $(t+1)(t-1)(t+6)(t+7)$ (h) $(m+5)(m-4)(m-2)$
4. (a) $(x+y)(x-y)^2$ (b) $(m-3n)(m+n)^2$ (c) $(2x-1)(x+3)(3x+2)$
- (d) $(5m-3)(2m+3)(3m-4)$ (e) $5(3r-4)(3r+4)(5r+1)$ (f) $(3x+2)(2x-3)(2x-5)$

Exercise 1-12

1. (a) 42 (b) 12 (c) 24
- (d) xy (e) $6x^3$ (f) $3x^3y^2$

- (g) $\frac{2(m+2)}{(m-1)(m+2)(m+3)}$ (h) $\frac{4(x-1)(x+2)}{(x+1)(x+2)(x-4)}$ (i) $\frac{x(x-3)}{t^2(t+5)(t+7)}$
2. (a) $\frac{8}{x}$ (b) $\frac{4}{y}$ (c) $\frac{8}{m}$ (d) $\frac{5}{x+1}$ (e) $\frac{3x-5}{x-4}$
- (f) $\frac{5x+5y}{8}$ (g) $\frac{2x^2}{(x+5)(x+3)}$ (h) $\frac{2m^2-3m+4}{m-5}$
3. (a) $\frac{5}{4}$ (b) $\frac{7}{24}$ (c) $2x$ (d) $\frac{8m+7n}{6}$ (e) $\frac{3x-3}{20}$
- (f) $\frac{8x^2+5}{30}$ (g) $\frac{70+2m+7n}{14}$ (h) $\frac{-18t-49}{24}$
4. (a) $\frac{17}{6x}$ (b) $\frac{49}{30x}$ (c) $\frac{2x^2-3x+4}{x^3}$ (d) $\frac{-9y^2+12y+10}{18y^3}$
- (e) $\frac{2t^2+3st-2s^2}{s^2t^2}$ (f) $\frac{-x+4y-1}{xy}$ (g) $\frac{my-x+y^2}{y}$ (h) $\frac{w^2-x+wx}{w}$
- (i) $\frac{3x^2+1}{x}$ (j) $\frac{4x-1+x^2}{x}$
5. (a) $\frac{7x-5}{(x+1)(x-3)}$ (b) $\frac{11w+20}{w(w+4)}$ (c) $\frac{3m}{(m-4)(m-2)}$ (d) $\frac{7}{t-2}$
- (e) $\frac{23x^2+115x-24}{12(x+5)}$ (f) $\frac{x+3y}{(x-y)(x+y)}$ (g) $\frac{10}{2r-1}$ (h) $\frac{-t^2+22t}{(t-4)(t+5)}$
- (i) $\frac{2x^2+10x+13}{(x+2)(x+3)}$ (j) $\frac{-6m+14}{(m-1)(m-3)}$ (k) $\frac{9x^2-4x-56}{(x-4)(x+2)}$ (l) $\frac{3x^2-xy+2y^2}{(x-y)(x+y)}$
6. (a) $\frac{7x-10}{(x-4)(x+3)(x+5)}$ (b) $\frac{2x+8}{x(x+1)(x+2)}$ (c) $\frac{-w^2-9w}{(w-1)(w-5)(w-6)}$
- (d) $\frac{x+6}{(x+4)(x+3)}$ (e) $\frac{-t+3}{(t+4)(t+1)(t+5)}$ (f) $\frac{3x+5}{x(x-5)(x+1)}$
- (g) $\frac{3x+16}{(x-4)(x+4)}$ (h) $\frac{-2m+18}{(m-3)(m+3)}$ (i) $\frac{4x+3y}{xy(x-y)}$
- (j) $\frac{w^2-wx+x^2}{(w-x)(w+x)}$
7. (a) $\frac{2x+4}{(x+3)(x+1)}$ (b) $\frac{-4}{(x-2)(x-1)}$ (c) $\frac{x^2+6x-15}{(x-3)(x+3)(x-1)}$
- (d) $\frac{3w+x-4y}{(w-x)(w-y)(x-y)}$ (e) $\frac{x^2-8x+6}{(x+3)(x-3)}$ (f) $\frac{x^3+3x^2-9x-5}{2(x+1)(x-1)}$
8. (a) $\frac{-1}{(2x-1)(x+2)(3x-2)}$ (b) $\frac{-t+3}{(2t-3)(t+5)}$ (c) $\frac{3x-4}{(x+1)(x-2)}$
- (d) $\frac{x^2+6x+1}{(x+5)(x-5)}$ (e) $\frac{x^2-2x+3}{2x^2-x-1}$ (f) $\frac{rst(r^2+s^2+t^2)}{t^2-r^2-s^2}$
- (g) $\frac{3x^2-2x}{3x+1}$ (h) $\frac{x^2+1}{2x}$

Review Exercise

1. (a) $x^2+18x+77$ (b) $4x^2+20x+25$ (c) $9x^2-1$ (d) $t^2+4t-21$ (e) m^2+m-72 (f) m^4-6m^2-16
2. (a) $5(x+6)$ (b) $4m(2m-1)$ (c) $(x-y)(3m+2n)$ (d) $(x+6)(x+3)$
- (e) $(m-7)^2$ (f) $(n-6)(n+4)$ (g) $(x-4)(x+4)$ (h) $(x-7)(x-4)$
- (i) $(y-5)(y+4)$ (j) $(2x-5)(2x+5)$ (k) $(t+4)^2$ (l) $(1-5w)(1+5w)$
3. (a) $m \leq 0$ (b) $x \leq 5$ (c) $n \leq -7$ (d) $x \leq 5, -2$
4. (a) $\frac{15}{x^2}$ (b) $\frac{8}{x}$ (c) $\frac{5}{3}$ (d) $\frac{2}{x}$ (e) $\frac{-1}{x+2}$ (f) $\frac{3}{4}$
- (g) $\frac{12}{(x+2)^2}$ (h) x^2 (i) $\frac{1}{x^2}$ (j) 1
5. (a) $5x+1$ (b) $7m-23n$ (c) $5x^2+x$
- (d) $12x^2+23x-24$ (e) $-21t^2+31t-4$ (f) $12x^4-41x^3+35x^2$
- (g) $x^4-6x^3+23x^2-42x+49$ (h) $6m^3+m^2-14m-8$ (i) $-6s^3+11s^2-14s+5$

- (j) $3w^4 - 7w^3 - 10w^2 + 19w - 5$ (k) $-22x^2 - 48x - 10$ (l) $18x^2 - 17xy + 32y^2$
 (m) $2x^3 - 36x^2 + 13x - 9$
6. (a) $(m+5)(m+3)$ (b) $2(x-4)(x+3)$ (c) $(2x-3)(x-7)$ (d) $(3x+7)(2x+9)$
 (e) $(5x-4)(3x-1)$ (f) $(3m+2)(4m-5)$ (g) $(4t-5)(3t-7)$ (h) $2(4n+5)(n-6)$
 (i) $(6r-1)(2r+5)$ (j) $(x+y)(m+n)$ (k) $(x-y)(m-4)$ (l) $(s+t)(w+5)$
 (m) $(4m-3n)(2m+5n)$ (n) $(5p-q)(4p+3q)$ (o) $(2x+5y)(3x+4y)$ (p) $3(3s-5t)(2s+5t)$
7. (a) $(5x-6y)(5x+6y)$ (b) $(1-10m)(1+10m)$
 (c) $(t^2-6xy)(t^2+6xy)$ (d) $(3x-5)^2$
 (e) $(4m-3n)^2$ (f) $(2r+5s)^2$
 (g) $(x+y-6)(x+y+6)$ (h) $(5m-x+y)(5m+x-y)$
 (i) $(m+2n-2r-2s)(m+2n+2r+2s)$ (j) $(x+2y-m-3n)(x+2y+m+3n)$
 (k) $(x-4y-6m)(x-4y+6m)$ (l) $(3t-x+3y)(3t+x-3y)$
8. (a) $6x+1$ (b) $3x^2-2x+1$ (c) $-5x^2y^4+6xy^6-7x^2y^5$
 (d) $1-3mn+4m^3n^2$ (e) $-4+3rt-5t^3$ (f) $n^2+2m^2-3m^3n^5$
9. (a) x^2-x-1 (b) x^2-3x-5 (c) m^3-3m+1 (d) t^2-3t+1 (e) $3x^2-2xy+y^2$
10. (a) 8 (b) 5 (c) -13 (d) 23
11. (a) $(x-2)(x-5)(x+3)$ (b) $(x+3)(x+4)(x-4)$ (c) $(x+2)(x+3)(x+5)$
 (d) $(x+3)(x-2)(3x+2)$ (e) $(x-4)(3x^2+x-5)$ (f) $(2x+3)(x-1)(x-5)$
 (g) $(x+3)(x-1)(x-2)$ (h) $(x+1)(x^2+2x-2)$
12. (a) $(x+3)(x^2-3x+9)$ (b) $(m-2)(m^2+2m+4)$
 (c) $(5t-1)(25t^2+5t+1)$ (d) $(x^3+3y^4)(x^6-3x^3y^4+9y^8)$
13. (a) $15x^2y$ (b) $\frac{2y^4}{x^4}$ (c) $\frac{x+4}{x+5}$ (d) 1 (e) $\frac{x}{x+2}$ (f) $\frac{x-6}{x-7}$ (g) $\frac{x+7}{x}$
14. (a) $\frac{2x+7}{(x+4)(x+3)}$ (b) $\frac{-t+1}{(t-3)(t-4)}$ (c) $\frac{-2x-33}{(2x+3)(2x-3)}$
 (d) $\frac{6x+10}{(x+1)(x+2)(x+3)}$ (e) $\frac{-m^2-m}{(m-1)(m-2)(m+3)}$ (f) $\frac{-x}{(x+2)(x-4)(x-3)}$
 (g) $\frac{10x+4}{(x+2)(x+2)(x-2)}$ (h) $\frac{11x+8}{(2x+1)(3x+2)(x-4)}$ (i) $\frac{7x+11}{(x+2)(x+1)}$
 (j) $\frac{3m-24}{(m+2)(m-4)}$ (k) $\frac{2x^2-7x+8}{(x-4)(x-2)}$
15. (a) $\frac{4x+7}{4x-1}$ (b) 1 (c) $\frac{x-5}{x+6}$
 (d) $\frac{x-4}{x+4}$ (e) $\frac{4x^2-x+3}{(2x-1)(3x+2)(4x+5)}$ (f) $\frac{-6x-1}{(3x+2)(2x+1)}$

REVIEW AND PREVIEW TO CHAPTER 2

Exercise 1

1. \$3.89 2. \$15.00 3. \$100.00 4. \$32.50 5. \$0.55 6. \$0.45
 7. \$26.95 8. \$18.40 9. \$102.00 10. \$24.86 11. \$42.00 12. \$835.25
 13. 10% 14. 5% 15. 25% 16. 5% 17. 215% 18. 167%
 19. 17% 20. 9% 21. 2% 22. 17% 23. 14% 24. 0.7%
 25. (a) \$1694, \$2180, \$2398 (b) \$188.16 26. (a) \$900 (b) \$1650 (c) \$122.50
 (d) \$2482 27. (a) \$49.60 (b) \$77.14 (c) \$31.57 (d) \$48.52

Exercise 2

1. 9 2. 150 3. 51 4. -52 5. 1
 6. 1 7. $\frac{1}{2}$ 8. $\frac{35}{48}$ 9. 130 10. 14
 11. 0 12. -4 13. 6.22 14. 2.23 15. 25.1
 16. 354 17. 0.453 18. 0.217 19. 3.56 20. 592
 21. 60.8 22. 0.0826 23. 11.9 24. -0.0796 25. 2.29

CHAPTER 2

Exercise 2-1

1. (a) 0.125 (b) 0.25 (c) 0.375 (d) 0.625 (e) 0.875
 (f) 0.42 (g) 0.1 (h) 0.45 (i) 0.538461 (j) 1.36

2. (a) $\frac{3}{10}$ (b) $\frac{43}{200}$ (c) $\frac{309}{500}$ (d) $\frac{7}{200}$ (e) $\frac{4}{11}$ (f) $\frac{5}{9}$ (g) $\frac{235}{999}$ (h) $\frac{523}{110}$
 (i) $\frac{1}{7}$ (j) $\frac{1}{7}$ (k) $\frac{4769}{999}$ (l) $\frac{1}{55}$
7. (a) $\frac{1}{12}$ (b) $\frac{1}{12}$ (c) $\frac{1}{11}$ (d) $\frac{1}{27}$ (e) $\frac{2}{9}$ (f) $\frac{4}{55}$
 8. (a) $\frac{1}{2}$ (b) $\frac{1}{2}$ (c) $\frac{7}{10}$ (d) $\frac{1}{4}$
 9. (a) $\frac{1}{3}$ (b) $\frac{1}{3}$ (c) $\frac{4}{11}$ (d) $\frac{52}{333}$

Exercise 2-2

1. (a) closure (+) (b) neutral element ($\times$) (c) neutral element (+)
 (d) commutative ($\times$) (e) associative (+) (f) commutative ($\times$)
 (g) commutative (+) (h) inverse element ($-$) (i) distributive
 (j) closure ($\times$) (k) distributive (l) closure ($\times$)
2. (a) trichotomy (b) symmetric (c) completeness (d) transitive
 (e) transitive (f) transitive (g) symmetric (h) symmetric
3. (a) 9, 6 (b) 0, 5 (c) 3, 6 (d) 3, 7 (e) -3, 2 (f) 0, 3
 (g) $\frac{3}{5}, \frac{1}{5}$ (h) $\frac{1}{4}, 1$ (i) 3.5, 0.8 (j) $a, 3$ (k) $b+2, 2$ (l) $a, \frac{a^2-1}{a}$
4. (a) $a < b$ (b) $a = a$ (c) $a > b$ (d) $3a+3b=3(a+b)$
 (e) $x+3=2x+1$ (f) $a+b > a-b$ (g) $-1 < 3c$ (h) $0.25 = \frac{1}{4}$
 5. (a) $y=0$ (b) $y < 0$ (c) $y > 0$ (d) $y=0$
 (e) $y > 0$ (f) $y=0$, also $y=1$
6. (a) $x > 2$ (b) $x < 5$ (c) $x < 0$ (d) $x < -4$ (e) $x > 4$ (f) $x < 4$ (g) $x < -3$
 (h) $x > -3$ (i) $x < -2$ (j) $x < 6$ (k) $x < -6$ (l) $x > -5$
7. (a) 2.5 (b) 0.5 (c) -3.5 (d) 33.5 (e) 1.45 (f) -2.55 (g) $\frac{5}{12}$ (h) $-\frac{17}{24}$
 8. (a) 2 (b) 6 (c) 4 (d) 3 (e) $-\frac{1}{2}$ (f) 7

Exercise 2-3

1. (a) no (b) no (c) yes (d) no (e) no (f) yes (g) yes
 (h) yes (i) no (j) no
2. (a) 3 (b) 4 (c) -21 (d) 8 (e) 4 (f) 14 (g) 8
 (h) $\frac{1}{2}$ (i) 7 (j) 11
3. (a) 3 (b) 6 (c) 4 (d) -1 (e) 18 (f) -3 (g) -4
 (h) 11 (i) 2.5 (j) 7.5
4. (a) 5 (b) -5 (c) $-\frac{1}{2}$ (d) -2 (e) 44 (f) 11 (g) $\frac{15}{7}$
 (h) 10 (i) -0.03 (j) 0.03
5. (a) $\frac{b+c}{a}$ (b) $\frac{b^2+bc}{a}$ (c) $\frac{a}{b+c}$ (d) $\frac{ac}{cb-1}$ (e) $\frac{ab+c}{a}$
 (f) $\frac{3a-5b+5c}{2}$ (g) $\frac{2ab+b-c}{2a}$ (h) $\frac{-a+b+c}{3(a-b)}$ (i) $\frac{3a-c}{3b}$ (j) $\frac{5a+4b+3c}{2a}$

Exercise 2-4

1. (a) $x > 2$ (b) $x < 3$ (c) $x \leq 3$ (d) $x \leq -5$ (e) $x \leq -2$ (f) $x < -1$ (g) $x < 0$
 (h) $x \geq 0$ (i) $x > 1$
2. (a) yes (b) yes (c) no (d) no (e) no (f) no (g) yes (h) yes
 3. (a) $x \leq 6$ (b) $x < -6$ (c) $x \geq 7$ (d) $x \leq 6$ (e) $x > 5$ (f) $x > -6\frac{2}{3}$ (g) $x \geq -\frac{1}{4}$
 (h) $x < 8$ (i) $x < 0$
4. Graphs are left for the student.
 (a) $x < 3$ (b) $x \leq 5$ (c) $x \leq -4$ (d) $x < 4$ (e) $x < 4\frac{2}{3}$ (f) $x \leq -1\frac{2}{3}$
 5. (a) $x \geq \frac{2}{3}$ (b) $x < 2\frac{2}{3}$ (c) $x \geq 3$ (d) $x < 1\frac{1}{3}$ (e) $x > 4$ (f) $x \leq 1$
6. (a) $x < \frac{c-b}{a}$ (b) $x \geq \frac{ac+bc}{ab}$ (c) $x \geq \frac{ab-bc}{ac}$ (d) $x > \frac{ab-c}{a-b}, a > b$
 (e) $x \leq \frac{ac-ab-b^2}{c}$ (f) $x < \frac{ab-a^2c}{(b-a)c}, b > a$
7. (a) $x > \frac{c-b}{a}$ (b) $x \geq \frac{ac+bc}{ab}$ (c) $x \geq \frac{ab-bc}{ac}$ (d) $x < \frac{ab-c}{a-b}, a > b$

Exercise 2-5

1. (a) 5 (b) 4 (c) 10 (d) 4 (e) -4 (f) 14 (g) 12 (h) -2
 2. (a) 5, -5 (b) 7, -7 (c) no value (d) 2, -4
 (e) $-6 \leq x \leq 6$ (f) $x < -4$ or $x > 4$ (g) $-2 < x < 2$ (h) $x < -5$ or $x > 5$

3. (a) $1, -2\frac{1}{3}$ (b) $2\frac{1}{3}, -1$ (c) $4, -2$ (d) $-1, 2$ (e) $2, -\frac{1}{2}$ (f) $1, 4$
 (g) $2\frac{1}{2}, 1\frac{1}{4}$ (h) $3, -1$ (i) $-\frac{1}{3}$
 4. (a) $-5 < x < 1$ (b) $x \leq 1$ or $x \geq 5$ (c) $-2 \leq x \leq 2$
 (d) $-2 \leq x \leq 2$ (e) $x < -2$ or $x > 3$ (f) $x \leq \frac{1}{3}$ or $x \geq 1$
 (g) $x \in R$ (h) $-4 \leq x \leq 4$ (i) $-2 \leq x \leq 5$
 5. (a) $5, -1\frac{2}{3}$ (b) $2, \frac{2}{5}$ (c) $1, \frac{1}{3}$ (d) $4, 2$ (e) 0 (f) $2\frac{1}{4}, 2\frac{1}{2}$
 (g) $-2, \frac{7}{2}$ (h) $-\frac{3}{4}, 3\frac{1}{2}$ (i) $4, -\frac{4}{5}$
 6. (a) $-a \leq x \leq a$ (b) $x \neq 0$ (c) $x \geq -a$
 (d) $b-a, -a-b$ (e) $\frac{b+c}{a}, \frac{b-c}{a}$ (f) $-a-b < x < b-a$

Exercise 2-6

1. (a) $2, -5$ (b) $5, -2$ (c) $0, -3$ (d) $3, -7$ (e) $10, -7$ (f) $7, 0$ (g) $a, -b$ (h) $1, 3, 5$
 2. (a) $2, 3$ (b) $3, -2$ (c) $-3, 2$ (d) $1, 6$ (e) $3, 4$ (f) $2, 5$ (g) $4, -3$
 (h) $5, -3$ (i) $-10, 2$
 3. (a) $-4, -\frac{3}{2}$ (b) $2, \frac{5}{3}$ (c) $-\frac{1}{2}, -1\frac{1}{2}$ (d) $-1, -3\frac{1}{3}$ (e) $-2, 1\frac{1}{3}$ (f) $1\frac{1}{3}, -\frac{2}{5}$
 (g) $4, -\frac{1}{4}$ (h) $-\frac{1}{3}, -1\frac{1}{2}$ (i) $4, -\frac{1}{4}$
 4. (a) $6, -1$ (b) $0, 3$ (c) $3, 4$ (d) $4, -3$ (e) $(0, 5)$ (f) $1, -5$ (g) $5, 5$
 (h) $7, -6$ (i) $2\frac{1}{2}, \frac{1}{2}$
 5. (a) $-3, 2, 3$ (b) $\frac{5}{2}, 3, 4$ (c) $-3, -1, 2$ (d) $1, 1, 2$ (e) $-2, -1, 1$
 (f) $1, 3, 4$ (g) $-\frac{1}{2}, -1, 2$
 6. (a) 3 (b) 1 (c) $-\frac{2}{3}$ (d) $3, -1$
 7. (a) $x^2 - 3x + 2 = 0$ (b) $x^2 - x - 6 = 0$ (c) $x^2 - 16 = 0$ (d) $x^2 - 5x = 0$
 (e) $x^3 - 8x^2 + 19x - 12 = 0$ (f) $x^3 - 3x^2 - 4x = 0$ (g) $x^3 - 3x^2 - 4x + 12 = 0$ (h) $x^3 + 6x^2 + 11x + 6 = 0$
 8. $2, -5$
 9. (a) $5, -3$ (b) $3\frac{1}{2}, -1$

Exercise 2-7

1. $1, \frac{2}{3}$ 2. $\frac{5 \pm \sqrt{193}}{12}$ 3. $\frac{1 \pm \sqrt{21}}{4}$ 4. no real roots
 5. $\frac{4 \pm \sqrt{10}}{2}$ 6. $\frac{2}{3}, \frac{3}{4}$ 7. $\frac{9 \pm \sqrt{33}}{8}$ 8. $2, \frac{3}{8}$
 9. $\frac{12}{7}, -1$ 10. $\frac{11 \pm \sqrt{229}}{18}$ 11. $5, -\frac{3}{2}$ 12. $-2, -\frac{4}{3}$
 13. $\frac{-7 \pm \sqrt{145}}{8}$ 14. $1, -\frac{3}{11}$ 15. no real roots 16. $5.24, 0.764$
 17. $0.506, -1.10$ 18. $1.01, -0.419$ 19. $1.16, 0.289$ 20. $0.431, -0.381$
 21. $0.309, -1.39$ 22. $7.05, 0.455$ 23. $2.10, 0.044$ 24. $0.655, -0.655$
 25. $-0.0383, -6.42$

Exercise 2-9

1. (a) $\sqrt{125}$ (b) $\sqrt{28}$ (c) $-\sqrt{99}$ (d) $\sqrt{32}$ (e) $\sqrt{63}$ (f) $\sqrt[3]{54}$ (g) $\sqrt[3]{-40}$ (h) $\sqrt[4]{48}$
 2. (a) $3\sqrt{3}$ (b) $7\sqrt{2}$ (c) $12\sqrt{2}$ (d) $5\sqrt{3}$ (e) $10\sqrt{2}$ (f) -3 (g) $-4\sqrt[3]{2}$ (h) $2\sqrt[4]{3}$
 3. (a) 5 (b) $2\sqrt{2}$ (c) 2 (d) 2 (e) $\frac{1}{8}$ (f) -3 (g) 2 (h) 2
 4. (a) $7\sqrt{3}$ (b) $\sqrt{5}$ (c) $4\sqrt{3}$ (d) $9\sqrt{6} - 3\sqrt{5}$
 (e) $-\frac{1}{2}\sqrt{2}$ (f) $8\sqrt{2} - 7\sqrt{3}$
 5. (a) $2\sqrt{3} - \sqrt{6}$ (b) $2\sqrt{15} - 4\sqrt{3}$ (c) $6\sqrt{5} - 3\sqrt{10}$ (d) $6 + 3\sqrt{2}$ (e) $\sqrt{ab} - 2\sqrt{a}$
 (f) $a + \sqrt{ab}$ (g) $4 - 2\sqrt{3}$ (h) $11 - 6\sqrt{2}$ (i) $2\sqrt{6} - 2\sqrt{15}$
 6. (a) $x \geq 1$ (b) $x \geq -3$ (c) $x \geq 0$ (d) $x \in R$
 (e) $-1 \leq x \leq 1$ (f) $x > 0$ or $x < -1$ (g) $x \geq 1.5$ (h) $x \leq 1.5$
 7. (a) $5|x|y^2$ (b) $6|x|\sqrt{(x+y)}$ (c) $4|a|\sqrt{a(a+b)}$ (d) $7|a|\sqrt{b(a-b)}$
 8. (a) $\sqrt{15ax}$ (b) $-\sqrt{18x^3}$ (c) $\sqrt{9x^3y^5}$ (d) $\sqrt{9x^6(x+4)}$
 9. (a) $15\sqrt{2}$ (b) $11\sqrt{5}$ (c) $\sqrt{3} + 8\sqrt{5}$
 (d) $26\sqrt{2} - 7\sqrt{3}$ (e) $18\sqrt{11} - 14\sqrt{3} - 2\sqrt{22}$ (f) $34\sqrt{2} - 5\sqrt{7}$

10. (a) $6\sqrt{2}-6\sqrt{3}$ (b) $10\sqrt{3}-15\sqrt{2}$ (c) $10\sqrt{2}-4\sqrt{5}$ (d) $19-6\sqrt{2}$ (e) $5-2\sqrt{6}$
 (f) $62-20\sqrt{6}$ (g) 17 (h) 11 (i) 62 (j) $53+36\sqrt{2}$
 (k) $-11+7\sqrt{2}$ (l) $\sqrt{10}-2$
11. (a) $|x+3|\sqrt{x+3}$ (b) $|x-3|$ (c) $|x-5|\sqrt{x}$ (d) $|x(x+4)|\sqrt{x}$
12. (a) $\sqrt{45x^3-60x^2+20x}$ (b) $\sqrt{9x^3-27x^2+27x-9}$ (c) $\sqrt{x^3-7x^2-5x+75}$
 13. (a) $-\sqrt{x}$ (b) $10x^2\sqrt{2}$ (c) $-3|x|\sqrt{x}$
 (d) $10\sqrt{x}-13\sqrt{y}$ (e) $2|x|\sqrt{2x+\sqrt{3x+2\sqrt{2x}}}$ (f) $6x^2\sqrt{xy}$
14. (a) $a-5\sqrt{a}$ (b) $6\sqrt{a}-10a$ (c) $4a-12\sqrt{ab}+9b$ (d) $a-4b$
 (e) $18a-12\sqrt{a}+2$ (f) $9a+6\sqrt{ax}-35x$
15. (a) $x-6\sqrt{x+9}$ (b) $x+3+4\sqrt{x-1}$ (c) $17+x-8\sqrt{x+1}$ (d) $2x+2\sqrt{x^2-1}$
 (e) $2x-1-2\sqrt{x^2-x}$ (f) $2x-3+2\sqrt{x^2-3x}$

Exercise 2-10

1. (a) 44 (b) $13+4\sqrt{3}$ (c) 2 (d) 2 (e) $a-b$ (f) $19+6\sqrt{2}$
2. (a) $\frac{3\sqrt{10}}{5}$ (b) $\frac{\sqrt{6}+\sqrt{10}}{2}$ (c) $\frac{\sqrt{2}}{3}$ (d) $\frac{\sqrt{30}}{9}$
3. (a) $\frac{6+3\sqrt{2}}{2}$ (b) $9+5\sqrt{3}$ (c) $3+2\sqrt{2}$ (d) $5-2\sqrt{6}$ (e) $\frac{-9\sqrt{5}-3\sqrt{10}}{7}$
 (f) $\frac{16-5\sqrt{10}}{3}$ (g) $\frac{8\sqrt{6}+4\sqrt{3}}{7}$ (h) $5-\sqrt{15}$ (i) $2\sqrt{14}-\sqrt{35}$
4. (a) 3.14 (b) 1.31 (c) 2.38 (d) 9.90 (e) 3.62 (f) 2.07 (g) -4.11
 (h) 0.82 (i) 1.49
5. (a) $\frac{\sqrt{a}+\sqrt{b}}{a-b}$ (b) $\frac{a+b+2\sqrt{ab}}{a-b}$ (c) $\frac{a\sqrt{a}-b\sqrt{b}+a\sqrt{b}-b\sqrt{a}}{a-b}$
 (d) $\frac{3a-6\sqrt{a}}{a-4}$ (e) $\frac{6a-19\sqrt{a}+15}{4a-9}$ (f) $\frac{a\sqrt{a}-a\sqrt{b}-b\sqrt{a}+b\sqrt{b}}{a-b}$
 (g) $\frac{\sqrt{x-1}-1}{x-2}$ (h) $\sqrt{x+3}-x$ (i) $\frac{5x\sqrt{x}-5x\sqrt{2x-5}}{5-x}$

Exercise 2-11

1. (a) $x \geq 0$ (b) $x \geq 2$ (c) $x \geq -3$ (d) $x \in \mathbb{R}$
 (e) $x \leq -1$ or $x \geq 1$ (f) $x \leq -2$ or $x \geq 2$ (g) $x > -2$ (h) $x > 3$
2. (a) 4 (b) 1 (c) no root (d) 5 (e) 4 (f) 4
3. (a) 48 (b) 8.5 (c) -1 (d) 12
4. (a) 7 (b) 1, 2 (c) 5 (d) 3
5. (a) 4 (b) 4, 0 (c) 0 (d) -1, 3
6. (a) 9, 25 (b) $2\frac{1}{8}$ (c) $1\frac{1}{3}$ (d) 9

Exercise 2-12

1. (a) $x+3$ (b) $3x$ (c) $3x+0.7$ (d) $80x$
2. (a) $-$ (b) $\div$ (c) $-$ (d) $-$
3. (a) 640 km (b) 3280 m (c) 375 km (d) 140 km
4. 13°C , 21°C
5. 07:32:31
6. 85%
7. (a) 7.64 cm^2 (b) 370 m^2 (c) 164 cm^2
8. 2, 3, 4 or -2, -3, -4
9. -4, -6, -8 or 4, 6, 8
10. 3, 5, 7
11. 10 cm
12. 10 m
13. 3 m, 4 m
14. 50 km/h
15. 75 km/h
16. 52 km/h, 1.44 m/s
17. 4 s
18. 10 s and 190 s
19. from 1 s to 3 s after throwing
20. 11 s

Review Exercise

- (a) $0.08\dot{3}$ (b) 0.15 (c) $0.5\dot{4}$ (d) $0.\dot{0}7692\dot{3}$
- (a) $\frac{325}{999}$ (b) $3\frac{8}{11}$ (c) $\frac{243}{990}$
- (a) 5 (b) $\frac{1}{6}$ (c) 4 (d) 2 (e) 4 (f) $\frac{c-b}{a}$ (g) $-\frac{5}{a+7}$
- (a) $x \leq 8$ (b) $x < -4$ (c) $x < \frac{19}{3}$ (d) $x < 3$ (e) $x \geq 8\frac{3}{5}$
- (f) $-5 < x < -1$ (g) $x \leq 4$ or $x \geq 6$ (h) $-2 \leq x \leq 3$
- (a) 5, -4 (b) 4, -4 (c) 1, -2 (d) $\frac{3}{5}, \frac{1}{2}$ (e) $\frac{3}{4}, -\frac{5}{2}$
- (f) $-3, -\frac{2}{3}$ (g) 2, 4, 5 (h) $\frac{1}{2}, 1, -6$
- (a) 2.35, 1.15 (b) 12.9, 1.48 (c) 0.765, -0.555 (d) 0.538, -2.38
- (e) 2.68, -0.811 (f) 0.785, -0.732
- (a) $3\sqrt{2}$ (b) $9\sqrt{3}$ (c) $6\sqrt{5}$ (d) $6\sqrt{5} + 15\sqrt{2}$ (f) $6\sqrt{2} - 2\sqrt{6}$
- (f) $1 - \sqrt{15}$ (g) $14 + 4\sqrt{6}$
- (a) 3.54 (b) 4.04 (c) 0.82 (d) 1.98 (e) 6.68
- (a) 9 (b) 3 (c) 4 (d) 4
- 100 km/h 13. 7, 8 14. 3 cm 15. -7, -5 or 5, 7

REVIEW AND PREVIEW TO CHAPTER 3

Exercise 1

- (a) $\frac{b-9}{b^2-9}$ (b) $\frac{3x^2+20x-32}{49-x^2}$
- $R = \frac{R_1 R_2}{R_1 + R_2}$, $R_1 = \frac{R_2 R}{R_2 - R}$, $R_2 = \frac{R_1 R}{R_1 - R}$
- $P_1 = \frac{p_2 V_2 T_1}{V_1 T_2}$, $V_1 = \frac{p_2 V_2 T_1}{p_1 T_2}$, $T_1 = \frac{p_1 V_1 T_2}{p_2 V_2}$
 $P_2 = \frac{p_1 V_1 T_2}{V_2 T_1}$, $V_2 = \frac{p_1 V_1 T_2}{p_2 T_1}$, $T_2 = \frac{p_2 V_2 T_1}{p_1 V_1}$
- $x = \frac{y-1}{y+1}$ 5. $y = \frac{z}{xz-1}$, $z = \frac{y}{xy-1}$ 6. $m = \frac{2E}{v^2}$, $v = \sqrt{\frac{2E}{m}}$
- $s = \frac{bc(m-1)}{mc-b}$, $b = \frac{msc}{s+c(m-1)}$, $c = \frac{sb}{b+m(s-b)}$

Exercise 2

- 1 2. $\frac{43}{23}$ 3. $\frac{2x+3}{x+2}$ 4. $1-x$ 5. $\frac{3x-1}{2x-1}$ 6. $\frac{x^2+x+1}{x}$
- $\frac{2(x-1)}{3(x+1)}$ 8. $\frac{x+2}{x^2}$ 9. $\frac{1}{x+1}$

CHAPTER 3

Exercise 3-1

- (a) 2 (b) -1 (c) 8 (d) 2.6 (e) -13 (f) 3 (g) 302 (h) -28
- (i) $2+3a$ (j) $2+3b$
- (a) -3 (b) 0 (c) 0 (d) -4 (e) 96 (f) 60 (g) 60 (h) 1
- (i) a^2-4 (j) a^2-4
- (a) 1 (b) 28 (c) -13 (d) 27 (e) -3 (f) 10 (g) 0 (h) 17 (i) 16
- (a) Domain = {Bill, Henry, Sue, Gord}, Range = {12, 14, 15, 17}
- (b) Domain = {Mary, Glen, Bob, Fred}, Range = {3, 10, 16}
- (c) Domain = {-1, 0, 1, 2}, Range = {-2, 0, 2, 3}
- (d) Domain = {-2, -1, 0, 1, 2}, Range = {1}
- (a) 13 (b) 5 (c) 260 (d) 5.69 (e) 40 (f) 488 (g) 9.76 (h) 5.0201
- (a) $-1\frac{1}{3}$ (b) $\frac{14}{17}$ (c) $\frac{6}{23}$ (d) $-\frac{7}{8}$ (e) $-\frac{16}{23}$ (f) $\frac{103}{195}$ (g) $\frac{7}{36}$ (h) $\frac{a+3}{2a-5}$

7. (a) 3 (b) 13 (c) 273 (d) 8.3125
 (e) 2451 (f) 1333 (g) $3 + \sqrt{2}$ (h) 21.362 508 01
8. (a) 2 (b) 3 (c) 216 (d) 1024 (e) 2 (f) -4
9. (a) $3a^2 + a - 1$ (b) $3a^2 + 7a + 3$ (c) $12a^2 + 2a - 1$
 (d) $3(a^2 + 2ab + b^2) + a + b - 1$ (e) $3a^2 - a - 1$ (f) $3x^2 - x - 1$
10. (a) 5 (b) 9 (c) 2.5
11. Range = {791, 793, 796}
12. (a) {1, 2, 5} (b) {-4, -2, 0, 2, 4} (c) {0, 1, 2}
13. (a) $\{x \in R \mid x \geq 1\}$ (b) R (c) $\{x \in R \mid x \geq 0\}$
14. (a) $\{x \in R \mid x \neq 3\}$ (b) $\{x \in R \mid x \neq -2\}$ (c) $\{x \in R \mid x \geq 2\}$ (d) $\{x \in R \mid -3 \leq x \leq 3\}$
15. (a) 50 m (b) After 20 s
16. (a) 19 602.22 (b) 2.013 673 (c) 4312.842 (d) 2.695 262
17. (a) 1 (b) 95 (c) -8
 (d) -4 (e) $3x^2 - 11$ (f) $9x^2 + 24x + 11$
18. (a) $\frac{1}{8}$ (b) $1\frac{1}{2}$ (c) $\frac{3}{7}$ (d) $1\frac{3}{4}$ (e) $\frac{1}{x+2}$ (f) $\frac{x+2}{x+1}$
19. (a) 3, -4 (b) 4, -5 (c) 6, -7
20. 1, 4

Exercise 3-2

1. (i) (a) 0 (b) 2 (c) -2 (d) -1 (e) 1
 (ii) (a) 2 (b) -1 (c) 0 (d) -1 (e) 0
 (iii) (a) -1 (b) -1 (c) -2 (d) 1 (e) 2

Exercise 3-4

1. (a) Domain = {elephant, monkey}, Range = {peanut, banana}
 (b) Domain = {3, 4, 5, 6}, Range = {17, 0, 1}
 (c) Domain = $\{\pi\}$, Range = {1, 2, 3}
 (d) Domain = {-2, 0, 2, 4}, Range = {0, -3}
 (e) Domain = {Sandra, Pat, Jane}, Range = {82, 84, 86}
 (b) and (d) are functions
2. (a) Domain = {-2, -1, 0, 1, 2}, Range = {1, 2, 3}
 (b) Domain = {-2, -1, 0, 1, 2}, Range = {1, 2, 3}
 (c) Domain = $\{x \in R \mid -2 \leq x \leq 2\}$, Range = $\{y \in R \mid -2 \leq y \leq 3\}$
 (d) Domain = $\{x \in R \mid -3 \leq x \leq 4\}$, Range = $\{y \in R \mid -2 \leq y \leq 3\}$
 (e) Domain = $\{x \in R \mid -2 \leq x \leq 3\}$, Range = $\{y \in R \mid -3 \leq y \leq 2\}$
 (f) Domain = $\{x \in R \mid -2 \leq x < 3\}$, Range = {-2, -1, 0, 1, 2}
 (a), (d) and (f) are functions.
3. (a) {(Jane, Joe), (Judy, Jane), (Judy, Joe), (Jim, Judy), (Jim, Jane), (Jim, Joe), (John, Jim), (John, Judy), (John, Jane), (John, Joe)}
 Domain = {Jane, Judy, Jim, John}
 Range = {Joe, Jane, Judy, Jim}
- (b) {(Jane, Joe), (Jane, Judy), (Jane, Jim), (Jane, John), (Judy, Joe), (Judy, Jane), (Judy, Jim), (Judy, John)}
 Domain = {Jane, Judy}
 Range = {Joe, Jane, Judy, Jim, John}
4. (a) {(4, 3), (6, 3), (6, 5), (8, 3), (8, 5), (8, 7), (10, 3), (10, 5), (10, 7), (10, 9)}
 (b) Domain = {4, 6, 8, 10}, Range = {3, 5, 7, 9}
5. (a) {(2, 2), (2, 4), (2, 6), (2, 8), (3, 3), (3, 6), (3, 9), (4, 4), (4, 8), (5, 5), (6, 6), (7, 7), (8, 8), (9, 9)}
7. (b), (d), (g), and (j) are functions.

Exercise 3-5

1. (a) "is loved by" (b) "is owned by" (c) "is a factor of"
 (d) "has a brother named" (e) "is eaten by" (f) "is older than"
2. (i) (a) {(2, 0), (3, 0), (6, 4), (2, 8)}
 (b) Domain = {2, 3, 6}, Range = {0, 4, 8}
- (ii) (a) {(-2, 1), (-1, 2), (0, 3), (1, 1)}
 (b) Domain = {-2, -1, 0, 1}, Range = {1, 2, 3}
- (iii) (a) {(1, -1), (2, -2), (3, -3)}
 (b) Domain = {1, 2, 3}, Range = {-1, -2, -3}

- (iv) (a) $\{(\frac{1}{3}, 3), (\frac{1}{4}, 4), (3, 3), (4, 4)\}$
 (b) Domain = $\{\frac{1}{3}, \frac{1}{4}, 3, 4\}$, Range = $\{3, 4\}$

3. yes

- (a) Ann (b) Paul (c) Jack (d) Jean (e) -2 (f) -1 (g) 3 (h) -2
 (i) 1 (j) 3 (k) 3 (l) 0

5. In (a) and (c), f^{-1} is a function.

6. In (a), (d), and (f), f^{-1} is a function

7. (a) $f^{-1}(x) = x - 1$ (b) $g^{-1}(x) = \sqrt{x}$

Exercise 3-6

7. $N = 2^t$

11. $D = K \sqrt[3]{N}$ (K a constant)

12. $m = C I^3$ (C a constant)

Review Exercise

1. (a) -3 (b) 7 (c) 11 (d) 84 (e) 0.4 (f) 0 (g) $3\frac{1}{4}$
 (h) 52 (i) $1 - 2a$ (j) $b^2 + 3$
 2. (a) 15 (b) Glen (c) Sue (d) 17 (e) -3 (f) 1 (g) -3
 (h) 2 (i) 16 (j) -1 (k) -1 (l) 17
 3. (a) Domain = $\{x \in R \mid -3 \leq x \leq 3\}$, Range = $\{y \in R \mid -3 \leq y \leq 3\}$
 (b) Domain = $\{x \in R \mid -2 \leq x \leq 2\}$, Range = $\{y \in R \mid -2 \leq y \leq 2\}$
 (c) Domain = R , Range = $\{y \in R \mid -1 \leq y \leq 2\}$
 (d) Domain = $\{x \in R \mid -3 \leq x \leq 2\}$, Range = $\{y \in R \mid -2 \leq y \leq 3\}$
 (e) Domain = $\{x \in R \mid -3 \leq x \leq 2\}$, Range = $\{y \in R \mid -2 \leq y \leq 3\}$
 (f) Domain = $\{x \in R \mid 1 \leq x \leq 2\}$, Range = $\{y \in R \mid 1 \leq y \leq 2\}$
 (c) and (e) represent functions.
 4. (a) 2 (b) $\frac{3}{5}$ (c) $-\frac{144}{145}$ (d) -194
 (e) 1472 (f) $\frac{99}{101}$ (g) $\pi^3 - 2\pi^2 + 3\pi - 4$ (h) $-\frac{3}{5}$
 5. (a) $2a^2 - 3a + 1$ (b) $2a^2 - 7a + 6$ (c) $2(a^2 - 2ab + b^2) - 3a + 3b + 1$
 (d) $18a^2 - 9a + 1$ (e) $2a^2 + 3a + 1$ (f) $2x^2 + 3x + 1$
 8. (a) $\{(2, 2), (4, 2), (4, 4), (6, 2), (6, 6), (8, 2), (8, 4), (8, 8), (10, 2), (10, 10), (12, 2), (12, 4), (12, 6), (12, 12)\}$
 10. In (a) and (c) f^{-1} is a function.
 12. (a) 0, 1 (b) 3, -2 (c) 5, -4
 13. (a) 2.647 059 (b) -2.690 731 (c) 0.008 205 311 (d) 3.645 720

REVIEW AND PREVIEW TO CHAPTER 4

Exercise 1

1. (a) $\frac{12}{7}$ (b) $\frac{4}{9}$ (c) $\frac{1}{8}$ (d) -2 (e) $\frac{3}{5}$ (f) $-\frac{3}{4}$
 (g) $\frac{3}{5}$ (h) $\frac{9}{10}$ (i) $-\frac{6}{5}$ (j) no slope (k) 0 (l) $\frac{10}{11}$
 (m) no slope (n) -1 (o) $-\frac{6}{7}$ (p) $\frac{7}{5}$ (q) $-\frac{10}{3}$ (r) $\frac{b-d}{a-c}$
 4. (a) 6 (b) 3 (c) -14 (d) 4 (e) 4 (f) -3
 6. (a) yes (b) no (c) no (d) yes (e) yes (f) no

Exercise 2

1. (a) (3, 4), (1, 6), (-3, 10), (9, -2) (b) (5, -7), (-5, 3), (-3, 1), (4, -6)
 (c) (4, 1), (9, 6), (-4, -7), (2, -1) (d) (6, 10), (1, 5), (-3, 1), (-9, -5)
 (e) (3, 7), (-1, 3), (-2, 2), (-7, -3) (f) (4, 11), (1, 5), (-2, -1), (-5, -7)
 (g) (5, -7), (-12, 10), (-8, 6), (10, -12) (h) (3, 2), (0, 4), (-3, 6), (15, -6)
 (i) (0, -4), (10, 4), (-5, -8), (20, -12) (j) (2, 0), (4, 3), (-4, -9), (-2, -6)
 (k) (2, -8), (-2, 2), (-4, 7), (0, -3)
 3. (a) yes (b) no (c) no (d) yes (e) no (f) no (g) yes (h) yes

CHAPTER 4

Exercise 4-1

2. (a) 3, 5 (b) 6, 2 (c) 4, -8 (d) 7, -4 (e) 3, -4 (f) -, -6 (g) -2, -
 (h) 5, -2 (i) 2, -1 (j) $\frac{1}{2}, -\frac{1}{3}$

5. $y=6$
 6. $x=4$
 7. (a) $3x-y-4=0$ (b) $3x-2y-7=0$ (c) $x-3y-8=0$
 (d) $2x-y-1=0$ (e) $4x+y+7=0$ (f) $x-2y-10=0$
 (g) $2x+6y+7=0$ (h) $4x-3y+72=0$ (i) $8x-15y-160=0$
 8. (a) $y=-2x+7$ (b) $y=-\frac{1}{2}x+\frac{3}{2}$ (c) $y=-3x+4$ (d) $y=2x-6$ (e) $y=5x-1$
 (f) $y=\frac{3}{2}x$ (g) $y=-\frac{5}{2}x+\frac{9}{2}$ (h) $y=\frac{1}{5}x-\frac{6}{5}$ (i) $y=\frac{2}{3}x-\frac{4}{3}$ (j) $y=-\frac{7}{3}$
 10. (a) 3 (b) 4 (c) 2 (d) 4 (e) 4 (f) 1, -1

Exercise 4-2

1. (a) $y-2=4(x-3)$ (b) $y-4=6(x-1)$ (c) $y-5=1(x+2)$ (d) $y+2=5(x+7)$
 (e) $y+4=-2(x+3)$ (f) $y=-4(x-6)$ (g) $y-5=2x$ (h) $y+2=0$
 2. (a) $y=2x+3$ (b) $y=-4x+5$ (c) $y=5x-3$ (d) $y=\frac{1}{2}x-7$ (e) $y=-\frac{2}{3}x$ (f) $y=-6$
 3. (a) 3, -7 (b) $-\frac{1}{2}, 5$ (c) -5, -4 (d) 7, 0 (e) $\frac{2}{3}, -\frac{1}{3}$ (f) 2, -7 (g) -3, 4
 (h) 2, -3 (i) -5, 2 (j) 7, -3 (k) 4, -5 (l) $-\frac{3}{2}, \frac{5}{2}$ (m) $-\frac{1}{3}, -\frac{2}{3}$ (n) 6, -4
 4. (a) $3x-y-5=0$ (b) $5x-y+19=0$ (c) $x+y+1=0$ (d) $7x-y+5=0$ (e) $6x+y-30=0$
 (f) $x-2y-10=0$ (g) $y-6=0$ (h) $x-4=0$ (i) $x+2y-2=0$ (j) $x-3y+4=0$
 5. (a) $2x-y+13=0$ (b) $4x+y+11=0$ (c) $3x-2y+22=0$ (d) $x+2y-6=0$ (e) $5x+2y+10=0$
 6. (a) $4x-3y=0$ (b) $4x-3y+18=0$ (c) $4x-3y+27=0$ (d) $4x-3y-29=0$ (e) $4x-3y-18=0$
 7. (a) $y=3x-1$ (b) $y=-2x-3$ (c) $y=2x-6$ (d) $y=-0.5x$ (e) $y=\frac{1}{2}x-3$
 (f) $y=-6$ (g) $y=\frac{3}{4}x+\frac{25}{4}$ (h) $y=0.2x+18$
 8. (a) $2x-y=0$ (b) $x-y+1=0$ (c) $x-y+1=0$ (d) $x+2y-7=0$
 (e) $x+6y+8=0$ (f) $6x+5y+16=0$ (g) $7x-13y+34=0$ (h) $9x+5y-43=0$
 (i) $4x-9y-21=0$ (j) $10x+7y+11=0$ (k) $12x-22y-19=0$ (l) $60x-44y-21=0$
 (m) $15x-50y+117=0$ (n) $10x+3y+27=0$ (o) $2x+y+8=0$ (p) $x-4=0$
 (q) $y+3=0$ (r) $x-y=0$
 9. (a) $y=-4x+7; -4, 7$ (b) $y=-\frac{3}{2}x+\frac{5}{2}; -\frac{3}{2}, \frac{5}{2}$ (c) $y=\frac{2}{3}x-\frac{1}{3}; \frac{2}{3}, -\frac{1}{3}$ (d) $y=\frac{1}{2}x-2; \frac{1}{2}, -2$
 (e) $y=\frac{1}{4}x+\frac{3}{4}; \frac{1}{4}, \frac{3}{4}$ (f) $y=-\frac{2}{3}x; -\frac{2}{3}, 0$ (g) $y=\frac{1}{2}x-\frac{3}{2}; \frac{1}{2}, -\frac{3}{2}$ (h) $y=7; 0, 7$
 10. (a) $3x-y-7=0$ (b) $2x-y=0$ (c) $4x-y+1=0$ (d) $x+2y-4=0$
 11. (a) $3x-y-12=0$ (b) $2x+y-10=0$ (c) $x+y+3=0$ (d) $5x-y+10=0$
 (e) $x-4y=0$ (f) $x-5y+1=0$ (g) $5x+10y-7=0$ (h) $65x+25y+338=0$
 12. (a) $2x+y-6=0$ (b) $2x-y+4=0$ (c) $5x-2y-10=0$ (d) $6x+y+6=0$
 (e) $x-2y+8=0$ (f) $7x-3y-21=0$
 13. $2x+y+2=0$
 14. $2x+3y+12=0$
 15. $2x+5y-6=0$
 16. $x+y-2=0, 3x+y=0$

Exercise 4-3

1. (a) $2, -\frac{1}{4}$ (b) $\frac{1}{4}, -4$ (c) $-\frac{3}{4}, \frac{4}{3}$ (d) $-4, \frac{1}{4}$ (e) $-\frac{3}{2}, \frac{2}{3}$
 (f) $\frac{5}{6}, -\frac{6}{5}$ (g) -1, 1 (h) 0, no slope
 2. (a) neither (b) parallel (c) perpendicular (d) neither
 (e) perpendicular (f) neither (g) perpendicular (h) parallel
 3. (a) parallel (b) perpendicular (c) neither (d) perpendicular
 (e) parallel (f) parallel (g) perpendicular (h) perpendicular
 4. (a) parallel (b) neither (c) perpendicular (d) neither
 5. (a) $3x-y-6=0$ (b) $2x+y+7=0$ (c) $x-3y+16=0$ (d) $x+2y+7=0$
 (e) $3x-5y-5=0$ (f) $3x-2y+27=0$ (g) $18x-24y+17=0$ (h) $5x+2y-25=0$
 (i) $x+4y-8=0$ (j) $x+1=0$ (k) $y+6=0$ (l) $5x-4y-20=0$
 6. (a) $7x-y+41=0$ (b) $3x-2y+6=0$ (c) $3x+2y-12=0$ (d) $x-5y-10=0$
 (e) $x+1=0$ (f) $y+2=0$
 10. (a) $\frac{2}{3}$ (b) $-\frac{3}{2}$ (c) $-\frac{8}{3}$
 11. (a) $-\frac{3}{2}$ (b) 15 (c) 10
 12. $4x-5y=0$ and $x+y-9=0$
 13. (a) $x-7y-11=0$ (b) $7x+y-27=0$ (c) $x-7y+39=0$
 14. $Ax+By-As-Bt=0$

Exercise 4-4

1. (c) $y=\frac{3}{4}x+30$ (d) 75
 2. (d) $v=\frac{4}{3}t+364$ (e) 448
 3. (a) $t=0.025p$ (b) 0.025 (c) \$1425 (d) \$36 000 (e) 35

4. (c) $f = 3s - 105$ (d) \$75 (e) 35 km/h
 5. 3.36 g
 6. \$280
 7. 45.5 months
 8. 61.7 cm^3
 9. 1027 kg
 10. (c) $L = -\frac{1}{6}d + 100$ (d) 60 L (f) 100 L (g) 600 km (h) $16\frac{2}{3} \text{ L/100 km}$
 11. 18.75Ω
 12. (c) $t = 11.5h + 11$ (d) 22.5°C
 13. (b) $c = \frac{1}{20}d + 84$ (c) \$124
 14. (b) $c = 0.3d + 2$ (c) \$14 (d) 90 km
 15. (b) $c = 0.3t + 1.60$ (c) \$3.70 (d) 10 min
 16. (b) $w = 9.60t, 0 \leq t \leq 40; w = 14.40t, t > 40$ (c) \$456
 17. (b) $c = 5n, 0 \leq n \leq 100; c = 4.5n, 101 \leq n \leq 200; c = 4n, n > 200$
 (c) (i) \$435 (ii) \$688.50 (iii) \$852
 18. (b) (i) \$12 (ii) \$72 (iii) \$132
 (c) $f = 2s, 1 \leq s \leq 10; f = 4s, 11 \leq s \leq 20; f = 6s, 21 \leq s \leq 30; f = 10s, s > 30$

Review Exercise

2. (a) $x - y + 2 = 0$ (b) $4x - y + 11 = 0$ (c) $2x + y - 3 = 0$ (d) $y + 2 = 0$
 (e) $4x - y + 27 = 0$ (f) $6x + y + 25 = 0$ (g) $x - 2y - 4 = 0$ (h) $x + 5y + 7 = 0$
 3. (a) $2x - y - 1 = 0$ (b) $8x - y - 26 = 0$ (c) $3x - y + 3 = 0$ (d) $y + 2 = 0$
 (e) $x + y - 1 = 0$ (f) $10x - y - 7 = 0$ (g) $x - 2 = 0$ (h) $6x - 7y - 1 = 0$
 4. (a) $x - 2y + 7 = 0$ (b) $5x + 11y + 8 = 0$ (c) $3x - y + 3 = 0$ (d) $x - 2y + 5 = 0$
 (e) $3x - 4y - 12 = 0$ (f) $2x - y - 10 = 0$ (g) $x - 2y + 4 = 0$
 5. (a) $3x + 2y + 9 = 0$ (b) $3x + 4y + 28 = 0$ (c) $3x + y - 15 = 0$ (d) $y - 6 = 0$
 (e) $x - 3 = 0$ (f) $3x + 2y + 16 = 0$ (g) $5x - y + 34 = 0$
 6. $k = -\frac{3}{4}$
 7. $k = 7$
 10. \$994
 11. (b) $c = 20n + 500$ (c) \$2320
 12. (c) \$846.30
 13. (a) $t = 0.037p$ (b) 0.037 (c) \$1961 (d) \$42 000

REVIEW AND PREVIEW TO CHAPTER 5

Exercise 1

1. (a) $7x + 7y + 1$ (b) $5a - 9b - 13$ (c) $8n - 10$
 (d) $-4t - 19$ (e) $15x - 7y + 8$ (f) $3b - 5$
 (g) $9x - 2y + 2$ (h) $3b - 7$ (i) $15m - 13$
 (j) $4a + 2b + 6x - 15$ (k) $3a + 5b + 3m - 6t - 4$ (l) $12x^2 + 3x - 11$
 2. (a) $x + 2y + 2$ (b) $a + b - 11$ (c) $-m - 5n - 11$ (d) $3x + 1$ (e) $-4a + 10$
 (f) $4m - 2n - 2$ (g) $3x - 10y + 6$ (h) $2a + 4b - 4$ (i) $4m + 6x + 5$ (j) $11a - 3b - 11$
 (k) $-4a + 9b + 6$ (l) $-x^2 - 11x + 5$
 3. (a) $4a - 25b - 4$ (b) $-4x + 6y + 7$ (c) $-8x + 15y - 8$ (d) $2x - 8y - 2$ (e) $6x^2 - 8x + 1$
 (f) $x^2 - 7x - 5$ (g) $2a + 5b + 1$

Exercise 2

1. (a) 9 (b) -1 (c) 3 (d) 5 (e) 6 (f) 15 (g) 10 (h) 30
 (i) 44 (j) 8 (k) -13 (l) $-6\frac{2}{3}$ (m) $-2\frac{1}{3}$ (n) $-3\frac{1}{4}$ (o) 0 (p) -10
 (q) 0 (r) 13
 2. (a) 30 (b) 17 (c) -2 (d) $3\frac{2}{3}$ (e) -79 (f) -34 (g) $-\frac{2}{9}$ (h) 10.5
 (i) 9.25 (j) -27 (k) $4\frac{6}{7}$ (l) 7 (m) -13 (n) $-6\frac{1}{2}$ (o) $\frac{1}{5}$ (p) $-\frac{2}{7}$
 (q) 3 (r) 21
 3. (a) $7 - y$ (b) $\frac{y+2}{3}$ (c) $\frac{3(y-2)}{2}$ (d) $\frac{5-4y}{3}$ (e) $2y - 1$ (f) $\frac{3(y+4)}{2}$
 (g) $\frac{2(3-y)}{3}$ (h) $\frac{2(2y+7)}{3}$ (i) $10 - y$

4. (a) $\frac{2(2+x)}{3}$ (b) $\frac{2(3-2x)}{3}$ (c) $3x+7$ (d) $\frac{2(x-3)}{3}$ (e) $2(x+6)$ (f) $15-x$
 (g) $\frac{2(14-x)}{3}$ (h) $\frac{x}{5}$ (i) $\frac{2x}{5}$

5. (a) $\frac{2A}{b}$ (b) $\frac{P-2I}{2}$ (c) $\sqrt{\frac{E}{RT}}$ (d) $\frac{v-u}{a}$ (e) $\frac{2(s-ut)}{t^2}$ (f) $\frac{y+ma}{m}$
 (g) $\frac{ay}{a-x}$ (h) $\frac{y-b}{x}$ (i) $\frac{2s}{n}-I$ (j) $\frac{P_2 V_2 T_1}{P_1 V_1}$

CHAPTER 5

Exercise 5-1

- (a) (7, 5) (b) (5, 4) (c) (8, 2) (d) (4, 2) (e) (2, 5) (f) (9, 2) (g) (3, 1) (h) (5, 5)
 (i) (3, 1) (j) (1, 4) (k) (2, 1) (l) (4, 2)
- (a) consistent, dependent (b) inconsistent (c) consistent, independent
 (d) consistent, independent (e) consistent, dependent (f) inconsistent
- (a) (2, 4) (b) (4, 11) (c) (-5, -7) (d) (-3, -6) (e) (2, 7) (f) (6, 3)
 (g) (3, 7) (h) (1, -6) (i) (-1, -4) (j) (-4, -5)
- (a) (-4, 3) (b) ϕ (c) (-7, 0)
 (d) (-3, 8) (e) (-11, -19) (f) $(-\frac{1}{2}, 5)$
 (g) $(-3, \frac{1}{3})$ (h) infinitely many solutions (i) $(\frac{1}{2}, -\frac{1}{5})$
 (j) (9, -2)
- (a) (2, 1) (b) (5, 2) (c) (-2, -3) (d) (2, -1) (e) (-2, -4) (f) (-2, 7)
- (a) (4, 8) (b) (2, -3) (c) (10, 6) (d) (0.2, 0.3) (e) (-0.6, 0.2)
 (f) (-0.5, -0.7) (g) (3, -4) (h) (4, 9) (i) (-2, -3) (j) (3, -1)
 (k) (3, 5) (l) (5, -2)
- (4, 3), (-4, -5), (-2, 5)
- (a) $(\frac{1}{2}, \frac{1}{3})$ (b) $(\frac{1}{4}, 3)$ (c) $(-1, \frac{1}{2})$ (d) (6, -2)

Exercise 5-2

- 471, 292
- 562, 385
- 97, 84
- 35, 24
- 81, 72
- 57, 33
- 121, 59
- \$4600 at 8%, \$7500 at 9%
- \$300 000 at 7%, \$35 000 at 10%
- 16
- \$3, -\$2
- $m = \$4$, $n = -\$3$
- 8 L of 40%, 12 L of 30%
- 70 L of 50%, 30 L of 40%
- 45 km/h, 585 km/h
- 30 km/h, 270 km/h
- 4 km/h, 20 km/h
- $A = 4$, $B = -2$
- $A = 4$, $B = -2$
- $m = 3$, $b = -4$
- $m = \frac{1}{2}$, $b = 4$
- $a = 3$, $c = -4$
- $\frac{m+n}{2}, \frac{m-n}{2}$
- 1 km/h

Exercise 5-3

- (a) $① \times 3$, $② \times 2$ (b) $① \times 3$, $② \times 2$ (c) $① \times e$, $② \times b$ (d) $① \times m$, $② \times n$
- (a) $(a, -b)$ (b) $(\frac{a+b}{2}, \frac{a-b}{2})$ (c) $(2a, 3a)$ (d) $(\frac{1}{a}, \frac{1}{b})$
 (e) $(a, -b)$ (f) $(-c, -d)$ (g) $(m-n, m+n)$ (h) $(a-b, -a-b)$
 (i) $(\frac{st-nw}{ms-nr}, \frac{rt-mw}{nr-ms})$ (j) $(\frac{b_2c_1-b_1c_2}{a_1b_2-a_2b_1}, \frac{a_2c_1-a_1c_2}{a_2b_1-a_1b_2})$
- (a) $(\frac{d}{ad-bc}, \frac{-c}{ad-bc})$ (b) $(\frac{-b}{ad-bc}, \frac{a}{ad-bc})$

Exercise 5-4

- (a) yes (b) yes (c) no (d) no (e) yes (f) no (g) no (h) yes
- (a) (1, 2, 3) (b) (3, 2, 1) (c) (-3, 2, 1) (d) (5, 6, -2)
 (e) (-3, 0, 7) (f) (3, -5, -2) (g) (6, -3, 0) (h) (-4, -3, -2)
 (i) (3, 5, -2) (j) (-2, -2, -2) (k) (-3, -1, 0) (l) (3, 0, -4)

3. (a) $(5, -3, -4)$ (b) $(\frac{1}{2}, 4, -3)$ (c) $(\frac{1}{2}, -\frac{1}{3}, \frac{1}{4})$ (d) $(3, 1, -2)$
 (e) $(6, -4, -2)$ (f) $(3, 2, 4)$ (g) $(0.1, -0.3, -0.2)$ (h) $(2, 1, 3, 2)$
 (i) $(-2, -1, -3, 1)$ (j) $(12, 6, -6)$
 4. $x = \$2, y = -\$3, z = \$1$ 5. $r = \$3, s = \$2, t = -\$3$ 6. $a = 1, b = -3, c = 2$
 7. $a = 2, b = 3, c = -5$ 8. 20 vans 9. $(2, 3, 4, -4, -3, 1)$

Exercise 5-5

1. (a) 2×3 (b) 3×2 (c) 3×1 (d) 1×4 (e) 1×1 (f) 2×2 (g) 1×2 (h) 3×3
 2. (a) $\begin{pmatrix} 8 & 3 \\ 6 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 4 \\ 4 & 5 \end{pmatrix}$ (c) $\begin{pmatrix} 6 & 10 \\ -4 & 8 \end{pmatrix}$ (d) $\begin{pmatrix} 8 & -10 \\ -1 & 4 \\ -4 & 1 \end{pmatrix}$
 (e) $\begin{pmatrix} -12 & -6 & 15 \\ -3 & 9 & 0 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & -2 \\ 4 & -5 \end{pmatrix}$
 3. (a) $\begin{pmatrix} 7 & 9 \\ 5 & 14 \end{pmatrix}$ (b) $\begin{pmatrix} 7 & 4 \\ 8 & -5 \\ 2 & 19 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & 14 & -2 \\ 20 & -1 & -5 \end{pmatrix}$ (d) $\begin{pmatrix} -4 & 38 \\ -13 & -5 \end{pmatrix}$
 (e) $\begin{pmatrix} -15 \\ 16 \\ -21 \end{pmatrix}$ (f) $\begin{pmatrix} -8 & -11 \\ 14 & -12 \end{pmatrix}$
 4. (a) $\begin{pmatrix} 13 & 13 \\ 21 & 11 \end{pmatrix}$ (b) $\begin{pmatrix} 17 \\ 6 \end{pmatrix}$ (c) $\begin{pmatrix} 6 & -2 \\ 3 & 4 \end{pmatrix}$ (e) (12)
 (f) $\begin{pmatrix} 7 & -4 & -10 \\ 10 & -7 & -17 \end{pmatrix}$ (g) $\begin{pmatrix} 18 & -3 & -9 \\ 26 & -4 & -4 \\ 30 & -3 & -9 \end{pmatrix}$ (h) $\begin{pmatrix} 5 & -2 \\ 4 & 3 \end{pmatrix}$ (j) $\begin{pmatrix} 2 & 3 & 1 \\ -1 & 0 & 2 \\ 4 & -3 & 5 \end{pmatrix}$
 (k) $\begin{pmatrix} 17 & -10 \\ 10 & -12 \end{pmatrix}$ (m) $\begin{pmatrix} -2 & -12 \\ 17 & -3 \end{pmatrix}$ (n) $\begin{pmatrix} 1 & 1 \\ -7 & -7 \end{pmatrix}$ (o) (2)
 5. (a) $\begin{pmatrix} 3 & -2 \\ -3 & -6 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ -2 & -7 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & -8 \\ -2 & 2 \end{pmatrix}, \begin{pmatrix} 3 & -8 \\ -2 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 5 & -6 \\ 3 & 6 \end{pmatrix}, \begin{pmatrix} 5 & -6 \\ 3 & 6 \end{pmatrix}$
 (d) $\begin{pmatrix} 1 & 9 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 5 \\ 0 & 5 \end{pmatrix}$ (e) $\begin{pmatrix} 15 & 5 \\ -5 & 0 \end{pmatrix}, \begin{pmatrix} 14 & 1 \\ -6 & 1 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & 5 \\ 5 & 26 \end{pmatrix}, \begin{pmatrix} 2 & 9 \\ 6 & 25 \end{pmatrix}$

Exercise 5-6

1. (a) 13 (b) 11 (c) 0 (d) 14 (e) -11 (f) -19 (g) -5 (h) -62
 2. (a) $\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{pmatrix}$ (c) $\begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$ (d) $\begin{pmatrix} \frac{2}{9} & 0 \\ -\frac{1}{9} & \frac{8}{9} \end{pmatrix}$ (g) $\begin{pmatrix} \frac{2}{5} & -\frac{3}{10} \\ -\frac{1}{5} & 0 \end{pmatrix}$ (h) $\begin{pmatrix} -3 & -2 \\ 1 & \frac{1}{2} \end{pmatrix}$
 3. (a) $\begin{pmatrix} \frac{3}{2} & -\frac{5}{2} \\ -1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
 4. (a) $\begin{pmatrix} 2 & -\frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Exercise 5-7

1. (a) $(1, 2)$ (b) $(3, 1)$ (c) $(-2, 1)$ (d) $(-3, 2)$ (e) $(\frac{1}{2}, -3)$ (f) $(-2, -1)$
 (g) $(2, 4)$ (h) $(3, -\frac{1}{2})$ (i) $(-2, -3)$
 2. (a) $(2, 1, 1)$ (b) $(1, 2, 1)$ (c) $(1, 1, 2)$ (d) $(-1, 2, 2)$ (e) $(1, 2, 1)$ (f) $(3, -2, 0)$

Exercise 5-8

1. (c) \$1645, \$1520, \$1590, \$1390 (d) \$6145
 2. (c) \$720, \$880, \$880, \$420, \$440 (d) \$3340
 3. (c) \$510, \$526, \$538, \$617, \$687, \$559, \$236 (d) \$3673
 4. 33 29 14 17 41 58 41 33 7 6 46 43
 5. GET OUT SOON
 6. 35 61 25 49 28 51 30 51 69 120 27 48 49 86
 7. GET ME OUT

Exercise 5-9

- (a) yes (b) no (c) yes (d) no (e) no (f) yes (g) yes (h) yes
(i) no (j) no (k) yes (l) no (m) yes (n) yes
- (a) $x \geq 1$ (b) $y \leq 2$ (c) $x < 2$ (d) $y > 1$ (e) $y \geq 0$ (f) $x \geq 0$ (g) $y \leq -2$
(h) $x < -1$ (i) $y \geq x$
- (a) $y \leq 3$ and $y \geq -2$ (b) $x < 1$ and $x \geq -4$ (c) $y < -1$ and $y > -3$ (d) $x \geq 1$ and $y \geq -2$
(e) $x \leq 4$ and $y < 2$ (f) $y \geq x$ and $y \leq -x$

Exercise 5-10

- (a) max at (4, 0), min at (0, 3)
(c) max at (6, 1), min at (0, 5)
(e) max at (-1, 4), min at (-3, -2), (2, -4)
(g) max at (-5, 0), min at (6, 0)
(j) max at (0, 15), min at (-10, -10)
- (a) min at (1, 3) (b) min at (4, -1)
- max at (10, 0) and min at (0, 0)
- (b) max at (6, 6), min at (0, 0)
(d) max at (4, 3), min at (-3, 2)
(f) max at (3, -2), min at (-6, 0)
(h) max at (10, 0), min at (-8, 0)
(j) max at (4, 0), min at (-4, 4)
3. max at (0, 8) and min at (0, 0)
5. max at (3, 0) and min at (0, 5)

Exercise 5-11

- 60 Amateur, 40 Professional
- 6 AM/FM, 3 AM
- 80 cars, 20 buses
- 6 Super I, 12 Super II
- 6 L of X, 4 L of Y
- 4 of A, 8 of B
- 2 loads of X, 10 loads of Y
- 3 Brand X, 1 Brand Y
- 1 Brand X, 3 Brand Y
- 100 ha of peas, 150 ha of beans

Review Exercise

- (a) (2, 3) (b) (2, 1) (c) (-3, 2) (d) (-4, -5) (e) $(\frac{1}{2}, -3)$ (f) $(-3, -\frac{1}{3})$
- (a) (8, -6) (b) (-4, -5) (c) (0.2, -0.3) (d) (-1, -3) (e) (-4, 6)
- 243, 175
- 11, 9
- 140 L of 40%, 60 L of 30%
- \$3, \$4
- 25 km/h
- (a) $a + b$, $a - b$ (b) $2a + b$, $a + 2b$
- 5 colour, 4 black-and-white
- 50 t of A, 60 t of B
- (a) 3, 2, -2 (b) -2, 0, 2 (c) 2, -1, -2 (d) 2, 1, -1, -2
- (c) May: \$9570, June: \$12 280, July: \$11 610 (d) \$33 460
- 35 22 57 38 33 24 31 17 43 28 44 28 30 25 14 9

REVIEW AND PREVIEW TO CHAPTER 6

Exercise 1

- (a) $2\sqrt{5}$ (b) $7\sqrt{2}$ (c) $10\sqrt{2}$ (d) $2\sqrt{17}$ (e) $\sqrt{277}$ (f) $2\sqrt{109}$
(g) $5\sqrt{5}$ (h) 17 (i) 10 (j) 8 (k) $5\sqrt{5}$ (l) $\sqrt{4.5}$
(m) $\sqrt{130}$ (n) $\sqrt{6.29}$ (o) $\sqrt{34}$ (p) $\sqrt{a^2 + b^2}$
- (a), (d) 6. (9, 0), (-3, 0) 7. (11, 0), (-13, 0) 8. (0, -3), (0, -11) 10. (0, -4)

CHAPTER 6

Exercise 6-1

- (a) (3, 5) (b) (5, 11) (c) (-4, -3) (d) (-4, 6₂) (e) $(-4\frac{1}{2}, -\frac{1}{2})$
(f) $(5, -6\frac{1}{2})$ (g) (-4, 5) (h) $(3, -2\frac{1}{2})$ (i) (1.3, -0.8) (j) $(-1.3, -4.8)$
- (-2, 9) 3. (-11, -4) 4. (3, -3)
- (-5, 3), (-2, -2), (1, -7) 6. $(2\sqrt{2}, 3\sqrt{3})$ 7. 6, 1
- $3x + 2y - 7 = 0$ 12. (4, 6), (6, 9)

Exercise 6-2

- (a) 10 (b) 30 (c) $20\frac{1}{2}$ (d) 22 (e) 34 (f) $22\frac{1}{2}$ (g) $21\frac{1}{2}$ (h) 30
(i) 34 (j) 62
- (b), (c)

4. (a) 56 (b) 62 (c) 89 (d) 94
 5. 8, $-\frac{4}{3}$ 6. 2, 18 7. $4\frac{2}{5}$ 8. 1, 5
 9. $2x+3y-33=0$, $2x+3y+23=0$ 10. (2, 2), (8, 0)

Exercise 6-3

1. (a) $1\frac{1}{5}$ (b) $\frac{7}{13}$ (c) $\frac{1}{2}$ (d) 1 (e) 3 (f) $\frac{4\sqrt{13}}{13}$
 2. (a) $1\frac{4}{5}$ (b) $\frac{5\sqrt{26}}{26}$ (c) $\frac{44\sqrt{41}}{41}$ (d) $\frac{15\sqrt{58}}{29}$ (e) $\frac{15\sqrt{34}}{34}$ (f) 0 (g) $\frac{4\sqrt{13}}{13}$ (h) $\frac{13\sqrt{17}}{17}$
 3. $\frac{19\sqrt{10}}{10}$
 4. (a) $\frac{11\sqrt{10}}{10}$ (b) 4 (c) $2\frac{2}{13}$
 5. $\frac{23\sqrt{34}}{34}$, $\frac{23\sqrt{37}}{37}$, $\frac{23\sqrt{17}}{17}$ 6. $\frac{3\sqrt{61}}{61}$ 7. $\frac{12\sqrt{13}}{13}$, $\frac{12\sqrt{13}}{13}$, 3
 8. $3x+4y+26=0$, $3x+4y-14=0$ 9. $9x-7y-11=0$

Exercise 6-4

1. (a) (9, 1) 2. (-5, 1), (1, -4), (7, 1), (1, 6)
 3. (a) (4, 4), (-2, 2), (3, -3) (b) 20
 4. (b) (-3, 6)
 6. (11, 6), (1, -2), (-5, 4)
 7. (a) $x-5y+6=0$ (b) $5x-7y-6=0$ (c) (4, 2)
 8. (a) (-1, 3) (b) $x-2y+7=0$ (c) $3x+4y-39=0$ (d) (5, 6)
 10. (a) $x+2y-2=0$, $4x-y+1=0$ (b) (0, 1)
 12. (a) $5x-4y=0$, $2x+5y-11=0$, $7x+y-11=0$
 13. (a) $4x-y-1=0$, $x-2y+1=0$, $2x+3y-3=0$
 14. (a) $x-y+1=0$ (b) $x+3y-3=0$ (c) $3x+y-1=0$
 15. (5, 8), (7, -4), (1, -2)
 16. (a) $2x-3y=0$, $x+2y-2=0$, $4x+y-4=0$, $(\frac{6}{7}, \frac{4}{7})$ (b) $5x-6y-6=0$, $x+3y-4=0$, $x-2=0$, $(2, \frac{2}{3})$
 (c) $4x+y-11=0$, $x+2y-4=0$, $2x-y-3=0$, $(2\frac{4}{7}, \frac{5}{7})$

Exercise 6-5

1. (a) $x^2+y^2=25$ (b) $x^2+y^2=81$ (c) $x^2+y^2=3$ (d) $x^2+y^2=20$ (e) $x^2+y^2=m^2$
 2. (a) 6 (b) 8 (c) 12 (d) 4 (e) $2\sqrt{6}$ (f) 1
 3. (a) $x^2+y^2<49$ (b) $x^2+y^2<4$ (c) $x^2+y^2<5$ (d) $x^2+y^2<18$ (e) $x^2+y^2<a^2$
 4. (a) $x^2+y^2>9$ (b) $x^2+y^2>100$ (c) $x^2+y^2>7$ (d) $x^2+y^2>48$ (e) $x^2+y^2>b^2$
 5. $x^2+y^2=25$ (b) $x^2+y^2=20$ (c) $x^2+y^2=36$ (d) $x^2+y^2=4$ (e) $x^2+y^2=24$
 8. $x^2+y^2=20$ 9. (c) 3 (f) 10
 10. $(x-2)^2+(y-1)^2=25$ 11. $(x-h)^2+(y-k)^2=r^2$

Review Exercise

1. (a) $\sqrt{85}$ (b) $2\sqrt{2}$ (c) $6\sqrt{2}$ (d) $\sqrt{13}$ (e) $\sqrt{2}$ (f) $\sqrt{109}$
 2. (a) (5, 4) (b) (5, 6) (c) (2, -1) (d) (-5, 1) (e) $(5\frac{1}{2}, -4\frac{1}{2})$ (f) $(\frac{1}{2}, -2\frac{1}{2})$
 3. (a) 6 (b) 36 (c) 4 (d) 46 (e) 69 (f) 79
 4. (a) $1\frac{2}{5}$ (b) $\frac{1}{13}$ (c) $\frac{23\sqrt{17}}{17}$ (d) $\frac{\sqrt{13}}{13}$ (e) $\frac{15\sqrt{41}}{41}$ (f) $\frac{11\sqrt{10}}{5}$
 5. (a) $x^2+y^2=25$ (b) $x^2+y^2=169$ (c) $x^2+y^2=58$ (d) $x^2+y^2=49$ (e) $x^2+y^2=16$
 7. 48.5 8. $\frac{26\sqrt{17}}{17}$, $\frac{13\sqrt{10}}{5}$, $\frac{16\sqrt{29}}{29}$ 9. (2, 6), (-6, 2), (4, -4) 10. b, c

REVIEW AND PREVIEW TO CHAPTER 7

Exercise 2

1. 30 2. 30 3. 30 4. 105 5. 90 6. 70

CHAPTER 7 Deductive proofs are not given in the answers.

Exercise 7-1

1. (a) $\triangle ABC \cong \triangle GHI$ (b) $\triangle AEB \cong \triangle CED$ (c) $\triangle ABC \cong \triangle ADC$ (d) $\triangle AED \cong \triangle BEC$
 (e) $\triangle AOB \cong \triangle COD$ (f) $\triangle ADC \cong \triangle AEB$

Exercise 7-2

1. (a) AC, AB, BC (b) DF, EF, ED (c) GH, HI, GI
 2. (a) AB (b) GH (c) KL
 3. (a) yes (b) no (c) yes (d) no (e) yes (f) yes (g) no (h) yes
 4. (a) $2 < s < 8$ (b) $8 < s < 22$ (c) $0 < s < 10$ (d) $1 < s < 23$
 (e) $0 < s < 16$ (f) $3 < s < 19$ (g) $10 < s < 20$ (h) $a - b < s < a + b$

Exercise 7-3

1. (a) $a = 110^\circ, c = 70^\circ$ (b) $a = 60^\circ, b = 60^\circ, c = 120^\circ$ (c) $a = c = 65^\circ, b = 115^\circ$
 (d) $a = 30^\circ, b = 70^\circ, c = 80^\circ$ (e) $a = 70^\circ, b = 30^\circ, c = 80^\circ$ (f) $a = b = 60^\circ, c = 120^\circ$
 (g) $a = 75^\circ, b = 55^\circ, c = 50^\circ$ (h) $a = 60^\circ, b = 80^\circ, c = 40^\circ$ (i) $a = 60^\circ, b = 80^\circ, c = 40^\circ$
 2. (a) yes (b) no (c) no (d) yes (e) no (f) yes

Exercise 7-5

1. (a) $16 \text{ cm}^2, 8 \text{ cm}^2, 8 \text{ cm}^2$ (b) $20 \text{ cm}^2, 10 \text{ cm}^2$ (c) $20.8 \text{ cm}^2, 10.4 \text{ cm}^2$
 2. $6.3 \text{ cm}^2, 12.6 \text{ cm}^2$ 3. $13.5 \text{ cm}^2, 30.6 \text{ cm}^2$ 4. $24 \text{ cm}^2, 8 \text{ cm}$ 5. $72 \text{ cm}^2, 144 \text{ cm}^2$

Exercise 7-6

1. (a) 10 (b) 13 (c) 17 (d) 25 (e) 12 (f) 15 (g) 21.2 (h) 10
 2. (a) yes (b) no (c) yes (d) yes
 3. (a) $\sqrt{3}$ (b) 2 (c) $\sqrt{2}$
 4. (a) 10 cm (b) 13.0 cm (c) 15.1 cm (d) 7.62 cm (e) 10.8 cm (f) 19.7 cm
 5. (a) 24 cm^2 (b) 30 m^2 (c) 60 cm^2 (d) 7.5 cm^2 (e) 15 m^2 (f) 7 m^2
 14. 11

Exercise 7-7

1. (a) 1440° (b) 1980° (c) 3240°
 2. (a) 60° (b) 90° (c) 108° (d) 120° (e) 171°
 6. 43
 7. (a) no (b) yes (c) no
 8. 5 9. no

Exercise 7-8

2. (a) $ABCD \parallel EFGH, BCFG \parallel ADHE, ABFE \parallel DCGH$ (b) AB is parallel to $DCGH$
 (c) FG is perpendicular to $DCGH$ (d) $BCGF \perp DCGH$ (e) $BCGF, DCGH, FGHE$
 3. (a) E and F, G and H, I and J, A and $D.$ (b) $E, F, G,$ and H (c) B
 (d) C and B (e) yes, in this diagram
 4. (a) 26 (b) 25 (c) 13
 5. (a) 320 m^2 (b) 204 m^2 (c) 198 m^2
 6. (a) 670 m^3 (b) 400 m^2 (c) 2560 m^2

Review Exercise

1. (a) $\triangle ACB \cong \triangle ECD, SAS$ (b) $\triangle ABC \cong \triangle DEC, SSS$ (c) $\triangle ABC \cong \triangle DEC, ASA$
 2. (a) $a = 59^\circ, b = 62^\circ, c = 59^\circ$ (b) $a = 115^\circ, b = 35^\circ, c = 30^\circ$ (c) $a = 36^\circ, b = 72^\circ, c = 108^\circ$
 3. (a) 20 (b) 10 (c) 10 (d) 30
 4. (a) 13.6 (b) 11.0 (c) 9.23 (d) 13.3
 5. (a) 3960° (b) 165°
 6. (a) 427, 4020 (b) 260, 400 (c) 204, 276
 12. (a) 3 (b) 22 (c) 12
 13. (a) 33 (b) 1.76 m
 14. 11.7

REVIEW AND PREVIEW TO CHAPTER 8

Exercise 3

1. yes 2. no 3. yes 4. yes 5. no 6. yes 7. no 8. yes 9. no

Exercise 4

1. no 2. no 3. yes 4. yes

CHAPTER 8

Exercise 8-1

1. (a) $\frac{4}{7}$ (b) 12 (c) 16 (d) $\frac{y}{7}$ (e) $\frac{11}{9}$ (f) $\frac{k-10}{10}$ (g) $\frac{1}{c}$
 (h) $4(x+1)$ (i) $\frac{6}{r}$ (j) $\frac{2}{7}$ (k) ± 3 (l) 4, 3
 2. (a) 15 (b) $\frac{27}{10}$ (c) 9 (d) 13 (e) $\frac{27}{4}$ (f) $\frac{28}{15}$ (g) $\frac{19}{3}$ (h) 4
 (i) 4 (j) $-\frac{17}{4}$ (k) $\frac{as}{r}$ (l) $\frac{1}{4}$ (m) ± 5 (n) -7 (o) $\frac{ac}{b+c}$
 3. (a) $\frac{10}{3}, 63$ (b) 90, 126 (c) $\frac{1}{5}, \frac{2}{15}$ (d) $\pm 16, \pm 2$
 4. (a) $\frac{2}{3}, 54$ (b) $\frac{9}{25}$ (c) $\frac{8}{11}$ (d) 7
 5. (a) $\frac{2}{3}, 9$ (b) $\frac{2}{7}$ (c) $\pm \frac{5}{4}$
 6. (a) ± 12 (b) ± 10 (c) ± 2 (d) $\pm 3\sqrt{7}$ (e) $\pm x^2$ (f) $\pm 6x$
 7. (a) $\frac{28}{3}$ (b) 20 (c) 1 (d) $\frac{bc}{a}$
 8. 15, 35 9. 40 10. $\frac{25}{168}$

Exercise 8-2

1. (a) 6, 9 (b) 3, 5 (c) 14, 2 (d) 4, 12 (e) 15, 6 (f) $1, \frac{3}{4}$ (g) 1, 15 (h) 4, 2
 2. (a) $\frac{2}{3}, 54$ (b) $6, \frac{63}{2}$ (c) $14, \frac{7}{9}$ (d) $-\frac{1}{3}, \frac{10}{3}$ (e) $\frac{2}{3}, 9$ (f) $\frac{35}{2}, 25, \frac{65}{2}$
 (g) 4, 11 (h) 4, -2 3. $\frac{2}{3}$ 4. 2.5 kg nitrogen, 1.5 kg phosphoric acid
 5. (a) 416 $\frac{2}{3}$ g peanuts, 166 $\frac{2}{3}$ g almonds (b) 250 g peanuts, 150 g cashews, 100 g almonds
 6. 42, 18, 12 7. 45, 60, 90, 135 8. 20°, 60°, 100°

Exercise 8-3

1. (a) $\frac{PQ}{QS}$ (b) $\frac{RT}{PR}$ (c) $\frac{PR}{PT}$ (d) $\frac{VY}{YZ}$ (e) $\frac{UX}{XZ}$ (f) $\frac{VY}{VZ}$
 2. (a), (d)
 3. (a) 5.25 (b) 9.6 (c) 18 (d) 9

4.

KL	LN	KN	KM	MO	KO
5	3	8	4	2.4	6.4
6	4	10	4.2	2.8	7
$3\frac{1}{3}$	2	$5\frac{1}{3}$	2.5	1.5	4
12	8	20	9	6	15
a	b	$a+b$	$\frac{ac}{b}$	c	$\frac{(a+b)c}{b}$
a	$\frac{ac}{b}$	$\frac{a(b+c)}{b}$	b	c	$b+c$

Exercise 8-4

1. (a) $\angle X = \angle S, \angle Y = \angle I, \angle Z = \angle T, \frac{XY}{SI} = \frac{YZ}{IT} = \frac{ZX}{TS}$
 (b) $\angle A = \angle K, \angle B = \angle L, \angle C = \angle M, \angle D = \angle N, \angle E = \angle O, \frac{AB}{KL} = \frac{BC}{LM} = \frac{CD}{MN} = \frac{DE}{NO} = \frac{EA}{OK}$

$$(c) \angle L = \angle P, \angle A = \angle Q, \angle M = \angle R, \frac{LA}{PQ} = \frac{AM}{QR} = \frac{ML}{RP}$$

$$(d) \angle S = \angle H, \angle O = \angle A, \angle F = \angle R, \angle T = \angle D, \frac{SO}{HA} = \frac{OF}{AR} = \frac{FT}{RD} = \frac{TS}{DH}$$

$$2. (a) OP \quad (b) A \quad (c) BE \quad (d) SU \quad (e) P \quad (f) BD, RA$$

$$3. \triangle ABC \sim \triangle YZX, \frac{AB}{YZ} = \frac{BC}{ZX} = \frac{CA}{XY} \quad \triangle PQR \sim \triangle PST, \frac{PQ}{PS} = \frac{QR}{ST} = \frac{RP}{TP}$$

$$\triangle DEF \sim \triangle KML, \frac{DE}{KM} = \frac{EF}{ML} = \frac{FD}{LK}$$

$$4. (a) 4_3^2 \quad (b) 8.75 \quad (c) 3\sqrt{2}$$

$$11. DE = 4.8 \text{ cm}, EF = 8 \text{ cm}, FD = 11.2 \text{ cm}$$

Exercise 8-5

1. (a) yes (SAS~) (b) yes (SSS~) (c) yes (SAS~)
2. (a), (b), (d) (SSS~)

Exercise 8-6

1. (a) $\frac{r}{s} = \frac{s}{t}$ (b) $\frac{d}{e} = \frac{e}{f}$ (c) $\frac{p}{r} = \frac{r}{s}$
2. (a) $\frac{c}{b} = \frac{b}{r}, \frac{c}{a} = \frac{a}{t}$ (b) $\frac{k}{n} = \frac{n}{d}, \frac{k}{m} = \frac{m}{f}$ (c) $\frac{c}{b} = \frac{b}{p}, \frac{c}{a} = \frac{a}{s}$
3. (a) 6 (b) 6.25 (c) 6 (d) 3.6 (e) $2\sqrt{15}$
5. (b) $\sqrt{6ab}$

Exercise 8-7

1. (a) 25:4 (b) 9:49 (c) 9:25 (d) 0.25:1
2. (a) 8:27 (b) 64:1 (c) 125:4 (d) 1:2
3. 85_7^5 cm^2 4. 1.2 cm 5. 3.2 cm^2 6. 390.625 cm^3
7. (a) 9:25 (b) 16:25 (c) 9:16
8. (a) 1:4 (b) 3:4 (c) 1:3
9. 8:25 11. (a) 1:4 (b) 1:12 14. 8 times as heavy

Exercise 8-8

1. (a) internally, 2:1 (b) externally, 3:2 (c) internally, 3:1 (d) externally, 2:5
(e) internally, 4:5 (f) internally, 6:5 (g) internally, 5:6 (h) internally, 1:2
(i) externally, 3:5 (j) externally, 6:5
2. (a) 1:3 (b) 2:9 (c) 3:1
3. 4 cm, 14 cm 4. 40 cm, 48 cm 5. 24 cm, 42 cm
9. (a) $(3, 2_3^2)$ (b) $(-1_5^3, -4)$ (c) $(2_3^3, -_4^1)$ (d) $(2_9^5, _9^5)$ (e) $(1_5^2, _5^3)$ (f) $(_5^6, -2_5^5)$
10. (a) (16, 7) (b) $(6_2^1, 4)$ (c) (-11, 4) (d) (24, -10) (e) $(-8_3^1, _3^2)$ (f) $(-4, 8_5^2)$
11. (-1, 0), (-4, -2) 12. (20, 7) 13. $\left(\frac{ax_2 + bx_1}{a+b}, \frac{ay_2 + by_1}{a+b}\right)$

Exercise 8-9

1. 20.9 m 2. 550 m 3. 25.5 m 4. 7.2 cm 5. yes 6. 27 m 7. $\sqrt{2}$ 8. $\sqrt[3]{2}$
9. (a) 4 times as strong (b) 8 times as heavy (c) the first animal

Review Exercise

1. (a) 18 (b) $\frac{8}{x}$ (c) $\frac{8}{9}$ (d) $4p$ (e) $\frac{b+8}{8}$ (f) $\frac{5}{3}$ (g) $\frac{3}{10}$ (h) $\frac{8}{r}$
2. (a) 6 (b) 2, 4 (c) 1 (d) 4, 2 (e) 5, 25 (f) 4, 8, 12
3. (a) $\frac{KM}{MO}$ (b) $\frac{KN}{KL}$ (c) $\frac{LN}{KL}$

4. (a) E (b) LV (c) CA (d) V (e) AB (f) LU, QR
5. (i) (a) $\frac{p}{q} = \frac{q}{r}$ (b) $\frac{c}{b} = \frac{b}{p}, \frac{c}{a} = \frac{a}{r}$ (ii) (a) $\frac{x}{y} = \frac{y}{z}$ (b) $\frac{f}{e} = \frac{e}{x}, \frac{f}{d} = \frac{d}{z}$
6. (a) $\frac{21}{2}$ (b) $\frac{6}{5}, \frac{9}{5}$ (c) $\frac{36}{5}$ (d) $\frac{8}{3}, \frac{15}{2}$ (e) 25 (f) $\frac{9}{2}, \frac{80}{3}$ (g) $\frac{6}{5}$ (h) ± 4
- (i) $\frac{bc}{a}$ (j) $-\frac{17}{4}$
7. (a) 2 (b) $\frac{3}{2}$ 8. (a) ± 25 (b) $\pm 4\sqrt{5}$ (c) $\pm 12x$
9. 46, 115 10. $30^\circ, 45^\circ, 105^\circ$ 11. (a) 4.5 (b) 19.2 (c) 6, 10
12. $12\frac{12}{13}, 8\frac{1}{13}$
14. (a) $\frac{4}{3}$ (b) 4.8, 10 (c) 6, 12 (d) 7.5, 10, 7.5
15. (a) $\triangle ABC \sim \triangle DAC$ (SSS~) (b) $\triangle PQR \sim \triangle STR$ (SAS~)
22. (a) $\frac{4}{9}$ (b) $\frac{4}{49}$
23. $\sqrt[3]{4}$ 24. 11.25 m 26. $(-1, -2), (2, 2), (5, 6)$ 27. $24^\circ \text{ cm}, 30^\circ \text{ cm}$
28. 35 cm, 15 cm 29. $(2\frac{4}{5}, 3\frac{1}{5})$ 30. $(-20, 16)$

REVIEW AND PREVIEW TO CHAPTER 9

Exercise 1

1. $\begin{pmatrix} 1 & 0 \\ 1 & 7 \end{pmatrix}$ 2. $\begin{pmatrix} 10 & -1 \\ 2 & -1 \end{pmatrix}$ 3. $\begin{pmatrix} 4 & -8 \\ 2 & 0 \end{pmatrix}$ 4. $\begin{pmatrix} 6.1 & 2.8 \\ -0.2 & 8.2 \end{pmatrix}$ 5. $\begin{pmatrix} 38 \\ 26 \end{pmatrix}$ 6. $\begin{pmatrix} -9 \\ 11 \end{pmatrix}$
7. $\begin{pmatrix} 16 & 2 \\ 18 & 0 \end{pmatrix}$ 8. $\begin{pmatrix} 5 & 6 \\ -1 & 0 \end{pmatrix}$ 9. $\begin{pmatrix} 3 & 0 \\ 5 & 2 \end{pmatrix}$ 10. $\begin{pmatrix} 5 & 22 \\ 10 & -5 \end{pmatrix}$ 11. $\begin{pmatrix} -1 & 2 \\ -2 & 26 \end{pmatrix}$ 12. $\begin{pmatrix} 5 & 12 \\ 0 & -20 \end{pmatrix}$

Exercise 2

1. -4, 8, 3, -2 2. 3, -2, 1, 4 3. -2, 3, 5, 1 4. 1, -1, 2, 4 5. 2, 4, 6, -2
6. 1, 4, 1, 11 7. -1, 4, 0, 6 8. -1, 2 9. 5, -4

CHAPTER 9

Exercise 9-1

1. (a) $\overline{AB} = [2, 1], \overline{CD} = [-1, -3], \overline{EF} = [1, -1], \overline{GH} = [5, 0], \overline{OP} = [2, 1], \overline{HP} = [-1, -3], \overline{KL} = [0, 3]$
(b) $\overline{AB} = \overline{OP}, \overline{CD} = \overline{HP}$
2. (a) (5, 2) (b) (7, 6) (c) (4, 8) (d) (12, 0) (e) $(x+5, y+2)$
3. (a) [5, 6] (b) [0, -4] (c) [5, 1] (d) [-1, 6] (e) [9, -8] (f) [-7, 4]
4. (i) $a=3, b=8$ (ii) $c=2, d=-1$ (iii) $e=2, f=2$
6. (b) (i) [3, -4] (ii) [-10, 5] (iii) [2, -7] (iv) [5, 6] (v) [-7, 1] (vi) [-3, 4]
(vii) [-5, -6] (viii) [8, 2]
- (c) (i) 5 (ii) $5\sqrt{5}$ (iii) $\sqrt{53}$ (iv) $\sqrt{61}$ (v) $5\sqrt{2}$ (vi) 5 (vii) $\sqrt{61}$ (viii) $2\sqrt{17}$
7. (a) (7, 5) (b) (7, 5) (c) (3, -3)
8. (a) 13 (b) $2\sqrt{17}$ (c) 10 (d) $2\sqrt{5}$ (e) $3\sqrt{5}$ (f) 6.5 (g) $4\sqrt{5}$ (h) $2\sqrt{5}$
9. (a) [-260, 0] (b) [0, -80] (c) $[50\sqrt{2}, 50\sqrt{2}]$ (d) $[30\sqrt{2}, -30\sqrt{2}]$
(e) [-200, $200\sqrt{3}$]

Exercise 9-2

1. (a) [8, 6] (b) [1, 7] (c) [-4, 3] (d) [18, -9] (e) [-6, 5]
(f) [-3, -8] (g) [9, 6.5] (h) $[p+r, q+s]$
2. (a) [3, 0] (b) [-2, -4] (c) [-8, 13] (d) [0, 0] (e) [6, 9.5]
(f) $[p-r, q-s]$
3. (a) $\vec{c} = \vec{a} + \vec{b}, \vec{a} = \vec{c} + (-\vec{b}), \vec{b} = (-\vec{a}) + \vec{c}, (\text{b}) \vec{u} = \vec{v} + \vec{w}, \vec{v} = \vec{u} + (-\vec{w}), \vec{w} = -\vec{v} + \vec{u}$
(c) $\vec{v} = \vec{e} + \vec{f}, \vec{e} = \vec{v} + (-\vec{f}), \vec{f} = -\vec{e} + \vec{v}, \vec{g} = \vec{v} + \vec{h}, \vec{h} = -\vec{v} + \vec{g}$
4. (a) $\overline{AC}$ (b) $\overline{AD}$ (c) $\overline{CA}$ (d) $\overline{BD}$ (e) $\overline{AD}$
5. (a) $\vec{v} = \vec{w} + \vec{s} + \vec{r} + \vec{u}$ (b) $\vec{v} = \vec{u} + (-\vec{w}) + \vec{c} + \vec{d} + \vec{e} + (-\vec{m}) + \vec{n}$

10. (i) (a) $[9, 4]$ (b) $[6, 2]$ (c) $[3, 6]$ (d) $[3, 7]$ (e) $[3.5, -3]$ (f) $[5, 8]$
 (g) $[8, -1]$ (h) $[12, -1]$
11. (a) $\vec{u} + \vec{v} = [-2, 9]$, $\vec{v} + \vec{w} = [-1, 2]$, $\vec{u} + \vec{w} = [3, 3]$ (b) $\sqrt{26}, 5, 2\sqrt{2}, \sqrt{85}, \sqrt{5}, 3\sqrt{2}$
12. equality occurs when $\vec{u}$ and $\vec{v}$ have the same direction
13. (a) $-\vec{u}$ (b) $\vec{w}$ (c) $-\vec{u} - \vec{v}$ (d) $\vec{w} - \vec{u}$ (e) $\vec{v} + \vec{w}$
 (f) $\vec{u} + \vec{v} + \vec{w}$ (g) $-\vec{v} - \vec{w} + \vec{u}$ (h) $\vec{v} + \vec{w}$
14. (a) $\overline{AC}$ (b) $\overline{GB}$ (c) $\overline{AC}$ (d) $\overline{HC}$ (e) $\overline{EB}$ (f) $\overline{AG}$ (g) $\overline{DF}$ (h) $\overline{FH}$ (i) $\overline{AC}$

Exercise 9-3

1. (a) $[24, -4]$ (b) $[10, 15]$ (c) $[-18, 48]$ (d) $[5, -3]$ (e) $[-15, 6]$ (f) $[2, 6]$
 (g) $[0, 0]$ (h) $[2, 0]$ (i) $[10, 16]$
2. $\vec{u} = \frac{3}{4}\vec{v}$, $\vec{w} = -\frac{1}{2}\vec{v}$, $\vec{r} = \frac{3}{2}\vec{v}$, $\vec{s} = -\frac{1}{4}\vec{v}$
3. (a) $[24, 58]$ (b) $[-19, 42]$ (c) $[7.5, -3]$ (d) $[44, 68]$ (e) $[30, -10]$ (f) $[13, -2]$
5. (a) $20\vec{u}$ (b) $7(\vec{u} + \vec{v})$ (c) $\frac{1}{2}\vec{u} + \frac{1}{2}\vec{v}$ (d) $6\vec{u} - 3\vec{v}$ (e) $4\vec{u}$ (f) $-\frac{1}{3}\vec{u}$
6. (a) $2\vec{v}$ (b) $\vec{u}$ (c) $2\vec{u}$ (d) $\vec{u} + \vec{v}$ (e) $2\vec{u} + 2\vec{v}$ (f) $\vec{u} + 2\vec{v}$ (g) $2\vec{v} - 2\vec{u}$
7. (a) $\overline{PC} - \overline{PA}$ (b) $\frac{1}{3}\overline{PC} - \frac{1}{3}\overline{PA}$ (c) $\frac{2}{3}\overline{PA} + \frac{1}{3}\overline{PC}$

Exercise 9-5

1. (a) $(4, 1)$ (b) $(5, -6)$ (c) $(0, 0)$ (d) $(1, -5)$ (e) $(3, -4)$
 (f) $(5, 5)$ (g) $(\frac{3}{2}, -\frac{3}{2})$ (h) $(a+3, b-4)$
2. (a) $(1, 9)$ (b) $(4, 5)$ (c) $(7, 2)$ (d) $(-5, -1)$ (e) $(-8, 13)$
 (f) $(7, 14)$ (g) $(3, 3)$ (h) $(a-2, b+5)$
4. (a) $A'(-6, 1)$, $B'(-3, 5)$, $C'(-2, 0)$ (b) $5, \sqrt{26}, \sqrt{17}, 5, \sqrt{26}, \sqrt{17}$
5. (a) $A'(-3, -1)$, $B'(-1, 5)$, $C'(2, 1)$ (b) $2\sqrt{10}, 5, \sqrt{29}, 2\sqrt{10}, 5, \sqrt{29}$
6. no, yes, $A\vec{X}$ 9. $3x+2y-9=0$

Exercise 9-6

1. (a) $180^\circ, -180^\circ$ (b) $140^\circ, -220^\circ$ (c) $90^\circ, -270^\circ$
2. (a) $180^\circ, 360^\circ$ (b) $90^\circ, 180^\circ, 270^\circ, 360^\circ$
- (c) $45^\circ, 90^\circ, 135^\circ, 180^\circ, 225^\circ, 270^\circ, 315^\circ, 360^\circ$ (d) $120^\circ, 240^\circ, 360^\circ$
- (e) every angle (f) 360°

5.	90°	180°	270°
$A(1, 0)$	$(0, 1)$	$(-1, 0)$	$(0, -1)$
$B(3, 2)$	$(-2, 3)$	$(-3, -2)$	$(2, -3)$
$C(2, -5)$	$(5, 2)$	$(-2, 5)$	$(-5, -2)$
$D(-1, 4)$	$(-4, -1)$	$(1, -4)$	$(4, 1)$
$P(x, y)$	$(-y, x)$	$(-x, -y)$	$(y, -x)$

6. $(1, 9), (2, 2), (9, 3)$ 7. $(0, 0), \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), (0, \sqrt{2}), \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

11. (a) R_{60} (b) R_{125} (c) $R_{\theta+\alpha}$

12.	$A(2, 2)$	$B(0, 5)$	$C(-1, 3)$	$P(x, y)$
$R \cdot$	$(-2, 2)$	$(-5, 0)$	$(-3, -1)$	$(-y, x)$
$T \circ R$	$(0, -3)$	$(-3, -5)$	$(-1, -6)$	$(-y+2, x-5)$
$R \circ T$	$(3, 4)$	$(0, 2)$	$(2, 1)$	$(-y+5, x+2)$
$S \circ T$	$(-2, -3)$	$(-4, 0)$	$(-5, -2)$	$(x-4, y-5)$
$R \circ S$	$(-2, -4)$	$(-5, -6)$	$(-3, -7)$	$(-y, x-6)$

13. The translation defined by the vector $\vec{u} + \vec{v}$.

Exercise 9-7

1. (a) $(3, -8)$ (b) $(6, 5)$ (c) $(4, 0)$ (d) $(-1, -1)$ (e) $(0, -7)$ (f) $(-2, 3)$
 (g) $(0, 0)$ (h) $(a, -b)$
2. (a) $(-3, 8)$ (b) $(-6, -5)$ (c) $(-4, 0)$ (d) $(1, 1)$ (e) $(0, 7)$ (f) $(2, -3)$
 (g) $(0, 0)$ (h) $(-a, b)$

3. (a) The points on the mirror line
 (b) The lines which are perpendicular to the mirror line.
4. (a) 2, 2 (b) 4, 4 (c) 1, 1 (d) 0, 1 (e) 2, 2 (f) 1, 1 (g) 2, 2 (h) 0, 2 (i) 6, 6
6. (b) $A'(-2, 0)$, $B(4, -1)$, $C(2, -7)$ (d) $\sqrt{37}$, $2\sqrt{10}$, $\sqrt{65}$, $\sqrt{37}$, $2\sqrt{10}$, $\sqrt{65}$
8. (a) (3, 4) (b) (7, 6) (c) (4, 7) (d) $(-4, -7)$

13.

	$A(1, 4)$	$B(-2, 3)$	$C(0, 7)$	$P(x, y)$
R	$(-1, -4)$	$(2, -3)$	$(0, -7)$	$(-x, -y)$
$F \circ G$	$(-1, -4)$	$(2, -3)$	$(0, -7)$	$(-x, -y)$
$F \circ T$	$(4, 0)$	$(1, 1)$	$(3, -3)$	$(x+3, -y+4)$
$T \circ F$	$(4, -8)$	$(1, -7)$	$(3, -11)$	$(x+3, -y-4)$
$R \circ T$	$(-4, 0)$	$(-1, 1)$	$(-3, -3)$	$(-x-3, -y+4)$
$T \circ G$	$(2, 0)$	$(5, -1)$	$(3, 3)$	$(-x+3, y-4)$

14. (a) $A, H, I, M, O, T, U, V, W, X, Y$ (b) $B, C, D, E, H, I, K, O, X$ (c) H, I, N, O, S, X, Z

Exercise 9-8

1. (a) (30, 18) (b) (60, -48) (c) (0, 36) (d) $(-12, -30)$ (e) (3, 0)
2. (a) $(-1, -\frac{3}{5})$ (b) $(-2, \frac{8}{5})$ (c) $(0, -\frac{6}{5})$ (d) $(\frac{2}{5}, 1)$ (e) $(-\frac{1}{10}, 0)$
3. (a) the centre of the dilatation
 (b) the lines which pass through the centre of the dilatation
5. (a) larger (b) same (c) smaller (d) smaller (e) same (f) larger
6. (a) $A'(3, 3)$, $B'(6, -9)$, $C'(-6, 0)$ (c) $\sqrt{17}$, 5, $\sqrt{10}$, $3\sqrt{17}$, 15, $3\sqrt{10}$ (e) 9
7. (a) $P'(-1, 0)$, $Q'(-2, -5)$, $R'(-3, -2.5)$ (c) $2\sqrt{26}$, $\sqrt{29}$, $\sqrt{41}$, $\sqrt{26}$, $\frac{1}{2}\sqrt{29}$, $\frac{1}{2}\sqrt{41}$
8. (a) 2.5 (b) 7.2 (c) 15
9. (a) $A'(4, -2)$, $B'(10, 4)$, $C'(13, 1)$ (b) $A'(6, -6)$, $B'(12, 0)$, $C'(15, -3)$
 (c) $A'(0, 3)$, $B'(-6, 9)$, $C'(-3, 12)$ (d) $A'(-3, 0)$, $B'(-9, 6)$, $C'(-12, 3)$
10. a dilatation with centre O and factor $k_1 k_2$.

Exercise 9-9

1. (a) $(x, y) \rightarrow (-x, y)$, reflection in the y -axis
 (b) $(x, y) \rightarrow (-x, -y)$, rotation through 180° about the origin
 (c) $(x, y) \rightarrow (-y, x)$, rotation through 90° about the origin
2. (a) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
3. (a) $(x, y) \rightarrow (2x, y)$ (b) $(x, y) \rightarrow (3x, 4y)$ (c) $(x, y) \rightarrow (x+y, y)$ (d) $(x, y) \rightarrow (x, -2x+y)$
4. $\begin{pmatrix} 6 & 1 \\ 3 & -1 \end{pmatrix}$
5. (a) $\begin{pmatrix} 0 & -4 \\ 4 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} -4 & 0 \\ 0 & 4 \end{pmatrix}$

Review Exercise

1. (a) $\overline{AB} = [2, 1]$, $\overline{CD} = [-2, 3]$, $\overline{CE} = [-5, 0]$, $\overline{OP} = [-2, 3]$, $\overline{QP} = [3, 0]$, $\overline{RS} = [2, 1]$, $\overline{ST} = [1, 2]$
 (b) $\overline{AB} = \overline{RS}$, $\overline{CD} = \overline{OP}$
2. (a) [11, 2] (b) [2, 7] (c) [5, 2] (d) $[-10, -4]$ (e) [12, -32] (f) [3, 4.5]
 (g) [10, 6.5] (h) $[-8, -6]$
3. $\vec{u} = \vec{v} + \vec{w}$, $\vec{v} = \vec{u} + (-\vec{w})$, $\vec{w} = (-\vec{v}) + \vec{u}$, $\vec{a} = \vec{d} + (-\vec{e})$, $\vec{b} = (-\vec{a}) + (-\vec{c})$, $\vec{c} = (-\vec{b}) + (-\vec{a})$, $\vec{d} = \vec{a} + \vec{e}$, $\vec{e} = -\vec{a} + \vec{d}$
4. (a) reflection in y -axis (b) dilatation with factor 7 (c) translation
 (d) reflection in $y = x$ (e) rotation through 180° (f) rotation through 90°
5. (i) (a) (4, 4) (b) (1, -1) (c) (9, -4) (d) (5, -0.5) (e) $(a+3, b-1)$
 (ii) (a) (5, 25) (b) $(-10, 0)$ (c) (30, -15) (d) (10, 2.5) (e) $(5a, 5b)$
 (iii) (a) (1, -5) (b) $(-2, 0)$ (c) (6, 3) (d) (2, -0.5) (e) $(a, -b)$
 (iv) (a) $(-1, -5)$ (b) (2, 0) (c) $(-6, 3)$ (d) $(-2, -0.5)$ (e) $(-a, -b)$
6. (a) none (b) centre of rotation (c) points on the mirror line
 (d) centre of dilatation
7. (a) 5, 5 (b) 2, 2 (c) 0, 2 (d) 0, 1 (e) 8, 8 (f) 4, 4
9. (b) $\sqrt{13}$, $\sqrt{17}$, $\sqrt{10}$, $\sqrt{10}$, $\sqrt{13}$, $\sqrt{17}$
10. (a) [7, 50] (b) [0, 0] (c) [15, 40] (d) [14, -26]

13.		$A(4, 3)$	$B(6, 0)$	$C(-1, -5)$	$P(x, y)$
	T	$(6, 8)$	$(8, 5)$	$(1, 0)$	$(x+2, y+5)$
	R	$(-3, 4)$	$(0, 6)$	$(5, -1)$	$(-y, x)$
	F	$(3, 4)$	$(0, 6)$	$(-5, -1)$	(y, x)
	D	$(2, 1.5)$	$(3, 0)$	$(-0.5, -2.5)$	$(\frac{1}{2}x, \frac{1}{2}y)$
	$R \circ T$	$(-8, 6)$	$(-5, 8)$	$(0, 1)$	$(-y-5, x+2)$
	$T \circ R$	$(-1, 9)$	$(2, 11)$	$(7, 4)$	$(-y+2, x+5)$
	$D \circ T$	$(3, 4)$	$(4, 2.5)$	$(0.5, 0)$	$(\frac{1}{2}x+1, \frac{1}{2}y+\frac{5}{2})$
	$F \circ R$	$(4, -3)$	$(6, 0)$	$(-1, 5)$	$(x, -y)$
	$F \circ D$	$(1.5, 2)$	$(0, 3)$	$(-2.5, -0.5)$	$(\frac{1}{2}y, \frac{1}{2}x)$
	$D \circ R$	$(-1.5, 2)$	$(0, 3)$	$(2.5, -0.5)$	$(-\frac{1}{2}y, \frac{1}{2}x)$
	$T \circ T$	$(8, 13)$	$(10, 10)$	$(3, 5)$	$(x+4, y+10)$

14. (a) (i) $(x, y) \rightarrow (2x, -3y)$ (ii) $(x, y) \rightarrow (x+y, x)$ (iii) $(x, y) \rightarrow (x+2y, y)$

(b) $\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$

REVIEW AND PREVIEW TO CHAPTER 10

Exercise 1

1. $7x - y = 9$ 2. $2x - 3y + 15 = 0$ 3. $2x + y + 6 = 0$ 4. $y = 3x$
 5. $2x - 3y = 6$ 6. $2x + 3y + 21 = 0$ 7. $3x - y + 4 = 0$ 8. $x - y + 2 = 0$
 9. $3x - y + 11 = 0$ 10. $x + y + 2 = 0$

Exercise 2

1. 70° 2. 71° 3. 91° 4. 45° 5. 50° 6. 40° 7. 75° 8. 70°

Exercise 3

- 1 and 11, 3 and 4, 5 and 7, 6 and 12, 10 and 13

CHAPTER 10

Exercise 10-1

1. (a) radius (b) chord (c) radius (d) diameter (e) major arc
 (f) central angle (g) isosceles (h) 90°
 5. $\sqrt{7}$ cm 6. 13 cm

Exercise 10-2

1. (a) $\angle ARC, \angle ASC$ (b) $\angle AKC, \angle ALC, \angle AMC$ (c) $\angle AYC$
 2. (a) 96° (b) $65^\circ, 65^\circ, 65^\circ$ (c) $30^\circ, 80^\circ, 70^\circ$ (d) $40^\circ, 25^\circ, 60^\circ$ (e) $100^\circ, 200^\circ$
 (f) $50^\circ, 70^\circ, 70^\circ, 20^\circ$
 3. $39^\circ, 98^\circ$ 4. $56^\circ, 49^\circ, 49^\circ, 105^\circ$
 8. (b) (i) 7.5 (ii) $2\sqrt{6}$ (iii) 3 or 4

Exercise 10-3

1. (a) $80^\circ, 110^\circ$ (b) $60^\circ, 120^\circ, 65^\circ$ (c) $80^\circ, 80^\circ, 20^\circ$ (d) $100^\circ, 80^\circ, 100^\circ$
 2. yes 3. yes
 4. (a) $AFPE, BDPF, CEPD, ABDE, BCEF, CAFD$ (b) $PQST$ (c) $JNLK$
 6. (b) 14°

Exercise 10-4

1. (a) 1 (b) 2 (c) 0
 2. (a) 4 (b) 5 (c) $\sqrt{2}$ (d) 8 (e) 9, $55^\circ, 35^\circ, 55^\circ$
 (f) 8, $70^\circ, 20^\circ, 20^\circ$ (g) 40° (h) 12 (i) $6\sqrt{2}$
 5. 24 cm

6. (a) yes (b) yes (c) yes (d) yes (e) yes (f) no
 8. 74° 13. $18\frac{6}{13}$ cm

Exercise 10-5

1. (a) $65^\circ, 40^\circ$ (b) $105^\circ, 75^\circ$ (c) $45^\circ, 35^\circ, 35^\circ, 80^\circ$ (d) $70^\circ, 50^\circ, 50^\circ, 60^\circ$
 (e) $40^\circ, 50^\circ, 40^\circ$ (f) $60^\circ, 60^\circ, 60^\circ, 50^\circ, 50^\circ, 70^\circ$
 2. $32^\circ, 88^\circ, 60^\circ$ 3. 79°
 10. (a) 6 (b) 2.25 (c) $6\sqrt{2}$
 11. 4 cm
 12. (b) 2

Exercise 10-6

1. (a) $3x + 4y - 25 = 0$ (b) $x - \sqrt{3}y + 2 = 0$ (c) $4x + 7y + 65 = 0$ (d) $4x - 2y - 5\sqrt{2} = 0$
 (e) $3x + y - 10 = 0$ (f) $x_1x + y_1y - r^2 = 0$
 2. (a) $\sqrt{7}$ (b) 2 (c) 8 (d) $2\sqrt{11}$ (e) $\sqrt{\frac{5}{3}}$
 (f) $\sqrt{x_1^2 + y_1^2 - r^2}$

Review Exercise

1. (a) 34° (b) $75^\circ, 35^\circ, 70^\circ$ (c) $75^\circ, 95^\circ$ (d) $90^\circ, 50^\circ, 35^\circ$ (e) $70^\circ, 110^\circ, 70^\circ$
 (f) $30^\circ, 40^\circ, 40^\circ, 60^\circ$
 2. (a) 10 (b) 4, 4 (c) $30^\circ, 60^\circ, 60^\circ$ (d) $30^\circ, 70^\circ$
 (e) $65^\circ, 60^\circ, 60^\circ, 55^\circ, 70^\circ$ (f) $60^\circ, 30^\circ, 30^\circ$
 3. $6\sqrt{3}$ 5. $42^\circ, 85^\circ$ 6. $29^\circ, 83^\circ, 37^\circ, 46^\circ, 68^\circ$
 14. (b) 12 cm
 18. (a) $\frac{2}{3}$ cm (b) $2\sqrt{10}$ cm
 19. $3x - y - 20 = 0$ 20. $2\sqrt{6}$

REVIEW AND PREVIEW TO CHAPTER 11

Exercise 1

1. 2.70 2. 2.14 3. 12.6 4. 8.10 5. 7.04 6. 3.14

Exercise 2

1. 9.43 2. 10.9 3. 15.0 4. 6.71 5. 24.0 6. 70.7 7. 6.37 8. 23.1 9. 1.98

Exercise 3

1. 2.2 2. 5.09 3. 8.87 4. 15.9 5. 10.5 6. 0.556 7. 9.71 8. 5.22 9. 1.70

CHAPTER 11

Exercise 11-1

1. (a) $\sin \theta = \frac{5}{\sqrt{41}}, \cos \theta = \frac{4}{\sqrt{41}}, \tan \theta = \frac{5}{4}, \csc \theta = \frac{\sqrt{41}}{5}, \sec \theta = \frac{\sqrt{41}}{4}, \cot \theta = \frac{4}{5}$
 (b) $\sin \theta = \frac{2}{\sqrt{13}}, \cos \theta = \frac{3}{\sqrt{13}}, \tan \theta = \frac{2}{3}, \csc \theta = \frac{\sqrt{13}}{2}, \sec \theta = \frac{\sqrt{13}}{3}, \cot \theta = \frac{3}{2}$
 (c) $\sin \theta = \frac{4}{7}, \cos \theta = \frac{\sqrt{33}}{7}, \tan \theta = \frac{4}{\sqrt{33}}, \csc \theta = \frac{7}{4}, \sec \theta = \frac{7}{\sqrt{33}}, \cot \theta = \frac{\sqrt{33}}{4}$
 (d) $\sin \theta = \frac{7}{\sqrt{74}}, \cos \theta = \frac{5}{\sqrt{74}}, \tan \theta = \frac{7}{5}, \csc \theta = \frac{\sqrt{74}}{7}, \sec \theta = \frac{\sqrt{74}}{5}, \cot \theta = \frac{5}{7}$
 (e) $\sin \theta = \frac{1}{\sqrt{2}}, \cos \theta = \frac{1}{\sqrt{2}}, \tan \theta = 1, \csc \theta = \sqrt{2}, \sec \theta = \sqrt{2}, \cot \theta = 1$

- (f) $\sin \theta = \frac{\sqrt{3}}{2}$, $\cos \theta = \frac{1}{2}$, $\tan \theta = 3$, $\csc \theta = \frac{2}{\sqrt{3}}$, $\sec \theta = 2$, $\cot \theta = \frac{1}{\sqrt{3}}$
- (g) $\sin \theta = \frac{3}{5}$, $\cos \theta = \frac{4}{5}$, $\tan \theta = \frac{3}{4}$, $\csc \theta = \frac{5}{3}$, $\sec \theta = \frac{5}{4}$, $\cot \theta = \frac{4}{3}$
- (h) $\sin \theta = \frac{8}{17}$, $\cos \theta = \frac{15}{17}$, $\tan \theta = \frac{8}{15}$, $\csc \theta = \frac{17}{8}$, $\sec \theta = \frac{17}{15}$, $\cot \theta = \frac{15}{8}$
2. (a) 10 , $\sin \theta = \frac{3}{5}$, $\cos \theta = \frac{4}{5}$, $\tan \theta = \frac{3}{4}$, $\csc \theta = \frac{5}{3}$, $\sec \theta = \frac{5}{4}$, $\cot \theta = \frac{4}{3}$
- (b) 13 , $\sin \theta = \frac{12}{13}$, $\cos \theta = \frac{5}{13}$, $\tan \theta = \frac{12}{5}$, $\csc \theta = \frac{13}{12}$, $\sec \theta = \frac{13}{5}$, $\cot \theta = \frac{5}{12}$
- (c) 15 , $\sin \theta = \frac{15}{17}$, $\cos \theta = \frac{8}{17}$, $\tan \theta = \frac{15}{8}$, $\csc \theta = \frac{17}{15}$, $\sec \theta = \frac{17}{8}$, $\cot \theta = \frac{8}{15}$
- (d) 15 , $\sin \theta = \frac{3}{5}$, $\cos \theta = \frac{4}{5}$, $\tan \theta = \frac{3}{4}$, $\csc \theta = \frac{5}{3}$, $\sec \theta = \frac{5}{4}$, $\cot \theta = \frac{4}{3}$
3. (a) 5 , $\sin A = \frac{4}{5}$, $\cos A = \frac{3}{5}$, $\tan A = \frac{4}{3}$, $\csc A = \frac{5}{4}$, $\sec A = \frac{5}{3}$, $\cot A = \frac{3}{4}$
 $\sin B = \frac{3}{5}$, $\cos B = \frac{4}{5}$, $\tan B = \frac{3}{4}$, $\csc B = \frac{5}{3}$, $\sec B = \frac{5}{4}$, $\cot B = \frac{4}{3}$
- (b) 12 , $\sin A = \frac{12}{13}$, $\cos A = \frac{5}{13}$, $\tan A = \frac{12}{5}$, $\csc A = \frac{13}{12}$, $\sec A = \frac{13}{5}$, $\cot A = \frac{5}{12}$
 $\sin B = \frac{5}{13}$, $\cos B = \frac{12}{13}$, $\tan B = \frac{5}{12}$, $\csc B = \frac{13}{5}$, $\sec B = \frac{13}{12}$, $\cot B = \frac{12}{5}$
- (c) 8 , $\sin B = \frac{8}{17}$, $\cos B = \frac{15}{17}$, $\tan B = \frac{8}{15}$, $\csc B = \frac{17}{8}$, $\sec B = \frac{17}{15}$, $\cot B = \frac{15}{8}$
 $\sin C = \frac{15}{17}$, $\cos C = \frac{8}{17}$, $\tan C = \frac{15}{8}$, $\csc C = \frac{17}{15}$, $\sec C = \frac{17}{8}$, $\cot C = \frac{8}{15}$
- (d) 8 , $\sin P = \frac{4}{5}$, $\cos P = \frac{3}{5}$, $\tan P = \frac{4}{3}$, $\csc P = \frac{5}{4}$, $\sec P = \frac{5}{3}$, $\cot P = \frac{3}{4}$
 $\sin R = \frac{3}{5}$, $\cos R = \frac{4}{5}$, $\tan R = \frac{3}{4}$, $\csc R = \frac{5}{3}$, $\sec R = \frac{5}{4}$, $\cot R = \frac{4}{3}$
4. $\cos \theta = \frac{\sqrt{3}}{2}$, $\tan \theta = \frac{1}{\sqrt{3}}$, $\csc \theta = 2$, $\sec \theta = \frac{2}{\sqrt{3}}$, $\cot \theta = \sqrt{3}$
5. $\sin \theta = \frac{12}{13}$, $\cos \theta = \frac{5}{13}$, $\tan \theta = \frac{12}{5}$, $\csc \theta = \frac{13}{12}$, $\cot \theta = \frac{5}{12}$
6. $\sin \theta = \frac{2}{\sqrt{5}}$, $\cos \theta = \frac{1}{\sqrt{5}}$, $\csc \theta = \frac{\sqrt{5}}{2}$, $\sec \theta = \sqrt{5}$, $\cot \theta = \frac{1}{2}$
7. $\sin \theta = \frac{1}{2}$, $\tan \theta = \frac{1}{\sqrt{3}}$, $\csc \theta = 2$, $\sec \theta = \frac{2}{\sqrt{3}}$, $\cot \theta = \sqrt{3}$
8. (a) $\sin B = \frac{b}{a}$, $\cos B = \frac{c}{a}$, $\tan B = \frac{b}{c}$, $\csc B = \frac{a}{b}$, $\sec B = \frac{a}{c}$, $\cot B = \frac{c}{b}$
 $\sin C = \frac{c}{a}$, $\cos C = \frac{b}{a}$, $\tan C = \frac{c}{b}$, $\csc C = \frac{a}{c}$, $\sec C = \frac{a}{b}$, $\cot C = \frac{b}{c}$
- (b) (i) $\cos B = \sin C$ (ii) $\sin B = \cos C$ (iii) $\tan C = \cot B$
9. (a) $\sin A = \frac{5}{\sqrt{26}}$, $\cos A = \frac{1}{\sqrt{26}}$, $\tan A = 5$, $\csc A = \frac{\sqrt{26}}{5}$, $\sec A = \sqrt{26}$, $\cot A = \frac{1}{5}$
 $\sin B = \frac{1}{\sqrt{26}}$, $\cos B = \frac{5}{\sqrt{26}}$, $\tan B = \frac{1}{5}$, $\csc B = \sqrt{26}$, $\sec B = \frac{\sqrt{26}}{5}$, $\cot B = 5$

Exercise 11-2

1. $\frac{\sqrt{6}}{8}$ 2. 1 3. $\frac{\sqrt{3}}{2}$ 4. 1 5. $\frac{\sqrt{3}}{2}$ 6. 10

Exercise 11-3

1. (a) 0.4226 (b) 3.732 (c) 1.000 (d) 2.130 (e) 2.790 (f) 0.7431 (g) 1.012 (h) 0.5878
 (i) 1.133 (j) 0.2588 (k) 3.078 (l) 0.9511 (m) 0.7536 (n) 1.236 (o) 0.6018 (p) 1.600
2. (a) 65° (b) 60° (c) 66° (d) 25° (e) 39° (f) 16° (g) 65° (h) 79°
 (i) 35° (j) 35° (k) 25° (l) 50°

Exercise 11-4

1. (a) 39° (b) 53° (c) 60° (d) 35° (e) 45° (f) 57° (g) 65° (h) 28°
2. (a) 142 (b) 13.4 (c) 35.5 (d) 89.9 (e) 51.8 (f) 23.5 (g) 11.2 (h) 161
3. (a) $\angle A = 39^\circ$, $\angle C = 51^\circ$, $b = 32$ (b) $\angle A = 45^\circ$, $a = 80.6$, $c = 80.6$ (c) $\angle A = 25^\circ$, $a = 6.34$, $b = 13.6$
 (d) $\angle B = 52^\circ$, $\angle C = 38^\circ$, $c = 11.7$ (e) $b = 5.39$, $\angle A = 50^\circ$, $\angle C = 40^\circ$ (f) $c = 3.91$, $\angle A = 40^\circ$, $\angle B = 50^\circ$
 (g) $\angle A = 27^\circ$, $a = 3.72$, $c = 7.31$ (h) $\angle B = 50^\circ$, $a = 8.87$, $b = 6.79$
4. (a) $\angle B = 47^\circ$, $\angle C = 43^\circ$, $c = 10.2$ (b) $\angle F = 55^\circ$, $d = 70.0$, $e = 122$
 (c) $\angle G = 10^\circ$, $g = 21$, $h = 118$ (d) $\angle J = 56^\circ$, $\angle K = 34^\circ$, $l = 33.2$
5. (a) $\angle B = 35^\circ$, $\angle C = 55^\circ$, $c = 3.07$ (b) $\angle A = 33^\circ$, $\angle C = 57^\circ$, $b = 39.2$
 (c) $\angle C = 52^\circ$, $a = 5.52$, $b = 3.40$ (d) $\angle B = 23^\circ$, $a = 9.85$, $b = 4.18$
6. (a) 165 (b) 90.2

Exercise 11-5

1. 166 m 2. 78° 3. 369 m 4. 187 m 5. 89 m 6. 57.1 m 7. 39°
8. (a) 29 m (b) 37 m 9. 196 m 10. 85 m 11. 7.5 km 12. 380 m 13. (a) 124 m
- (b) 13.8 m 14. 150 km 15. 2412 m 16. 111 m 17. 21 m 18. 285 m 19. 35 m
20. 79.7 m

Review Exercise

1. (a) $\sin \theta = 0.8$, $\cos \theta = 0.6$, $\tan \theta = 1.33$, $\csc \theta = 1.25$, $\sec \theta = 1.67$, $\cot \theta = 0.75$
 (b) $\sin \theta = \frac{15}{17}$, $\cos \theta = \frac{8}{17}$, $\tan \theta = \frac{5}{4}$, $\csc \theta = \frac{17}{15}$, $\sec \theta = \frac{17}{8}$, $\cot \theta = \frac{4}{5}$
 (c) $\sin \theta = \frac{5}{\sqrt{29}}$, $\cos \theta = \frac{2}{\sqrt{29}}$, $\tan \theta = 2.5$, $\csc \theta = \frac{\sqrt{29}}{5}$, $\sec \theta = \frac{\sqrt{29}}{2}$, $\cot \theta = 0.4$
2. $\sin \theta = \frac{12}{13}$, $\cos \theta = \frac{5}{13}$, $\tan \theta = \frac{12}{5}$, $\sec \theta = \frac{13}{5}$, $\cot \theta = \frac{5}{12}$
3. (a) 1 (b) 1 (c) $\frac{\sqrt{2} + \sqrt{6}}{2}$ (d) $\frac{2}{3}$ (e) $\frac{1}{4}$
4. (a) $\angle A = 52^\circ$, $b = 94$, $c = 152$ (b) $\angle B = 38^\circ$, $\angle C = 52^\circ$, $b = 71.6$ (c) $\angle C = 23^\circ$, $a = 25.6$, $b = 23.6$
 (d) $\angle B = 45^\circ$, $\angle C = 45^\circ$, $b = 1000$ (e) $\angle A = 20^\circ$, $a = 45$, $c = 133$ (f) $\angle C = 52^\circ$, $a = 905$, $b = 1158$
5. 81 m 6. 385 m 7. 6040 m 8. 173.4 m 9. 1520 m 10. 425 m

REVIEW AND PREVIEW TO CHAPTER 12

Exercise 1

1. 3 2. 10 3. 13 4. 25 5. $\sqrt{58}$ 6. $\sqrt{29}$ 7. 17 8. $\sqrt{74}$
9. $\sqrt{74}$ 10. $\sqrt{130}$

Exercise 2

1. (a) (1.7, 1.0) (b) (-2.57, 3.06) (c) (-5.64, -2.05) (d) (1.71, -2.46)
2. (a) 63 (b) 127 (c) 59 (d) 315

CHAPTER 12

Exercise 12-1

1. (a) $2\sqrt{13}$, $\sin \theta = \frac{2}{\sqrt{13}}$, $\cos \theta = \frac{3}{\sqrt{13}}$, $\tan \theta = \frac{2}{3}$, $\csc \theta = \frac{\sqrt{13}}{2}$, $\sec \theta = \frac{\sqrt{13}}{3}$, $\cot \theta = \frac{3}{2}$
 (b) $\sqrt{13}$, $\sin \theta = \frac{3}{\sqrt{13}}$, $\cos \theta = -\frac{2}{\sqrt{13}}$, $\tan \theta = -\frac{3}{2}$, $\csc \theta = \frac{\sqrt{13}}{3}$, $\sec \theta = -\frac{\sqrt{13}}{2}$, $\cot \theta = -\frac{2}{3}$
 (c) $\sqrt{13}$, $\sin \theta = -\frac{2}{\sqrt{13}}$, $\cos \theta = -\frac{3}{\sqrt{13}}$, $\tan \theta = \frac{2}{3}$, $\csc \theta = -\frac{\sqrt{13}}{2}$, $\sec \theta = -\frac{\sqrt{13}}{2}$, $\cot \theta = \frac{3}{2}$
 (d) $\sqrt{41}$, $\sin \theta = -\frac{5}{\sqrt{41}}$, $\cos \theta = \frac{4}{\sqrt{41}}$, $\tan \theta = -\frac{5}{4}$, $\csc \theta = -\frac{\sqrt{41}}{5}$, $\sec \theta = \frac{\sqrt{41}}{4}$, $\cot \theta = -\frac{4}{5}$
 (e) $6\sqrt{2}$, $\sin \theta = -\frac{1}{\sqrt{2}}$, $\cos \theta = \frac{1}{\sqrt{2}}$, $\tan \theta = -1$, $\csc \theta = -\sqrt{2}$, $\sec \theta = \sqrt{2}$, $\cot \theta = -1$
 (f) 10, $\sin \theta = \frac{3}{5}$, $\cos \theta = -\frac{4}{5}$, $\tan \theta = -\frac{3}{4}$, $\csc \theta = \frac{5}{3}$, $\sec \theta = -\frac{5}{4}$, $\cot \theta = -\frac{4}{3}$
2. (a) $\sin \theta = \frac{4}{5}$, $\cos \theta = \frac{3}{5}$, $\tan \theta = \frac{4}{3}$, $\csc \theta = \frac{5}{4}$, $\sec \theta = \frac{5}{3}$, $\cot \theta = \frac{3}{4}$
 (b) $\sin \theta = \frac{12}{13}$, $\cos \theta = -\frac{5}{13}$, $\tan \theta = -\frac{12}{5}$, $\csc \theta = \frac{13}{12}$, $\sec \theta = -\frac{13}{5}$, $\cot \theta = -\frac{5}{12}$
 (c) $\sin \theta = -\frac{8}{17}$, $\cos \theta = -\frac{15}{17}$, $\tan \theta = \frac{8}{15}$, $\csc \theta = -\frac{17}{8}$, $\sec \theta = -\frac{17}{15}$, $\cot \theta = \frac{15}{8}$
 (d) $\sin \theta = -\frac{7}{25}$, $\cos \theta = \frac{24}{25}$, $\tan \theta = -\frac{7}{24}$, $\csc \theta = -\frac{25}{7}$, $\sec \theta = \frac{25}{24}$, $\cot \theta = -\frac{24}{7}$
 (e) $\sin \theta = \frac{9}{\sqrt{106}}$, $\cos \theta = \frac{5}{\sqrt{106}}$, $\tan \theta = \frac{9}{5}$, $\csc \theta = \frac{\sqrt{106}}{9}$, $\sec \theta = \frac{\sqrt{106}}{5}$, $\cot \theta = \frac{5}{9}$

- (f) $\sin \theta = -\frac{7}{\sqrt{58}}, \cos \theta = \frac{3}{\sqrt{58}}, \tan \theta = -\frac{7}{3}, \csc \theta = -\frac{\sqrt{58}}{7}, \sec \theta = \frac{\sqrt{58}}{3}, \cot \theta = -\frac{3}{7}$
- (g) $\sin \theta = \frac{7}{\sqrt{65}}, \cos \theta = -\frac{4}{\sqrt{65}}, \tan \theta = -\frac{7}{4}, \csc \theta = \frac{\sqrt{65}}{7}, \sec \theta = -\frac{\sqrt{65}}{4}, \cot \theta = -\frac{4}{7}$
- (h) $\sin \theta = -\frac{2}{\sqrt{29}}, \cos \theta = -\frac{5}{\sqrt{29}}, \tan \theta = \frac{2}{5}, \csc \theta = -\frac{\sqrt{29}}{2}, \sec \theta = -\frac{\sqrt{29}}{5}, \cot \theta = \frac{5}{2}$
3. $(-4, -3), \frac{3}{4}$ 4. $(1, -1), -\frac{1}{\sqrt{2}}, \sqrt{2}$
5. $x = 12, -12, \cos \theta = \frac{12}{13}, -\frac{12}{13}, \tan \theta = \frac{5}{12}, -\frac{5}{12}, \csc \theta = \frac{13}{5}, \sec \theta = \frac{13}{12}, -\frac{13}{12}, \cot \theta = \frac{12}{5}, -\frac{12}{5}$
6. $r = 5, 10, 15, \sin \theta = \frac{4}{5}, \cos \theta = \frac{3}{5}, \tan \theta = \frac{4}{3}, \csc \theta = \frac{5}{4}, \sec \theta = \frac{5}{3}, \cot \theta = \frac{3}{4}$ for all points.
7. (a) $\sin \theta = \frac{b}{\sqrt{a^2 + b^2}}, \cos \theta = \frac{a}{\sqrt{a^2 + b^2}}$ (b) 1 (c) $\frac{b}{a}$

Exercise 12-2

1. (a) 0.5000 (b) -11.4301 (c) 0.9397 (d) -0.9397 (e) -0.6428 (f) -0.1736
 (g) -1.4281 (h) 0.4226 (i) -0.5000 (j) 0.1736 (k) -0.7002 (l) -0.1736
 (m) -5.6713 (n) -0.7660 (o) -0.7071
2. (a) -0.6691 (b) -0.5736 (c) -0.3907 (d) -1.000 (e) -0.9925 (f) -0.9004
 (g) -0.6157 (h) -0.7660 (i) 1.000 (j) -0.8660 (k) -0.9063 (l) -0.9397
 (m) 3.2709 (n) 0.7431 (o) -0.3420
3. (a) $61^\circ, 119^\circ$ (b) $50^\circ, 230^\circ$ (c) $200^\circ, 340^\circ$ (d) $127^\circ, 233^\circ$ (e) $135^\circ, 315^\circ$ (f) $30^\circ, 330^\circ$
 (g) $82^\circ, 98^\circ$ (h) $100^\circ, 280^\circ$ (i) $141^\circ, 219^\circ$ (j) $225^\circ, 315^\circ$

Exercise 12-3

1. (a) 49.5 (b) 46.9 (c) 12.9 (d) 37.8 (e) 8.85 (f) 59.2
 2. (a) 56.5 (b) 17.8 (c) 40.3 (d) $a = 94.1, b = 56.2$ (e) $\angle A = 48^\circ$
 3. (a) $\angle A = 23^\circ, b = 253, c = 247$ (b) $\angle C = 66^\circ, a = 65.4, c = 76.9$ (c) $\angle C = 117^\circ, a = 101, b = 54.3$
 (d) $\angle A = 63^\circ, b = 11.6, c = 12.8$ (e) $\angle B = 48^\circ, a = 1.53, c = 0.721$ (f) $\angle B = 25^\circ, \angle C = 45^\circ, AC = 6.88$

Exercise 12-4

1. (a) 7.70 (b) 5.18 (c) 9.62
 2. (a) 30° (b) 32° (c) 100°
 3. (a) 27.6 (b) 3.35 (c) 405 (d) 41° (e) 104°
 4. (a) $\angle A = 70^\circ, b = 9.37, \angle C = 61^\circ$ (b) $\angle A = 57^\circ, \angle B = 79^\circ, \angle C = 44^\circ$ (c) $\angle A = 34^\circ, \angle B = 25^\circ, c = 14.6^\circ$
 (d) $\angle A = 49^\circ, b = 19.3, \angle C = 63^\circ$ (e) $\angle A = 63^\circ, \angle B = 35^\circ, \angle C = 82^\circ$

Exercise 12-5

1. 5° 2. 7° 3. 6.84 cm 4. 49 m 5. 748 m
 6. (a) 210 km (b) 197 km
 7. 335 100 8. 52 m 9. 11° 10. 69 cm

Exercise 12-6

1. (a) 135° (b) 120° (c) 240° (d) 150° (e) 270°
 2. (a) $\frac{\pi}{3}$ (b) $\frac{5\pi}{6}$ (c) $\frac{\pi}{2}$ (d) $\frac{11\pi}{3}$ (e) $\frac{-4\pi}{3}$
 3. 0.0175
 4. (a) 286° (b) 138° (c) 212° (d) 8.3° (e) $12\ 200^\circ$
 5. (a) 0.611 (b) 1.22 (c) 2.97 (d) 3.84 (e) 8.29
 6. (a) 1.14 (b) 23.6 (c) 4 (d) 15.4 (e) 1.90 (f) 1
 7. 2π m

Exercise 12-8

1. $\frac{\pi}{3}, \frac{2\pi}{3}$ 2. $\frac{\pi}{3}, \frac{5\pi}{3}$ 3. $\frac{7\pi}{6}, \frac{11\pi}{6}$ 4. $\frac{\pi}{4}, \frac{5\pi}{4}$ 5. $\frac{3\pi}{4}, \frac{7\pi}{4}$
 6. $0, \pi, 2\pi$ 7. $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ 8. $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ 9. $0, \pi, \frac{3\pi}{2}, 2\pi$ 10. $\frac{\pi}{2}, \frac{3\pi}{2}$

Review Exercise

- (a) $\frac{12}{13}$ (b) $-\frac{5}{12}$ (c) $-\frac{12}{5}$ (d) $-\frac{13}{5}$
- (a) $\frac{5}{4}$, $-\frac{5}{4}$
- (a) $\frac{1}{2}\pi$ (b) $\frac{\sqrt{2}}{3}\pi$ (c) -1 (d) 1
- (a) $\frac{1}{3}$ (b) $\frac{4}{3}$ (c) 1.31 (d) 0.49 (e) 9.95
- (a) 115° (b) 201° (c) 720° (d) 330° (e) 630°
- (a) 1 (b) $\frac{12}{\pi}$ (c) $\frac{7\pi}{3}$
- (a) 10.7 (b) 14.7 (c) 72° (d) 27.1 (e) 37.6 (f) 48°
- 138 km
- (b) $45^\circ, 225^\circ$
- (a) $0, \pi, 2\pi$ (b) $0, \frac{\pi}{2}, 2\pi$ (c) $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ (d) $0, \frac{\pi}{2}, \pi, 2\pi$

CHAPTER 13

Exercise 13-1

- (a) 6 (b) 4 (c) $\frac{2}{3}$ (d) $\frac{1}{3}$
- (a) $\frac{2}{5}$ (b) $\frac{3}{5}$
- The outcomes are not equally likely.
- (a) $\{6, 8, 9, 3, 5, 1, 2\}$ (b) $\{9, 3, 5, 1\}$ (c) $\frac{4}{7}$ (d) $\frac{3}{7}$
- (a) $\{S, T, A, T, I, S, T, I, C, S\}$ (b) $\frac{3}{10}$ (c) $\frac{7}{10}$
- (a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $\frac{1}{6}$ (d) $\frac{2}{3}$ (e) $\frac{1}{2}$ (f) $\frac{5}{6}$
- (a) $\{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$ (b) (i) $\frac{1}{8}$ (ii) $\frac{3}{8}$ (iii) $\frac{1}{2}$
- $\frac{1}{9}$
- (a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{3}{4}$ (d) $\frac{1}{13}$ (e) $\frac{1}{52}$ (f) $\frac{1}{26}$ (g) $\frac{12}{13}$ (h) $\frac{3}{13}$
- (a) $\frac{1}{2}$ (b) $\frac{1}{2}$ (c) $\frac{1}{5}$ (d) $\frac{1}{4}$ (e) $\frac{1}{10}$ (f) $\frac{3}{10}$
- (a) $\frac{1}{2500}$ (b) $\frac{1}{25000}$ (c) $\frac{1}{12500}$

Exercise 13-2

- (a) 0.983 (b) 0.766
- (a) 0.981 (b) 0.95 (c) 0.882 (d) 0.739
- (a) (i) 0.915 (ii) 0.596 (iii) 0.064 (b) (i) 0.955 (ii) 0.764 (iii) 0.137

Exercise 13-3

- (a), (d), (e)
- (a) $\frac{1}{4}$ (b) $\frac{5}{12}$ (c) $\frac{7}{12}$ (d) $\frac{5}{6}$ (e) $\frac{3}{4}$
- (a) $\frac{2}{13}$ (b) $\frac{15}{52}$
- (a) $\frac{2}{9}$ (b) $\frac{1}{12}$
- (a) $\frac{5}{18}$ (b) $\frac{1}{12}$ (c) $\frac{5}{12}$
- (a) 0 (b) $\frac{1}{216}$ (c) $\frac{7}{72}$ (d) $\frac{1}{36}$ (e) $\frac{5}{108}$
- (a) $\frac{3}{8}$ (b) $\frac{1}{4}$ (c) $\frac{11}{16}$

Exercise 13-4

- $\frac{1}{8}$
- (b) $\frac{1}{4}$
- (a) $\frac{1}{36}$ (b) $\frac{1}{18}$
- $\frac{1}{36}$
- (a) $\frac{25}{676}$ (b) $\frac{441}{676}$
- $\frac{1}{1296}$
- (a) $\frac{36}{169}$ (b) $\frac{9}{169}$ (c) $\frac{18}{169}$ (d) $\frac{36}{169}$ (e) $\frac{49}{169}$
- (a) $\frac{5}{26}$ (b) $\frac{1}{26}$ (c) $\frac{3}{26}$ (d) $\frac{3}{13}$ (e) $\frac{7}{26}$
- (a) $\frac{1}{16}$ (b) $\frac{1}{16}$ (c) $\frac{1}{8}$ (d) $\frac{1}{169}$ (e) $\frac{2}{169}$ (f) $\frac{1}{2704}$ (g) $\frac{1}{52}$
- (a) $\frac{3}{51}$ (b) $\frac{13}{204}$ (c) $\frac{13}{102}$ (d) $\frac{1}{221}$ (e) $\frac{2}{221}$ (f) 0 (g) 0
- (a) $\frac{1}{1250}$ (b) $\frac{3}{6248750}$ (c) $\frac{1}{5205208750}$
- $\frac{1}{2}$

Exercise 13-7

2. (a) mean: 11.4, median: 12, Mode: 12 (b) mean: 12.3, median: 12.5, modes: 10, 13
 (c) mean: 15, median: 15.5 (d) mean: 4.8, median: 5, mode: 2
 5. (a) mean: \$42 916.67, median: \$35 000, mode: \$30 000
 6. (a) mean: 10.9, median: 10.5, mode: 9

Exercise 13-8

1. (a) 21 (b) 43 (c) 53
 2. (a) 2.8 (b) 4.4
 3. (a) 16% (b) 48%
 4. (a) 2% (b) 16%
 5. (a) 48% (b) 2% (c) 480
 6. (a) 84% (b) 2% (c) 12 (d) 408
 8. (a) mean: 5, $\sigma = 1.4$ (b) mean: 7, $\sigma = 0.58$

Review Exercise

1. (a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) 0 (d) 1
 2. (a) $\frac{3}{10}$ (b) $\frac{3}{8}$ (c) $\frac{1}{8}$ (d) $\frac{1}{5}$ (e) $\frac{27}{40}$ (f) $\frac{4}{5}$ (g) $\frac{1}{2}$ (h) 0 (i) $\frac{7}{10}$
 3. (a) 123, 132, 213, 231, 312, 321 (b) (i) 1 (ii) $\frac{2}{3}$ (iii) $\frac{1}{2}$ (iv) 0
 4. (a) $\frac{1}{2}$ (b) $\frac{5}{6}$ (c) $\frac{7}{8}$ (d) $\frac{12}{13}$ (e) $\frac{5}{9}$ (f) $\frac{5}{26}$
 5. $\frac{3}{11}$
 6. (a) $\frac{1}{5}$ (b) $\frac{1}{125}$ (c) $\frac{64}{125}$
 7. $\frac{1}{1024}$
 8. (a) $\frac{1}{676}$ (b) $\frac{1}{17576}$ (c) $\frac{1}{456976}$
 9. (a) $\frac{1}{650}$ (b) $\frac{1}{15600}$ (c) $\frac{1}{358800}$
 10. (a) mean: 10.3, median: 10, mode: 5 (b) mean: 16.2, median: 15, mode: 12
 (c) mean: 11.5, median: 10.5, modes: 9, 10
 11. (a) range: 6, :2 (b) range: 15, :4.6
 14. (a) 98% (b) 84% (c) 378 (d) 216
 15. (a) mean: 50, :1.9

TRIGONOMETRIC RATIOS

θ°	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\operatorname{cosec} \theta$
0	0.0000	1.0000	0.0000	—	1.0000	—
1	0.0175	0.9999	0.0175	57.290	1.0001	57.299
2	0.0349	0.9994	0.0349	28.636	1.0006	28.654
3	0.0523	0.9986	0.0524	19.081	1.0014	19.107
4	0.0698	0.9976	0.0699	14.301	1.0024	14.335
5	0.0872	0.9962	0.0875	11.430	1.0038	11.474
6	0.1045	0.9945	0.1051	9.5144	1.0055	9.5668
7	0.1219	0.9926	0.1228	8.1443	1.0075	8.2055
8	0.1392	0.9903	0.1405	7.1154	1.0098	7.1853
9	0.1564	0.9877	0.1584	6.3137	1.0125	6.3924
10	0.1737	0.9848	0.1763	5.6713	1.0154	5.7588
11	0.1908	0.9816	0.1944	5.1445	1.0187	5.2408
12	0.2079	0.9782	0.2126	4.7046	1.0223	4.8097
13	0.2250	0.9744	0.2309	4.3315	1.0263	4.4454
14	0.2419	0.9703	0.2493	4.0108	1.0306	4.1336
15	0.2588	0.9659	0.2680	3.7320	1.0353	3.8637
16	0.2756	0.9613	0.2867	3.4874	1.0403	3.6279
17	0.2924	0.9563	0.3057	3.2708	1.0457	3.4203
18	0.3090	0.9511	0.3249	3.0777	1.0515	3.2361
19	0.3256	0.9455	0.3443	2.9042	1.0576	3.0715
20	0.3420	0.9397	0.3640	2.7475	1.0642	2.9238
21	0.3584	0.9336	0.3839	2.6051	1.0711	2.7904
22	0.3746	0.9272	0.4040	2.4751	1.0785	2.6695
23	0.3907	0.9205	0.4245	2.3558	1.0864	2.5593
24	0.4067	0.9136	0.4452	2.2460	1.0946	2.4586
25	0.4226	0.9063	0.4663	2.1445	1.1034	2.3662
26	0.4384	0.8988	0.4877	2.0503	1.1126	2.2812
27	0.4540	0.8910	0.5095	1.9626	1.1223	2.2027
28	0.4695	0.8830	0.5317	1.8807	1.1326	2.1300
29	0.4848	0.8746	0.5543	1.8040	1.1433	2.0627
30	0.5000	0.8660	0.5774	1.7320	1.1547	2.0000
31	0.5150	0.8572	0.6009	1.6643	1.1666	1.9416
32	0.5299	0.8481	0.6249	1.6003	1.1792	1.8871
33	0.5446	0.8387	0.6494	1.5399	1.1924	1.8361
34	0.5592	0.8290	0.6745	1.4826	1.2062	1.7883
35	0.5736	0.8192	0.7002	1.4281	1.2208	1.7434
36	0.5878	0.8090	0.7265	1.3764	1.2361	1.7013
37	0.6018	0.7986	0.7536	1.3270	1.2521	1.6616
38	0.6157	0.7880	0.7813	1.2799	1.2690	1.6243
39	0.6293	0.7772	0.8098	1.2349	1.2867	1.5890
40	0.6428	0.7660	0.8391	1.1917	1.3054	1.5557
41	0.6561	0.7547	0.8693	1.1504	1.3250	1.5242
42	0.6691	0.7431	0.9004	1.1106	1.3456	1.4945
43	0.6820	0.7314	0.9325	1.0724	1.3673	1.4663
44	0.6947	0.7193	0.9657	1.0355	1.3902	1.4395
45	0.7071	0.7071	1.0000	1.0000	1.4142	1.4142

TRIGONOMETRIC RATIOS

0°	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\operatorname{cosec} \theta$
46	0.7193	0.6947	1.0355	0.9657	1.4395	1.3902
47	0.7314	0.6820	1.0724	0.9325	1.4663	1.3673
48	0.7431	0.6691	1.1106	0.9004	1.4945	1.3456
49	0.7547	0.6561	1.1504	0.8693	1.5242	1.3250
50	0.7660	0.6428	1.1917	0.8391	1.5557	1.3054
51	0.7772	0.6293	1.2349	0.8098	1.5890	1.2867
52	0.7880	0.6157	1.2799	0.7813	1.6243	1.2690
53	0.7986	0.6018	1.3270	0.7536	1.6616	1.2521
54	0.8090	0.5878	1.3764	0.7265	1.7013	1.2361
55	0.8192	0.5736	1.4281	0.7002	1.7434	1.2208
56	0.8290	0.5592	1.4826	0.6745	1.7883	1.2062
57	0.8387	0.5446	1.5399	0.6494	1.8361	1.1924
58	0.8481	0.5299	1.6003	0.6249	1.8871	1.1792
59	0.8572	0.5150	1.6643	0.6009	1.9416	1.1666
60	0.8660	0.5000	1.7320	0.5774	2.0000	1.1547
61	0.8746	0.4848	1.8040	0.5543	2.0627	1.1433
62	0.8830	0.4695	1.8807	0.5317	2.1300	1.1326
63	0.8910	0.4540	1.9626	0.5095	2.2027	1.1223
64	0.8988	0.4384	2.0503	0.4877	2.2812	1.1126
65	0.9063	0.4226	2.1445	0.4663	2.3662	1.1034
66	0.9136	0.4067	2.2460	0.4452	2.4586	1.0946
67	0.9205	0.3907	2.3558	0.4245	2.5593	1.0864
68	0.9272	0.3746	2.4751	0.4040	2.6695	1.0785
69	0.9336	0.3584	2.6051	0.3839	2.7904	1.0711
70	0.9397	0.3420	2.7475	0.3640	2.9238	1.0642
71	0.9455	0.3256	2.9042	0.3443	3.0715	1.0576
72	0.9511	0.3090	3.0777	0.3249	3.2361	1.0515
73	0.9563	0.2924	3.2708	0.3057	3.4203	1.0457
74	0.9613	0.2756	3.4874	0.2867	3.6279	1.0403
75	0.9659	0.2588	3.7320	0.2680	3.8637	1.0353
76	0.9703	0.2419	4.0108	0.2493	4.1336	1.0306
77	0.9744	0.2250	4.3315	0.2309	4.4454	1.0263
78	0.9782	0.2079	4.7046	0.2126	4.8097	1.0223
79	0.9816	0.1908	5.1445	0.1944	5.2408	1.0187
80	0.9848	0.1737	5.6713	0.1763	5.7588	1.0154
81	0.9877	0.1564	6.3137	0.1584	6.3924	1.0125
82	0.9903	0.1392	7.1154	0.1405	7.1853	1.0098
83	0.9926	0.1219	8.1443	0.1228	8.2005	1.0075
84	0.9945	0.1045	9.5144	0.1051	9.5668	1.0055
85	0.9962	0.0872	11.430	0.0875	11.474	1.0038
86	0.9976	0.0698	14.301	0.0699	14.335	1.0024
87	0.9986	0.0523	19.081	0.0524	19.107	1.0014
88	0.9994	0.0349	28.636	0.0349	28.654	1.0006
89	0.9999	0.0175	57.290	0.0175	57.299	1.0001
90	1.0000	0.0000	—	0.0000	—	1.0000

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